

t -norm (λ, μ) -Fuzzy Left h-Ideals of Hemirings

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Abstract

In this paper we introduce the concept of (λ, μ) - fuzzy left h-ideals of a hemiring, t-norm (λ, μ) - fuzzy left h-ideals of a hemiring and imaginable t-norm (λ, μ) -fuzzy left h-ideals of a hemiring which can be regarded as a generalization of fuzzy left h-ideals of a hemiring.

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1. Introduction

Using the notion of a fuzzy subset introduced by Zadeh[5] in 1965. W. Liu[10] introduced the concept of fuzzy ideals of rings. Since then many scholars have studied the theories of fuzzy subring and fuzzy ideal of rings. In particular, S.K. Bhakat and P. Das introduced the concept of $(\in, \in \vee q)$ - fuzzy subgroups[9, 7], $(\in, \in \vee q)$ - fuzzy subrings and $(\in, \in \vee q)$ - fuzzy ideals[8]. B. Yao introduced the notions of (λ, μ) -fuzzy normal subgroup and (λ, μ) -fuzzy quotient subgroup of a group[1] and (λ, μ) -fuzzy subrings, (λ, μ) -fuzzy ideals of rings[2] which can be regarded as a generalization of W. Liu's and S.K. Bhakat and P. Das's correspondence concepts. In this paper we introduce (λ, μ) -fuzzy left h-ideals and t-norm (λ, μ) -fuzzy left h-ideals of hemirings.

2. Preliminaries

Recall that a semiring is an algebraic system $(S, +, \cdot)$ consisting of a non empty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that $(S, +)$ and (S, \cdot) are semigroups and the following distributive laws $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ are satisfied for all $a, b, c \in S$.

By zero of a semiring $(S, +, \cdot)$ we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring with zero and a commutative semigroup $(S, +)$ is called a hemiring. A left ideal of a semiring is a subset A of S closed with respect to the addition and such that $SA \subseteq A$. A left ideal of a hemiring S is called a left-h-ideal if for any $x, z \in S$ and $a, b \in A$ from $x + a + z = b + z$, it follows $x \in A$. We now recall some fuzzy logic concepts. A fuzzy set is a function $F : S \rightarrow [0, 1]$.

Definition 2.1. [13] A fuzzy set F of a hemiring S is called a fuzzy left (resp. right) h-ideal if it satisfies:

- (i) $\forall x, y \in S, F(x + y) \geq \min \{F(x), F(y)\}$
- (ii) $\forall x, y \in S, F(xy) \geq F(y)$ (resp., $F(xy) \geq F(x)$),
- (iii) $\forall a, b, x, z \in S, x + a + z = b + z$ implies that $F(x) \geq \min\{F(a), F(b)\}$.

Note that a fuzzy left (resp. right) h-ideal F of a hemiring S satisfies the inequality $F(0) \geq F(x)$ for all $x \in S$.

Definition 2.2. Let S be a hemiring and Let F be a fuzzy subset of S . For any $\alpha \in [0, 1]$, then the set $F_\alpha = \{x \in S / F(x) \geq \alpha\}$ is called a level subset of S with respect to F .

Theorem 2.3. [13] A fuzzy set F of a hemiring S is a fuzzy left (resp. right) h-ideal of S if and only if each non empty level subset is a left (resp. right) h-ideal of S .

Definition 2.4. A homomorphism of a hemiring S into a hemiring S' such that $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in S$ is defined.

Definition 2.5. [13] A triangular norm, t-norm is a function $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying, for each $a, b, c, d \in [0, 1]$, the following conditions:

- (i) $t(0, 0) = 0, t(a, 1) = a;$
- (ii) $t(a, b) \leq t(c, d)$, whenever $a \leq c, b \leq d$;
- (iii) $t(a, b) = t(b, a)$; and
- (iv) $t(t(a, b), c) = t(a, t(b, c))$.

