International Journal of Fuzzy Mathematics and Systems.
ISSN 2248-9940 Volume 3, Number 3 (2013), pp. 233-241

http://www.ripublication.com

Fuzzy Volterra Spaces and Functions

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Abstract

In this paper some results concerning functions that preserve fuzzy Volterra spaces in the context of images and preimages are obtained. Several examples are given to illustrate the concepts introduced in this paper.

KEYWORDS: Fuzzy G_δ-set,, Fuzzy dense, Fuzzy nowhere dense, Fuzzy continuous, Some what fuzzy continuous, Fuzzy open and Some what fuzzy open functions

2000 AMS Classification: 54 A 40, 03 E 72.

INTRODUCTION
The concept of fuzzy sets and fuzzy set operations were first introduced by L.A.ZADEH in his classical paper [14] in the year 1965. Thereafter the paper of C.L.CHANG [3] in 1968 paved the way for the subsequent tremendous growthof thenumerous fuzzy topological concepts. Since then much attention hasbeenpaidto generalize the basic concepts of General Topology in fuzzy settingandthus a moderntheory of fuzzy topology has been developed. The concepts of Volterra spaceshave been studied extensively in classical topology in [4] , [5] , [6] [7] and [8] .The conceptofVolterra spaces in fuzzy setting was introduced and studied by the authors in [12] .In this papersome results concerning functions that preservefuzzy Volterra spaces in the context of images and preimages are obtained. Several examplesare given to illustrate the concepts introduced inthis paper.

2. PRELIMINARIES
By a fuzzy topological space we shall mean a non – empty set X together with a fuzzy

topology T (in the sense of Chang) and denote it by (X, T).

DEFINTION 2.1: Let λ and μ be any two fuzzy sets in (X, T). Then we define $\lambda \vee \mu$: $X \to [0, 1]$ as follows: $(\lambda \vee \mu)(x) = \text{Max}\{\lambda(x), \mu(x)\}$. Alsowe define $\lambda \wedge \mu$: $X \to [0, 1]$ as follows: $(\lambda \wedge \mu)(x) = \text{Min}\{\lambda(x), \mu(x)\}$.

DEFINTION 2.2: Let (X, T) be any fuzzy topological space and λ be any fuzzy set in (X, T). We define $Cl(\lambda) = \wedge\{\mu / \lambda \leq \mu, 1-\mu \in T\}$ and $int(\lambda) = \vee\{\mu / \mu \leq \lambda, \mu \in T\}$.

For any fuzzy set in a fuzzy topological space (X, T), it is easy to see that $1 - cl(\lambda) = int(1 - \lambda)$ and $1 - int(\lambda) = cl(1 - \lambda)$ [1] .

DEFINTION 2.3: Let (X, T) and (Y, S) be any two fuzzy topological spaces. Let f be a function from the fuzzy topological space (X, T) to the fuzzy topological space (Y, S). Let λbe a fuzzysetin (Y, S).The inverse image of λ under f written as $f^{-1}(\lambda)$is the fuzzy set in (X, T) defined by $f^{-1}(\lambda)(x) = \lambda(f(x))$ for all $x \in X$. Also the image of λin (X, T) under fwritten as $f(\lambda)$ is the fuzzy set in (Y, S)defined by

$$f(\lambda)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \lambda(x) & if\ f^{-1}(y) \text{ is non} - empty; \\ 0 & otherwise. \end{cases} \quad \text{for each } y \in Y.$$

Lemma 2.1 [3] : Letf: $(X, T) \to (Y, S)$ be a mapping. For fuzzy sets λ and μ of (X, T) and(Y, S) respectively, the following statements hold.

1. $ff^{-1}(\mu) \leq \mu$;
2. $f^{-1}f(\lambda) \geq \lambda$;
3. $f(1 - \lambda) \geq 1 - f(\lambda)$;
4. $f^{-1}(1 - \mu) = 1 - f^{-1}(\mu)$;
5. If f is injective, then $f^{-1}f(\lambda) = \lambda$;
6. If f is surjective, then $ff^{-1}(\mu) = \mu$;
7. If f is bijective, then $f(1 - \lambda) = 1 - f(\lambda)$

DEFINTION 2.4 [9] : A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy dense if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$.

