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Fuzzy L-Quotient Ideals

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Abstract

In this paper, fuzzy L-ideal, f-invariant fuzzy L- ideal and fuzzy L-quotient ideal are defined. Also some theorems using f-invariant and fuzzy L-quotient are derived.

Keywords: Fuzzy L-ideals, fuzzy L-coset, fuzzy L- quotient ideals and f-invariant fuzzy L- ideal.

Introduction

L.A.Zadeh [1]. Introduced the concept of fuzzy sets in 1965. Also fuzzy group was introduced by Rosenfield [2]. Yuan and Wu [3] applied the concepts of fuzzy sets in lattice theory. The idea of fuzzy sublattice was introduced by Ajmal [4]. In paper [5], the definition of fuzzy L-ideal, level fuzzy L-ideal, union and intersection of fuzzy L-ideals, theorems, propositions and examples are given. In this present paper, fuzzy L-ideal, f-invariant fuzzy L-ideal and fuzzy L-quotient ideal are introduced. Some homomorphism theorems and lemmas are derived. Some more results related to this topic are also established.

Preliminaries

Fuzzy L-ideal, level fuzzy L-ideal are defined and examples are given.

Definition: 2.1 A fuzzy subset $\mu : L \to [0,1]$ of L is called a fuzzy L-ideal of L if $\forall$ x, y $\in$ L,
(i) $\mu(x \vee y) \geq \min \{ \mu(x), \mu(y)\}$
(ii) $\mu(x \wedge y) \geq \max \{ \mu(x), \mu(y)\}$.

Example: 2.2 Let $L = \{ 0,a,b,1\}$. Let $\mu : L \to [0,1]$ is a fuzzy set in L defined by $\mu(0)$

$= 0.9$, $\mu(a) = 0.5$, $\mu(b) = 0.5$, $\mu(c) = 0.5$, $\mu(1) = 0.5$. Then μ is a fuzzy L-ideal of L.

Definition: 2.3 Let μ be any fuzzy L-ideal of a lattice L and let $t \in [0,1]$. Then $\mu_t = \{ x \in L / \mu(x) \geq t\}$ is called level fuzzy L-ideal of μ.

Example : 2.4 Let L = { 0,a,b,1}. Let $\mu : L \rightarrow [0,1]$ is a fuzzy set in L defined by $\mu(0) = 0.7$, $\mu(a) = 0.5$, $\mu(b) = 0.5$, $\mu(c) = 0.5$, $\mu(1) = 0.5$. Then μ is a fuzzy L-ideal of L. In this example, let t = 0.5.

Then $\mu_t = \mu_{0.5} = \{ a, b, c, 1 \}$.

Fuzzy L-quotient ideals

In this section, some definitions, lemma and theorems on fuzzy L-quotient ideals are derived.

Definition: 3.1

Let μ be any fuzzy L-ideal of a lattice L. Then the fuzzy subset μ_x^* of L, where $x \in L$, defined by $\mu_x^*(y) = \mu[y \wedge x]$, for all $y \in L$, is termed as the fuzzy L-coset determined by x and μ.

Remark: 3.2

If μ is constant, then $L_\mu = \mu^*(0)$.

Theorem: 3.3

Let μ be any fuzzy L-ideal of a lattice L. Then μ_x^*, for all $x \in L$, the fuzzy L-coset of μ in L is also fuzzy L-ideal of L.

Proof:

Given μ be any fuzzy L-ideal of L and μ_x^* is a fuzzy L-coset of x in L/μ.

To prove: μ_x^* is a fuzzy L-ideal.

That is to prove,

i. For all $y, z \in L$,

$$\begin{aligned} \mu_x^*(y \vee z) &= \mu[(y \vee z) \wedge x], \text{ by definition} \\ &= \mu[(y \wedge x) \vee (z \wedge x)] \\ &\geq \min\{\mu(y \wedge x), \mu(z \wedge x)\} \\ &\geq \min\{\mu_x^*(y), \mu_x^*(z)\}. \end{aligned}$$

ii. $\mu_x^*(y \wedge z) = \mu[(y \wedge z) \wedge x]$, by definition

$$\begin{aligned} &= \mu[(y \wedge x) \wedge (z \wedge x)] \\ &\geq \max\{\mu(y \wedge x), \mu(z \wedge x)\} \\ &\geq \max\{ \mu_x^*(y), \mu_x^*(z) \}. \end{aligned}$$

Hence μ_x^* is a fuzzy L-ideal of L.