Example 2.6. A function $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$, defined as $t(a, b) = ab$ is a t-norm.

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Example 2.7. A function $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$, defined as $t(a, b) = a \wedge b$ is a t-norm.

For a t-norm on $[0, 1]$, it is denoted by $\Delta_t = \{\alpha \in [0, 1] / t(\alpha, \alpha) = \alpha\}$, It is clear that every t-norm has the following property: $t(\alpha, \beta) \leq \min\{\alpha, \beta\}$.

Definition 2.8. Let $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be a t-norm. Then the fuzzy set F in S is satisfying the imaginable property if $Im F \subseteq \Delta_t$.

3. (λ, μ) -fuzzy left h-ideals in hemirings

Based on the notions of (λ, μ) - fuzzy subrings and (λ, μ) - fuzzy ideals introduced by B. Yao [2], we introduce the following concepts which are the generalization of fuzzy left h-ideals of hemirings. Throughout this paper λ and μ are arbitrary and $0 \leq \lambda < \mu \leq 1$.

Definition 3.1. A fuzzy set F of S is said to be a (λ, μ) -fuzzy left (resp. right) h-ideal of S if for all $x, y \in S$

- (i) $F(x + y) \vee \lambda \geq F(x) \wedge F(y) \wedge \mu$
- (ii) $F(xy) \vee \lambda \geq F(y) \wedge \mu$
- (iii) $\forall a, b, x, z \in S, x + a + z = b + z \Rightarrow F(x) \vee \lambda \geq F(a) \wedge F(b) \wedge \mu$.

Remark 3.2. A fuzzy left h-ideal is a (λ, μ) -fuzzy left h-ideal with $\lambda = 0$ and $\mu = 1$ and $(\in, \in Vq)$ - fuzzy left h-ideal is a (λ, μ) -fuzzy left h-ideal with $\lambda = 0$ and $\mu = 0.5$. Thus every fuzzy left h-ideal and $(\in, \in Vq)$ - fuzzy left h-ideal are a (λ, μ) -fuzzy left h-ideal of S . However, the converse is not necessarily true as shown in the following example.

Example 3.3. Let S be a hemiring consists of positive rational numbers and 0. Let

$$F(x) = \begin{cases} 0.8 & \text{if } x = 7 \\ 0.4 & \text{if } x \text{ is an integer and } x \neq 7 \\ 0.6 & \text{if } x \text{ is rational} \end{cases}$$

Clearly F is a $(0.1, 0.4)$ -fuzzy left h-ideal. But F is not a fuzzy left h-ideal. Since $F(14) = F(7+7) < F(7) \wedge F(7)$. Also F is not a $(\in, \in Vq)$ - fuzzy left h-ideal, since $F(\frac{4}{5} + \frac{1}{5}) = F(1) < F(\frac{4}{5}) \wedge F(\frac{1}{5}) \wedge 0.5$.

Theorem 3.4. Let F and G are (λ, μ) -fuzzy left(resp. right) h-ideal of S , then $F \cap G$ is a (λ, μ) -fuzzy left(resp. right) h-ideal of S .

Proof. We only consider the case of (λ, μ) -fuzzy left h-ideals and the proof of (λ, μ) -fuzzy right h-ideals is similar.

Let $x, y \in S$

$$\begin{aligned}
 ((F \cap G)(x + y)) \vee \lambda &= (F(x + y) \wedge G(x + y)) \vee \lambda \\
 &= (F(x + y) \vee \lambda) \wedge (G(x + y) \vee \lambda) \\
 &\geq (F(x) \wedge F(y) \wedge \mu) \wedge (G(x) \wedge G(y) \wedge \mu) \\
 &= F(x) \wedge G(x) \wedge F(y) \wedge G(y) \wedge \mu \\
 &= (F \cap G)(x) \wedge (F \cap G)(y) \wedge \mu
 \end{aligned}$$

$$\begin{aligned}
 ((F \cap G)(xy)) \vee \lambda &= (F(xy) \wedge G(xy)) \vee \lambda \\
 &= (F(xy) \vee \lambda) \wedge (G(xy) \vee \lambda) \\
 &\geq (F(x) \wedge F(y) \wedge \mu) \wedge (G(x) \wedge G(y) \wedge \mu) \\
 &= F(x) \wedge G(x) \wedge F(y) \wedge G(y) \wedge \mu \\
 &= (F \cap G)(x) \wedge (F \cap G)(y) \wedge \mu
 \end{aligned}$$