DEFINTION 2.5 [10] : A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy nowhere dense if there exists no non – zero fuzzy open set μ in (X, T)such that $\mu < cl(\lambda)$. That is, $int\ cl(\lambda) = 0$.

DEFINITION 2.6 [9] : A fuzzyset λin a fuzzy topological space (X, T) is called fuzzy firstcategoryif $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where λ_i's are fuzzy nowhere dense sets in (X, T).

Any other fuzzy setin (X, T) is said to be of secondcategory.

DEFINTION 2.7 [2] : A fuzzy set λ in a fuzzy topological space(X, T) is called a fuzzyG_δ-set in (X, T) if $\boldsymbol{\lambda} = \wedge_{i=1}^{\infty} \boldsymbol{\lambda_i}$ where $\lambda_i \in T$ for $i \in I$.

Lemma 2.2 [1] : For a familyof $\{\lambda_{\alpha}\}$ of fuzzy sets of afuzzy topological space (X, T), $\vee$ cl $(\lambda_{\alpha}) \leq$ cl $(\vee\lambda_{\alpha})$. In case $\acute{A}$ is a finite set, $\vee$ cl $(\lambda_{\alpha})=$ cl $(\vee\lambda_{\alpha})$. Also $\vee$int$(\lambda_{\alpha}) \leq$int $(\vee\lambda_{\alpha})$.

DEFINITION 2.8 [9] : A function f: (X, T) → (Y, S) from a fuzzy topological space(X, T) into another fuzzy topological space (Y, S) is called somewhat fuzzy continuous if $\lambda \in S$ and $f^{-1}(\lambda) \neq 0$ implies that there exist a fuzzy open set δ in (X, T) such that $\delta \neq 0$ and $\delta \leq f^{-1}(\lambda)$.

DEFINITION 2.9 [9] : A function f: (X, T) → (Y, S) from a fuzzy topological space (X, T) intoanother fuzzy topological space (Y, S) is called somewhat fuzzy open if $\lambda \in T$ and $\lambda \neq 0$ implies that there exists a fuzzy open set η in (Y, S) such that $\eta \neq 0$ and $\eta \leq f(\lambda)$.

3. FUZZY VOLTERRA SPACES

DEFINITION 3.1 [12] : A fuzzy topological space(X, T) is called a fuzzy Volterraspace if$cl(\wedge_{i=1}^{N} (\lambda_i)) = 1$,where λ_i's are fuzzy denseandfuzzy G_δ sets in (X, T) .

DEFINITION 3.2 [12] : A fuzzy topological space (X, T) is called a fuzzy weakly Volterra space if $cl(\wedge_{i=1}^{N} (\lambda_i)) \neq 0$, where λ_i' s are fuzzy dense and fuzzy G_δ-sets in (X, T).

THEOREM 3.1 [11] : If λ is a fuzzy dense and fuzzy G_δ-set in a fuzzy topological space (X, T), then $1 - \lambda$ is a fuzzy first category set in (X, T).

THEOREM 3.2 [10] : Let (X, T) be a fuzzy topological space. Then the following are equivalent:

1. (X, T) is a fuzzy Baire space.
2. Int $(\lambda) = 0$ for every fuzzy first category setλ in (X, T).
3. cl $(\mu) = 1$ for every fuzzy residualset μ in (X, T).

PROPOSITION 3.1: A fuzzy topological space(X, T) is a fuzzy Volterra space, ifand only if int $(\vee_{i=1}^{N}(1 - \lambda_{i.})) = 0$, where λ_i's (i = 1 to N) are fuzzy dense andfuzzy G_δ sets in (X, T).

PROOF: : Let (X, T) be a fuzzy Volterra space and λ_i's (i = 1 to N) be fuzzy denseandfuzzy G_δ sets in (X, T). Thenwe have $cl(\wedge_{i=1}^{N} (\lambda_i)) = 1$.Now int $(\vee_{i=1}^{N}(1 - \lambda_{i.}))$ = int $(1- \wedge_{i=1}^{N}(\lambda_i)) = 1 -$cl $(\wedge_{i=1}^{N}(\lambda_i)) = 1 - 1 = 0$.