Lemma: 3.4
If μ is any fuzzy L-ideal of a lattice L, then the following holds:
$\mu(x) = \mu(0) \Leftrightarrow \mu_x^* = \mu_0^*$, where $x \in L$.

Proof:
Let $\mu(x) = \mu(0)$. -------------------- (1)
$\forall\ y \in L,\ \mu(y) \leq \mu(0)$ -------------------- (2)

From (1) and (2), we have $\mu(y) \leq \mu(x)$.

Case (i):
If $\mu(y) < \mu(x)$, then
$\mu(y \wedge x) \geq \max \{ \mu(y), \mu(x) \}$
$= \mu(x)$.

Case (ii):
If $\mu(y) = \mu(x)$, then $x, y \in \mu_t$, where $t = \mu(0)$.

Hence
$\mu(y \wedge x) \geq \max \{ \mu(y), \mu(x) \}$
$= \mu(x)$
$= \mu(0)$.

Therefore
$\mu(y \wedge x) = \mu(0) = \mu(y) = \mu(x)$.

Thus in either case, $\mu(y \wedge x) = \mu(x)$, $\forall\ y \in L$.
(i.e) $\mu_x^*(y) = \mu(x) = \mu_x^*(0)$.

Therefore
$\mu_x^* = \mu_0^*$.

The converse is straight forward.

Lemma: 3.5
If μ is a fuzzy L-ideal of a lattice L, then $L/\mu_t \cong L_\mu$, where $t = \mu(0)$.

Proof:
To Prove $f: L \rightarrow L_\mu$ is a map defined by $f(x)=\mu_x^*$, for all $x \in L$ is an onto homomorphism.
(i.e) to prove
(i) $f(x \wedge y) = \mu_{x \wedge y}^*$
$= \mu_{x \wedge y}^*(z)$
$= \mu[(x \wedge y) \wedge z]$

$= \mu[\,(x\wedge z) \wedge \mu(y\wedge z)\,]$
$= \mu(x\wedge z) \wedge \mu(y\wedge z)$
$= \mu_x^* \wedge \mu_y^*.$

(ii) $f(x\vee y) = \mu_{x\vee y}^*$
$= \mu_{x\vee y}^*(z)$
$= \mu[\,(x\vee y) \wedge z\,]$
$= \mu[\,(x\wedge z) \vee \mu(y\wedge z)\,]$
$= \mu(x\wedge z) \vee \mu(y\wedge z)$
$= \mu_x^* \vee \mu_y^*.$

Therefore f is an onto homomorphism.
Now, $f(x) = \mu_x^* \Leftrightarrow \mu_x^* = \mu_0^*$.
$\Leftrightarrow \mu(x) = \mu(0)$, by lemma 3.4
This shows that kerrnal of f equal μ_t .
Therefore $L/\mu_t \cong L_\mu$.

Theorem: 3.6
Let f be a homomorphism from a lattice L onto a lattice L' and let μ be any f-invariant fuzzy L-ideal of L. Then $L_\mu \cong L'_{f(\mu)}$.

Proof:
Since μ is f-invariant, $K_f \subseteq \mu_t$,where $t= \mu(0)$.
Now, $[f(\mu)]\,(0') = t$, because
$$[\,f(\mu)\,](0') = \sup_{x\in f^{-1}(0')} \mu(x)$$
$= \mu(0)$, since $f(0) = 0'$ and $\mu(x) \le \mu(0)$, $\forall\, x\in L$.
Next,
$[\,f(\mu)\,]_t = f(\mu_t)$, since
$f(x) \in [\,f(\mu)\,]_t \Leftrightarrow [\,f(\mu)\,(f(x))\,] \ge t$
$[\,f^{-1}(\,f(\mu)\,)\,](x) \ge t$
$\mu(x) \ge t$, as $f^{-1}(\,f(\mu)\,) = \mu$,
$x \in \mu_t$
$f(x) \in f(\mu_t)$, because $K_f \subseteq \mu_t$.

Therefore, by theorem 3.5,
$L_\mu \cong L / \mu_t$ and $L'_{f(\mu)} \cong L' / [\,f(\mu)\,]_t$
Also, note that $L / \mu_t \cong L'_{f(\mu t)}$.
From this, it can be shown that
$L_\mu \cong L / \mu_t \cong L'_{f(\mu t)} \cong L' / [\,f(\mu)\,]_t \cong L'_{f(\mu)}$.
$L_\mu \cong L'_{f(\mu)}$.