Hence $(F \cap G)$ is a (λ, μ) -fuzzy left ideal of S . Let $a, b, x, z \in S$ be such that $x + a + z = b + z$, then

$$\begin{aligned}
 ((F \cap G)(x)) \vee \lambda &= (F(x) \wedge G(x)) \vee \lambda \\
 &= (F(x) \vee \lambda) \wedge (G(x) \vee \lambda) \\
 &\geq (F(a) \wedge F(b) \wedge \mu) \wedge (G(a) \wedge G(b) \wedge \mu) \\
 &= F(a) \wedge G(a) \wedge F(b) \wedge G(b) \wedge \mu \\
 &= (F \cap G)(a) \wedge (F \cap G)(b) \wedge \mu
 \end{aligned}$$

Hence $(F \cap G)$ is a (λ, μ) -fuzzy left h-ideal of S . ■

Theorem 3.5. A non empty subset I of a hemiring S is a left(resp. right) h-ideal of S if and only if F_I is a (λ, μ) -fuzzy left(resp. right) h-ideal of S .

Proof. Let I be a left h-ideal of a hemiring S . Then F_I is a fuzzy left h-ideal of S and by Remark 3.2 F_I is a (λ, μ) -fuzzy left h-ideal of S .

Conversely, let F_I be a (λ, μ) -fuzzy left h-ideal of S . For any $x, y \in I$ we have

$$\begin{aligned}
 F_I(x + y) \vee \lambda &\geq F_I(x) \wedge F_I(y) \wedge \mu = 1 \wedge 1 \wedge \mu = \mu \\
 F_I(x + y) \vee \lambda &\geq \mu \Rightarrow F_I(x + y) \geq \mu \quad (\text{since } \lambda < \mu) \\
 &\Rightarrow x + y \in I.
 \end{aligned}$$

Now

$$\begin{aligned}
 F_I(xy) \vee \lambda &\geq F_I(y) \wedge \mu = 1 \wedge \mu = \mu \\
 F_I(xy) \vee \lambda &\geq \mu \Rightarrow F_I(xy) \geq \mu \quad (\text{since } \lambda < \mu) \\
 &\Rightarrow xy \in I.
 \end{aligned}$$

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For any $x, z \in S$ and $a, b \in I$ such that $x + a + z = b + z$, we have

$$\begin{aligned} F_I(x) \vee \lambda &\geq F_I(a) \wedge F_I(b)) \wedge \mu = 1 \wedge 1 \wedge \mu = \mu \\ F_I(x) \vee \lambda &\geq \mu \Rightarrow F_I(x) \geq \mu \quad (\text{since } \lambda < \mu) \\ \Rightarrow x &\in I. \end{aligned}$$

Hence I is a fuzzy left h-ideal of the hemiring S . ■

Theorem 3.6. A fuzzy set F of S is a (λ, μ) -fuzzy left(resp. right) h-ideal of S if and only if the level subset F_α is a left (resp. right) h-ideal of S for all $\lambda < \alpha \leq \mu$.

Proof. We only consider the case of (λ, μ) -fuzzy left h-ideal and the proof of (λ, μ) -fuzzy right h-ideal is similar. Let F be a (λ, μ) -fuzzy left h-ideal of S and $\lambda < \alpha \leq \mu$. Let $x, y \in F_\alpha$ then $F(x) \geq \alpha$ and $F(y) \geq \alpha$.