Conversely let int $(\vee_{i=1}^{N}(1 - \lambda_{i.})) = 0$, where λ_i's (i = 1 to N) are fuzzy denseandfuzzy G_δ sets in (X, T). Then, int $(1 - [\wedge_{i=1}^{N} (\lambda_i)]) = 0$, which implies that $(1 - cl\,[\wedge_{i=1}^{N} (\lambda_i)]) = 0$. Then $cl(\wedge_{i=1}^{N} (\lambda_i)) = 1$.Hence (X, T) is a fuzzyVolterra space.

PROPOSITION 3.2: Ifevery fuzzy first category set in (X, T) is formed from thefuzzydense andfuzzy G_δ sets ina fuzzy Volterra space, then (X, T) is a fuzzy Baire space.

PROOF: Let λ_i' s (i = 1 to N) be fuzzy denseandfuzzy $G_\delta$sets in (X, T).
Since (X, T) is a fuzzy Volterra space, by proposition 3.1, int $(\vee_{i=1}^{N}(1 - \lambda_{i.}))$ =0, By Lemma 2.2 [1] , we have $\vee_{i=1}^{N}$ int $(1-\lambda i))$≤int $(\vee_{i=1}^{N}(1-\lambda i)$, which implies that$\vee_{i=1}^{N}$ int $(1-\lambda i))$ = 0 .Then int $(1-\lambda_i)$ = 0. Since λ_i' s (i = 1 to N) are fuzzy denseandfuzzy $G_\delta$sets in (X, T), by theorem 3.1, $(1-\lambda_i)$'s are fuzzy first category sets in (X, T). By theorem 3.2, (X, T) is a fuzzy Baire space. Hence ifevery fuzzy first category set in (X, T) is formed from the fuzzydense andfuzzy G_δ sets ina fuzzy Volterra space, then (X, T) is a fuzzy Baire space.

4. FUZZY VOLTERRA SPACES AND FUNCTIONS

Let f be a function from the fuzzy topological space (X, T) to the fuzzy topological space (Y, S).*Underwhat Conditions on " f " may we assert that if (X, T)is a fuzzy Volterraspace, then (Y, S) is a fuzzy Volterraspace? It may be noticed that the fuzzy continuous image of a fuzzy Volterraspace may fail to be a fuzzy Volterraspace. For, consider the following example:*

EXAMPLE 4.1: Let X ={ a, b, c }. The fuzzysets λ, μ, υ, α, β and $\eta$are defined onX as follows:
λ: X → [0, 1] is defined as λ(a) = 1; λ(b) = 0.2; λ(c) = 0.9.
μ: X → [0, 1] is defined as μ(a) = 0.3 ; μ(b) =1; μ(c) = 0.2.
υ: X → [0, 1] is defined as υ(a) = 0.7 ; υ(b) = 0.4; υ(c) = 1.
α: X → [0, 1] is defined as α (a) = 0.9; α (b) = 1; α (c) =0.2.
β: X → [0, 1] is defined as β (a) = 0.2; β (b) =0.3;β (c) = 1.
η: X → [0, 1] is defined as η (a) = 1; η (b) = 0.7; η (c) = 0.4.

Then, T={0, λ, μ, υ, (λ∨μ), (λ∨υ), (μ∨υ), (λ∧ μ), (λ ∧υ), (μ∧υ), (λ ∨ [μ∧υ]), (μ∨ [λ∧υ]), (υ∧ [λ∨ μ), 1} and S= { 0, α, β, η, $(\alpha \vee \beta)$, $(\alpha \vee \eta)$, $(\beta \vee \eta)$, $(\alpha \wedge \beta)$, $(\alpha \wedge \eta)$, $(\beta \wedge \eta)$, $(\alpha \vee [\beta \wedge \eta]$), $(\beta \vee [\alpha \wedge \eta]$), $(\eta \wedge [\alpha \vee \beta]$), 1} are fuzzy topologies on X.