Definition: 3.7

Let μ be any fuzzy L-ideal of L. The fuzzy L-quotient ideal μ^* of L_μ ($= L/\mu_t$) is defined by $\mu^*(x\vee\mu_t) = \mu(x)$, $\forall\ x\in L$, where $\mu_t = \{ x / \mu(x) = \mu(0) = t \}$.

Theorem: 3.8

If μ is any fuzzy L-ideal of a lattice L, then the fuzzy subset μ^* of L_μ defined by $\mu^*(x\vee\mu_t) = \mu(x)$, where $x\in L$, is a fuzzy L-ideal of L_μ.

Proof:

Given that μ is a fuzzy L-ideal of a lattice L.

To show that the fuzzy subset μ^* of L_μ defined by $\mu^*(x\vee\mu_t) = \mu(x)$, where $x\in L$, is a fuzzy L-ideal of L.

For this, let $x, y \in L$.

Then

i. $\mu^*[(x\vee\mu_t)\vee(y\vee\mu_t)] = \mu^*(x\vee y\vee\mu_t)$

$= \mu(x\vee y)$

$\geq \min\{\mu(x),\mu(y)\}$.

ii. $\mu^*[(x\vee\mu_t)\wedge(y\vee\mu_t)] = \mu^*(x\wedge y\vee\mu_t)$

$= \mu(x\wedge y)$

$\geq \max\{\mu(x),\mu(y)\}$.

Therefore μ^* is a fuzzy L-ideal of L_μ.

Theorem: 3.9

i. Let μ be any fuzzy L-ideal of a lattice L and let $t = \mu(0)$. Then the fuzzy subset μ^* of L/μ_t defined by $\mu^*(x\vee\mu_t) = \mu(x)$, for all $x\in L$ is a fuzzy L-ideal of L / μ_t.

ii. If A is a ideal of L and θ is a fuzzy L-ideal of L/A such that $\theta(x\vee A) = \theta(A)$ only when $x\in A$, then there exists a fuzzy L-ideal μ of L such that $\mu_t = A$ [$t = \mu(0)$] and $\theta = \mu^*$.

Proof:

i. Since μ is a fuzzy L-ideal of L, μ_t is an ideal of L.

Now,

$x\vee\mu_t = y\vee\mu_t$

$\Rightarrow x\wedge y \in \mu_t$

$\Rightarrow \mu(x\wedge y) = t = \mu(0)$

$\Rightarrow \mu(x) = \mu(y)$

$\Rightarrow \mu^*(x\vee\mu_t) = \mu^*(y\vee\mu_t)$.

Therefore μ^* is well defined.

Next, for all $x, y \in L$,

$$\mu^*[\,(x\vee\mu_t)\wedge(y\vee\mu_t)\,] = \mu^*[\,(x\wedge y)\vee\mu_t\,]$$
$$= \mu(x\wedge y)$$
$$\geq \max\{\mu(x), \mu(y)\}$$
$$= \max\{\mu^*(x\vee\mu_t), \mu^*(y\vee\mu_t)\}.$$

$$\mu^*[(x\vee\mu_t)\vee(y\vee\mu_t)] = \mu^*[(x\vee y)\vee\mu_t]$$
$$= \mu(x\vee y)$$
$$\geq \min\{\mu(x), \mu(y)\}$$
$$= \min\{\mu^*(x\vee\mu_t), \mu^*(y\vee\mu_t)\}.$$

Therefore μ^* is a fuzzy L-ideal of L/μ_t.

ii. Define $\mu: L \to [0,1]$ by $\mu(x) = \theta(x\vee A)$ for all $x\in L$.

Then $\mu(x\vee y) = \theta(x\vee y\vee A)$
$$\geq \min\{\theta(x\vee A), \theta(y\vee A)\}$$
$$= \min\{\mu(x), \mu(y)\}.$$

$$\mu(x\wedge y) = \theta(x\wedge y\vee A)$$
$$\geq \max\{\theta(x\vee A), \theta(y\vee A)\}$$
$$= \max\{\mu(x), \mu(y)\}.$$

Therefore μ is a fuzzy L-ideal.

Also, $\mu_t = A$, because

$$x\in\mu_t \Leftrightarrow \mu(x) = \mu(0)$$
$$\Leftrightarrow \theta(x\vee A) = \theta(A)$$
$$\Leftrightarrow x\in A.$$

Now,

$$\mu^*(x\vee\mu_t) = \mu(x)$$
$$= \theta(x\vee A)$$
$$= \theta(x\vee\mu_t).$$

Hence $\mu^* = \theta$.