Now we have

$$\begin{aligned} F(x + y) \vee \lambda &\geq F(x) \wedge F(y) \wedge \mu \\ &\geq \alpha \wedge \alpha \wedge \mu = \alpha \\ \Rightarrow F(x + y) &\geq \alpha, \quad (\text{since } \lambda < \alpha) \\ \Rightarrow x + y &\in F_\alpha. \end{aligned}$$

Now, for every $x \in F_\alpha$ and $r \in S$, we have

$$\begin{aligned} F(rx) \vee \lambda &\geq F(x) \wedge \mu \\ &\geq \alpha \wedge \mu = \alpha \\ \Rightarrow F(rx) &\geq \alpha, \quad \text{since } \lambda < \alpha \end{aligned}$$

Therefore, F_α is a left ideal of S . Now let $x, z \in S$ and $a, b \in F_\alpha$ be such that $x + a + z = b + z$, then

$$\begin{aligned} F(x) \vee \lambda &\geq F(a) \wedge F(b) \wedge \mu \\ &\geq \alpha \wedge \alpha \wedge \mu = \alpha \\ \Rightarrow F(x) &\geq \alpha, \quad \text{since } \lambda < \alpha \\ \Rightarrow x &\in F_\alpha. \end{aligned}$$

Therefore, F_α is a left h-ideal of S . Conversely, let F be a fuzzy set of S such that F_α is a left h-ideal of S , for all $\lambda < \alpha \leq \mu$. Let $x, y \in S$. Suppose $F(x + y) \vee \lambda < F(x) \wedge F(y) \wedge \mu$. Choose α such that $F(x + y) \vee \lambda < \alpha < F(x) \wedge F(y) \wedge \mu$. Now $F(x) \wedge F(y) \wedge \mu > \alpha$. Then $F(x) \geq \alpha, F(y) \geq \alpha, \mu \geq \alpha$. Hence $x, y \in F_\alpha$ and F_α being left h-ideal of S , $x + y \in F_\alpha$. Thus $F(x + y) \geq \alpha > \lambda$ and this implies $F(x + y) \vee \lambda \geq \alpha$, a contradiction. Thus $F(x + y) \vee \lambda \geq F(x) \wedge F(y) \wedge \mu$. Suppose, $F(xy) \vee \lambda < F(y) \wedge \mu$. Choose α such that $F(xy) \vee \lambda < \alpha < F(y) \wedge \mu$. Now $F(y) \wedge \mu > \alpha$. Then $F(y) \geq \alpha, \mu \geq \alpha$. Hence $y \in F_\alpha$. Since F_α is a left h-ideal of S , $xy \in F_\alpha$. Thus $F(xy) \geq \alpha > \lambda$

and this implies $F(xy) \vee \lambda \geq \alpha$, a contradiction. Thus $F(xy) \vee \lambda \geq F(y) \wedge \mu$. Finally let $a, b, x, z \in S$ be such that $x + a + z = b + z$. Suppose $F(x) \vee \lambda < F(a) \wedge F(b) \wedge \mu$. Choose α such that $F(x) \vee \lambda < \alpha < F(a) \wedge F(b) \wedge \mu$. Now $F(a) \wedge F(b) \wedge \mu > \alpha$. Then $F(a) \geq \alpha$, $F(b) \geq \alpha$, $\mu \geq \alpha$. Hence $a, b \in F_\alpha$ and F_α being left h-ideal of S , $x + a + z = b + z \in F_\alpha$ and this implies that $x \in F_\alpha$. Thus $F(x) \geq \alpha > \lambda$ and this implies $F(x) \vee \lambda \geq \alpha$, a contradiction. Thus $F(x) \vee \lambda \geq F(a) \wedge F(b) \wedge \mu$. Therefore F is a (λ, μ) -fuzzy left h-ideal of S . \blacksquare

4. t-norm (λ, μ) -fuzzy left h-ideals in hemirings

Definition 4.1. A fuzzy set F of S is said to be a t-norm (λ, μ) -fuzzy left (resp. right) h-ideal of S if for all $x, y \in S$

- (i) $F(x + y) \vee \lambda \geq t(F(x), F(y)), \mu$.
- (ii) $F(xy) \vee \lambda \geq t(F(y), \mu)$.
- (iii) $\forall a, b, x, z \in S, x + a + z = b + z \Rightarrow F(x) \vee \lambda \geq t(t(F(a), F(b)), \mu)$.