Now λ ∧υ= { (μ ∨ [λ ∧υ]) ∧(υ∧ [λ ∨ μ])∧(υ)∧(λ) } and (υ∧ [λ ∨ μ])= { (λ ∨ μ) ∧(λ∨ [μ∧υ]) ∧(λ ∨υ) }. Then (λ ∧υ) and (υ∧ [λ ∨ μ]) are fuzzy G_δ sets in(X, T) and cl(λ ∧υ) = 1 andcl (υ∧ [λ ∨ μ]) = 1. Since cl([λ ∧υ] ∧ (υ∧ [λ ∨ μ) } =1, the fuzzy topological space(X, T) is a fuzzy Volterra pace. Now α ={ $(\alpha) \wedge (\alpha \vee \beta) \wedge$ $(\alpha \vee [\beta \wedge \eta]$)}, $(\alpha \wedge \eta)$ ={$(\eta) \wedge (\alpha \vee \eta) (\alpha \wedge \eta)$ $\wedge (\eta \wedge [\alpha \vee \beta]$)} and β = { $(\beta) \wedge (\beta \vee \eta)$ $\wedge (\beta \vee [\alpha \wedge \eta]$)} Then $\alpha, (\alpha \wedge \eta)$ *and* β are fuzzy $G_\delta$sets in(X, S) andcl(α) =1, cl(β) =1, cl($\alpha \wedge \eta$)=1. Since cl{ $(\alpha \wedge \beta \wedge (\alpha \wedge \eta)\}$ =1 $-(\beta \wedge \eta) \neq 1$, the fuzzy topological space(X, S) is not a fuzzy Volterraspace.

Define a function f: (X, T) → (X, S) by f (a)=b and f (b)=c and f (c)=a. Clearly f is a fuzzycontinuous function from the fuzzy Volterra space (X, T) tothe fuzzy topological space (X, S), which is not afuzzyVolterra space. *It may be also noticedthat the image of a fuzzy Volterraspace under a fuzzy open function may fail to be a fuzzy Volterra space. For, consider the following example.*

EXAMPLE 4.2: Let X = { a, b, c }. The fuzzysets λ, μ, υ and δ are defined on X as follows:
λ: X → [0, 1] is defined as $\lambda(a) = 1$; $\lambda(b) = 0.2$; $\lambda(c) = 0.9$.
μ: X → [0, 1] is defined as $\mu(a) = 0.3$; $\mu(b) =1$; $\mu(c) = 0.2$.
υ: X → [0, 1] is defined as $\upsilon(a) = 0.7$; $\upsilon(b) = 0.4$; $\upsilon(c) = 1$.
δ: X → [0, 1] is defined as $\delta(a) = 0.5$; $\delta(b) = 0.6$; $\delta(c) =0.4$.

Then, T={ 0, λ, μ, υ, $\lambda \vee \mu$, $\lambda \vee\upsilon$, $\mu\vee\upsilon$, $\lambda \wedge \mu$, $\lambda \wedge\upsilon$, $\mu\wedge\upsilon$, $\lambda \vee [\mu\wedge\upsilon]$, $\mu\vee [\lambda \wedge\upsilon]$, $\upsilon\wedge [\lambda \vee \mu]$, 1}and S= { 0, λ, μ, υ, δ, $\lambda \vee \mu$, $\lambda \vee\upsilon$, $\lambda \vee \delta$, $\mu\vee\upsilon$, $\mu\vee\delta$, $\upsilon\vee\delta$, $\lambda \wedge \mu$, $\lambda \wedge\upsilon$, $\lambda \wedge\delta$, $\mu\wedge\upsilon$, $\mu\wedge\delta$, $\upsilon\wedge\delta$, $\lambda \vee [\mu\wedge\upsilon]$, $\mu\vee [\lambda \wedge\upsilon]$, $\delta\vee [\lambda \wedge\upsilon]$, $\upsilon\wedge [\lambda \vee \mu]$, 1} are fuzzy topologies on X. Now $\lambda \wedge\upsilon= \{(\mu\vee [\lambda\wedge\upsilon])\wedge\upsilon\wedge [\lambda \vee \mu] \wedge\upsilon\wedge\lambda\}$ and $\upsilon\wedge [\lambda \vee \mu] = \{ (\lambda \vee \mu)\wedge(\lambda \vee [\mu\wedge\upsilon]) \wedge(\lambda \vee\upsilon) \}$. Then $\lambda \wedge\upsilon$and $(\upsilon\wedge [\lambda \vee \mu])$ are fuzzy G_δ sets in(X, T) and $cl(\lambda \wedge\upsilon) = 1$ andcl $(\upsilon\wedge [\lambda \vee \mu]) = 1$. Since $cl([\lambda \wedge\upsilon] \wedge (\upsilon\wedge [\lambda \vee \mu) \} = 1$, the fuzzy topological space(X, T) is a fuzzy Volterra space.