Theorem: 3.10

Let L be any lattice. Let μ^* be any fuzzy L-ideal of the quotient lattice L/K, where K is any subset of L. Then corresponding to μ^* in L/K, there exists a fuzzy L-ideal in L.

Proof:

Let μ^* be any fuzzy L-ideal of L/K.

Define the fuzzy subset θ of L by $\theta(x) = \mu^*(x\vee k)$, $\forall\, x\in L$.

To prove: θ is a fuzzy L-ideal of L:

$$\theta(x\vee y) = \mu^*[\,(x\vee y)\vee k\,]$$

$= \mu^{*}[(x \vee k) \vee (y \vee k)]$
$\geq \min\{\mu^{*}(x \vee k), \mu^{*}(y \vee k)\}$
$= \min\{\theta(x), \theta(y)\}$.

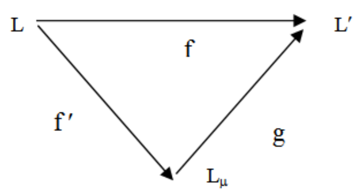
Therefore $\theta(x \vee y) \geq \min\{\theta(x), \theta(y)\}$.
$\theta(x \wedge y) = \mu^{*}[(x \wedge y) \vee k]$
$= \mu^{*}[(x \vee k) \wedge (y \vee k)]$
$\geq \max\{\mu^{*}(x \vee k), \mu^{*}(y \vee k)\}$
$= \max\{\theta(x), \theta(y)\}$.

Therefore $\theta(x \wedge y) \geq \max\{\theta(x), \theta(y)\}$
Hence θ is a fuzzy L-ideal of L.

Theorem: 3.11
Let f be a homomorphism from a lattice L onto a lattice L′ and let μ be any fuzzy L-ideal of L such that $\mu_t \subseteq K_f$, where $t = \mu(0)$. Then there exists a unique homomorphism f' from L_μ onto L′ with the property that $f = f' \circ g$ where $g(x) = \mu_x^{*}$, $\forall\, x \in L$.

Proof:
Define a function $f' : L_\mu \to L'$ by $f'(\mu_x^{*}) = f(x)$, $\forall\, x \in L$.
Now,
$\mu_x^{*} = \mu_y^{*}$
$\Rightarrow \mu_{x \wedge y}^{*} = \mu_0^{*}$
$\Rightarrow \mu(x \wedge y) = \mu(0) = t$
$\Rightarrow x \wedge y \in \mu_t \subseteq K_f$
$\Rightarrow f(x) = f(y)$
$\Rightarrow f'(\mu_x^{*}) = f'(\mu_y^{*})$.

Therefore f' is well defined.
Since f is onto, f' is also onto.
Therefore f' is homomorphism.
Now,
$f(x) = f'(\mu_x^{*})$
$= f'[g(x)]$
$= [f' \circ g](x)$, $\forall\, x \in L$.



Finally, to show that this factorization of f is unique.

Suppose that $f = h\circ g$ for some function $h: L_\mu \to L'$.

Then $f'(\mu_x^*) = f(x)$

$= [h \circ g](x)$

$= h[g(x)]$

$= h(\mu_x^*), \forall x\in L.$

$\Rightarrow f' = h.$

Hence there is a unique homomorphism f' from L_μ onto L' with the property that $f = f'\circ g$, where $g(x) = \mu_x^*, \forall x\in L.$

Corollary: 3.12

The induced f' is an isomorphism iff μ is f – invariant.

Proof:

Let f' be one - one.

Claim: μ is f-invariant

Let $x, y\in L$.

$f(x) = f(y)$

$\Rightarrow f'(\mu_x^*) = f'(\mu_y^*)$

$\Rightarrow \mu_x^* = \mu_y^*$

$\Rightarrow \mu_{x\wedge y}^* = \mu_0^*$

$\Rightarrow \mu(x\wedge y) = \mu(0)$

$\Rightarrow \mu(x) = \mu(y).$

On the other hand, let μ be f-invariant.

Claim: f′ is one – one.

$\mu(x) = \mu(y)$

$\Rightarrow f'[\mu(x)] = f'[\mu(y)]$

$\Rightarrow f'(\mu_x^*) = f'(\mu_y^*)$

$\Rightarrow f(x) = f(y)$

$\Rightarrow \mu(x) = \mu(y)$, since f is invariant

$\Rightarrow \mu_x^* = \mu_y^*$

$\Rightarrow$ f is one – one.

Conclusion

In this paper, the definition,lemma and some homomorphism theorems in fuzzy L-quotient ideals are given. Using these, various results can be developed under the topic fuzzy L-Quotient ideals.

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