Remark 4.2. A fuzzy left h-ideal is a t-norm (λ, μ) -fuzzy left h-ideal with $\lambda = 0$ and $\mu = 1$ and $t(\alpha, \beta) = \min(\alpha, \beta)$ and an $(\in, \in Vq)$ -fuzzy left h-ideal is a t-norm (λ, μ) -fuzzy left h-ideal with $\lambda = 0$ and $\mu = 0.5$ and $t(\alpha, \beta) = \min(\alpha, \beta)$. Thus every fuzzy left h-ideal and $(\in, \in Vq)$ -fuzzy left h-ideal are a t-norm (λ, μ) -fuzzy left h-ideal of S . However, the converse is not necessarily true as shown in the following example.

Example 4.3. Let S be a hemiring consists of positive rational numbers and 0. Let

$$F(x) = \begin{cases} 0.8 & \text{if } x = 7 \\ 0.4 & \text{if } x \text{ is an integer and } x \neq 7 \\ 0.6 & \text{if } x \text{ is rational.} \end{cases}$$

Let t-norm be defined as $t(\alpha, \beta) = \min(\alpha, \beta)$, for all $\alpha, \beta \in [0, 1]$. Clearly F is a t-norm $(0.1, 0.4)$ -fuzzy left h-ideal. But F is not a fuzzy left h-ideal. Since $F(14) = F(7 + 7) < F(7) \wedge F(7)$. Also F is not a $(\in, \in Vq)$ -fuzzy left h-ideal, since $F\left(\frac{4}{5} + \frac{1}{5}\right) = F(1) < F\left(\frac{4}{5}\right) \wedge \left(\frac{1}{5}\right) \wedge 0.5$.

Theorem 4.4. A non empty subset I of a hemiring S is a left(resp. right) h-ideal of S if and only if F_I is a t-norm (λ, μ) -fuzzy left(resp. right) h-ideal of S .

Proof. Let I be a left h-ideal of a hemiring S . Then F_I is a fuzzy left h-ideal of S and by Remark F_I is a t-norm (λ, μ) -fuzzy left h-ideal of S .

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Conversely, let F_I be a t-norm (λ, μ) -fuzzy left h-ideal of S . For any $x, y \in I$ we have

$$\begin{aligned} F_I(x + y) \vee \lambda &\geq t(t(F_I(x), F_I(y)), \mu) = \mu \\ F_I(x + y) \vee \lambda &\geq \mu \Rightarrow F_I(x + y) \geq \mu \quad (\text{since } \lambda < \mu) \\ &\Rightarrow x + y \in I. \\ F_I(xy) \vee \lambda &\geq t(F_I(y), \mu) = \mu \\ F_I(xy) \vee \lambda &\geq \mu \Rightarrow F_I(xy) \geq \mu \quad (\text{since } \lambda < \mu) \\ &\Rightarrow xy \in I. \end{aligned}$$

For any $x, z \in S$ and $a, b \in I$ such that $x + a + z = b + z$, we have

$$\begin{aligned} F_I(x) \vee \lambda &\geq t(t(F_I(a), F_I(b)), \mu) = \mu \\ F_I(x) \vee \lambda &\geq \mu \Rightarrow F_I(x) \geq \mu \quad (\text{since } \lambda < \mu) \\ &\Rightarrow x \in I. \end{aligned}$$

Hence I is a fuzzy left h-ideal of the hemiring S . ■

Theorem 4.5. A fuzzy set F of S is a t-norm (λ, μ) -fuzzy left(resp. right) h-ideal of S if and only if the level subset F_α is a left (resp. right) h-ideal of S for all $\lambda \leq \alpha \leq \mu$.

Proof. We only consider the case of t-norm (λ, μ) -fuzzy left h-ideal and the proof of t-norm (λ, μ) -fuzzy right h-ideal is similar. Let F be a t-norm (λ, μ) -fuzzy left h-ideal of S and $\lambda \leq \alpha \leq \mu$. Let $x, y \in F_\alpha$ then $F(x) \geq \alpha$ and $F(y) \geq \alpha$.

Now we have

$$\begin{aligned} F(x + y) \vee \lambda &\geq t(t(F(x), F(y)), \mu) \geq t(t(\alpha, \alpha), \mu) = \alpha \\ &\Rightarrow F(x + y) \geq \alpha \Rightarrow x + y \in F_\alpha. \end{aligned}$$

Now, for every $x \in F_\alpha$ and $r \in S$, we have

$$\begin{aligned} F(rx) \vee \lambda &\geq t(F(x), \mu) \geq \alpha \wedge \mu = \alpha \\ &\Rightarrow F(rx) \vee \lambda \geq \lambda \quad [\text{since } \lambda < \alpha] \end{aligned}$$

Therefore, F_α is a left ideal of S . Now let $x, z \in S$ and $a, b \in F_\alpha$ be such that $x + a + z = b + z$, then

$$\begin{aligned} F(x) \vee \lambda &\geq t(t(F(a), F(b)), \mu) \\ &\geq t(t(\alpha, \alpha), \mu) = \alpha \\ &\Rightarrow F(x) \geq \alpha \Rightarrow x \in F_\alpha. \end{aligned}$$

Therefore, F_α is a left h-ideal of S . Conversely, let F be a fuzzy set of S such that F_α is a left h-ideal of S . For all $\lambda < \alpha \leq \mu$, Suppose $F(x + y) \vee \lambda < t(t(F(x), F(y)), \mu) = \alpha$

then $F(x + y) < \alpha$ [since $\lambda < \alpha$]

$$\Rightarrow x + y \notin F_\alpha \text{ for } x, y \in F_\alpha.$$

Which is a contradiction to that F_α is a fuzzy left h-ideal of S . Hence $F(x + y) \vee \lambda \geq t(t(F(x), F(y)), \mu)$. Suppose, $F(xy) \vee \lambda < F(x) \wedge \mu = \alpha$
then $F(xy) < \alpha$ [since $\lambda < \alpha$]

$$\Rightarrow xy \notin F_\alpha \text{ for all } x, y \in F_\alpha,$$

Which is a contradiction.

So $F(xy) \vee \lambda \geq F(x) \wedge \mu$. Finally let $a, b, x, z \in S$ be such that $x + a + z = b + z$.

$$\text{Suppose } F(x) \vee \lambda < t(t(F(a), F(b)), \mu) = \alpha$$

$$\Rightarrow F(x) < \alpha \Rightarrow x \notin F_\alpha,$$

which is a contradiction. Therefore $F(x) \vee \lambda \geq t(t(F(a), F(b)), \mu)$. Therefore F is a fuzzy left h-ideal of S . \blacksquare

Definition 4.6. If F is a fuzzy set in a hemiring S and θ is a mapping from S into itself, we define a mapping $F(\theta) : S \rightarrow [0, 1]$ by $F(\theta)(x) = F(\theta(x))$ for all $x \in S$.

Theorem 4.7. If F is a t-norm (λ, μ) fuzzy left h-ideal of a hemiring S and θ is an endomorphism of S , then $F(\theta)$ is a t-norm (λ, μ) fuzzy left h-ideal of S .

Proof. For any $x, y \in S$, we have

$$\begin{aligned} F(\theta)(x + y) \vee \lambda &= F(\theta(x + y)) \vee \lambda \\ &= F(\theta(x) + \theta(y)) \vee \lambda \\ &\geq t(t(F(\theta(x)), F(\theta(y))), \mu) \\ &= t(t(F(\theta)(x), F(\theta)(y)), \mu) \\ F(\theta)(xy) \vee \lambda &= F(\theta(xy)) \vee \lambda \\ &= F(\theta(x).\theta(y)) \vee \lambda \\ &\geq F(\theta(y)) = F(\theta)(y) \end{aligned}$$

Hence $F(\theta)$ is a t-norm (λ, μ) fuzzy left ideal of S . Let $x, z, a, b \in S$ be such that $x + a + z = b + z$. Then $\theta(x + a + z) = \theta(b + z)$ and so $\theta(x) + \theta(a) + \theta(z) = \theta(b) + \theta(z)$
It follows that

$$\begin{aligned} F(\theta)(x) \vee \lambda &= F(\theta(x)) \vee \lambda \\ &\geq t(t(F(\theta(a)), F(\theta(b))), \mu) \\ &= t(t(F(\theta)(a), F(\theta)(b)), \mu) \end{aligned}$$

Therefore $F(\theta)$ is a t-norm (λ, μ) fuzzy left h-ideal of a hemiring S . \blacksquare

Definition 4.8. Let f be a mapping defined on a hemiring S . If B is a fuzzy set in $f(S)$, then the fuzzy set $A = B \circ f$ (ie, the fuzzy set defined by $A(x) = B[f(x)]$ for all $x \in S$) is called the preimage of B under f .

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Theorem 4.9. An onto homomorphic preimage of a t-norm (λ, μ) fuzzy left h-ideal of a hemiring S is a t-norm (λ, μ) fuzzy left h-ideal.

Proof. Let $f : S \rightarrow S'$ be an onto homomorphism of hemirings, and let B be a t-norm (λ, μ) fuzzy left h-ideal of S' , and A be the preimage of B under f . Then we have

$$\begin{aligned} A(x + y) \vee \lambda &= B[f(x + y)] \vee \lambda \\ &= B[f(x) + f(y)] \vee \lambda. \\ &\geq t(t(B[f(x)], B[f(y)]), \mu) \\ &= t(t(A(x), A(y)), \mu) \\ A(xy) \vee \lambda &= B[f(xy)] \vee \lambda \\ &= B[f(x)f(y)] \vee \lambda. \\ &\geq t(B[f(y)], \mu) = t(A(y), \mu) \end{aligned}$$

Hence, A is a t-norm (λ, μ) fuzzy left ideal of S . Let $x, z, a, b \in S$ be such that $x + a + z = b + z$ then

$$f(x + a + z) = f(b + z)$$

and so

$$f(x) + f(a) + f(z) = f(b) + f(z)$$

It follows that

$$\begin{aligned} A(x) \vee \lambda &= B[f(x)] \vee \lambda \\ &\geq t(t(B[f(a)], B[f(b)]), \mu) \\ &= t(t(A(a), A(b)), \mu) \end{aligned}$$

Therefore, A is a t-norm (λ, μ) fuzzy left h-ideal of S . ■

Definition 4.10. A t-norm (λ, μ) fuzzy left h-ideal of a hemiring S is said to be imaginable if it satisfies the imaginable property.

Example 4.11. Let S be the set of natural numbers including 0, and S is a hemiring with usual addition and multiplication. Define a fuzzy set $F : S \rightarrow [0, 1]$ by

$$F(x) = \begin{cases} 1 & \text{if } x \text{ is even or } 0 \\ 0 & \text{otherwise} \end{cases}$$

and let $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be a function defined by $t(\alpha, \beta) = \max\{\alpha + \beta - 1, 0\}$ for all $\alpha, \beta \in [0, 1]$. Then t is a t-norm. By routine calculations, we have that F is an imaginable t-norm $(0, 1)$ - fuzzy left h-ideal of a hemiring S .

Theorem 4.12. Every imaginable t-norm (λ, μ) -fuzzy left h-ideal F of a hemiring S is a (λ, μ) -fuzzy left h-ideal of S .

Proof. Assume that F is an imaginable t-norm (λ, μ) -fuzzy left h-ideal of a hemiring S , then we have for all $x, y \in S$,

- (i) $F(x + y) \vee \lambda \geq t(t(F(x), F(y)), \mu)$.
- (ii) $F(xy) \vee \lambda \geq t(F(x), \mu)$.
- (iii) Let $a, b, x, z \in S$, be such that $x + a + z = b + z \Rightarrow F(x) \vee \lambda \geq t(t(F(a), F(b)), \mu)$.

Since F is imaginable, we have

$$\begin{aligned} F(x) \wedge F(y) &= t(F(x) \wedge F(y), F(x) \wedge F(y)) \\ &\leq t(F(x), F(y)) \leq F(x) \wedge F(y) \end{aligned}$$

and so $t(F(x), F(y)) = F(x) \wedge F(y)$. It follows that

$$\begin{aligned} F(x + y) \vee \lambda &\geq t(t(F(x), F(y)), \mu) \\ &= t((F(x) \wedge F(y)), \mu) = (F(x) \wedge F(y)) \wedge \mu \\ F(xy) \vee \lambda &\geq t(F(y), \mu) = F(y) \wedge \mu \end{aligned}$$

Hence F is a (λ, μ) fuzzy left ideal of S . Let $a, b, x, z \in S$, be such that $x + a + z = b + z$ then $F(x) \vee \lambda \geq t(t(F(a), F(b)), \mu)$, since F is a t-norm (λ, μ) -fuzzy left h-ideal of S . It follows that

$$\begin{aligned} F(a) \wedge F(b) &= t(F(a) \wedge F(b), F(a) \wedge F(b)) \\ &\leq t(F(a), F(b)) \leq F(a) \wedge F(b) \end{aligned}$$

and so $t(F(a), F(b)) = F(a) \wedge F(b)$, since F is imaginable. Therefore,

$$\begin{aligned} F(x) \vee \lambda &\geq t(t(F(a), F(b)), \mu), \\ &= (F(a) \wedge F(b)) \wedge \mu \end{aligned}$$

Thus F is a (λ, μ) -fuzzy left h-ideal of S . ■

Remark 4.13. The converse of the above theorem is not true. The following example shows that there exists a t-norm such that a (λ, μ) -fuzzy left h-ideal of S may not be an imaginable t-norm (λ, μ) -fuzzy left h-ideal of S .

Example 4.14. Let S be a semiring of set of natural numbers with zero. Define a fuzzy subset $F : S \rightarrow [0, 1]$ such that

$$F(x) = \begin{cases} \frac{1}{2} & \text{if } x \text{ is even or 0} \\ \frac{1}{5} & \text{otherwise.} \end{cases}$$

t-norm (λ, μ) -Fuzzy Left h-Ideals of Hemirings

Now F is a (.1, .8) fuzzy left h-ideal of S . Let $\gamma \in [0, 1]$ and define *t – norm* on $[0, 1]$ as follows:

$$t(\alpha, \beta) = \begin{cases} \alpha \cap \beta & \text{if } \max\{\alpha, \beta\} = 1 \\ 0 & \text{if } \max\{\alpha, \beta\} < 1, \alpha + \beta \leq 1 + 0.6 \\ 0.6 & \text{otherwise.} \end{cases}$$

Check that F is a t-norm (.1, .8) - fuzzy left h-ideal of S . But

$$t_\gamma(F(0), F(0)) = t_\gamma\left(\frac{1}{2}, \frac{1}{2}\right) = 0 \neq F(0).$$

Hence F is not an imaginable t-norm (λ, μ) -fuzzy left h-ideal of S .

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