Now consider the following fuzzy setsin the fuzzy topological space(X, S). $\lambda = \{ (\lambda)\wedge (\lambda \vee \mu)\wedge (\lambda \vee\upsilon) \wedge [\lambda \vee\delta] \wedge(\lambda\vee [\mu\wedge\upsilon]) \}$, $\mu= \{ (\mu)\wedge (\mu\vee\upsilon) \wedge [\mu\vee\delta] \wedge (\mu\vee [\lambda\wedge\upsilon])\}$ and$(\lambda\wedge\upsilon) = \{ (\upsilon) \wedge(\upsilon\wedge\delta)\wedge(\lambda\wedge\upsilon) \wedge(\delta\vee [\lambda \wedge\upsilon]) \wedge(\upsilon\wedge [\lambda \vee \mu])\}$.

Then λ, μ, $(\lambda \wedge\upsilon)$ are fuzzy $G_\delta$sets in(X, S) and$cl(\lambda)=1$, $cl(\mu)=1$, $cl(\lambda\wedge\upsilon)=1$.

Since $cl(\lambda\wedge\mu\wedge [\lambda\wedge\upsilon]) =1- \delta \neq 1$, the fuzzy topological space(X, S) is not a fuzzyVolterraspace.

Define afunction f: (X, T) → (X, S)byf (a)=a, f (b)=band f (c)=c.

Clearly *f is an fuzzy open function from the fuzzy Volterraspace (X, T) to the fuzzytopological space (X, S), which is not afuzzyVolterra Space.*

PROPOSITION 4.1: Ifa function f: (X, T) → (Y, S) from a fuzzy topological space (X, T) into another fuzzy topological space (Y, S) is fuzzy continuous, 1-1 and if δ is a fuzzy dense set in (X, T), then $f(\delta)$ is a fuzzy dense set in (Y, S).

PROOF: Suppose f (δ) is not a fuzzy dense set in (Y, S). Then there exists a fuzzyclosed set η in (Y, S) suchthat $f(\delta) < \eta < 1$. Then $f^{-1} f(\delta) < f^{-1}(\eta) < f^{-1}(1)$.

Since f is 1-1, $f^{-1}f(\delta) = \delta$. Hence we have $\delta < f^{-1}(\eta) < 1$. Since f is fuzzy continuous and η is a fuzzy closed set in (Y, S), $f^{-1}(\eta)$ is a fuzzy closed set in (X, T). Then cl $(\delta) \neq 1$, which is acontradiction toδ being a fuzzy dense set in (X, T). Therefore $f(\delta)$ is a fuzzy dense set in (Y, S).

THEOREM 4.1 [13] : Letf: (X, T) → (Y, S) be a fuzzy open function. Then for every fuzzy set β in (Y, S), $f^{-1}(cl(\beta)) \leq cl(f^{-1}(\beta))$.

PROPOSITION 4.2: If a function f: (X, T) → (Y, S) from a fuzzy topological space (X, T) onto another fuzzy topological space (Y, S) is fuzzy openand if λ is a fuzzy dense set in (Y, S) then $f^{-1}(\lambda)$ is a fuzzy dense set in (X, T).

PROOF: Let λ be a fuzzydense set in(Y, S). Then we have cl $(\lambda) = 1$. Since f is a fuzzy open function, by theorem 4.1, $f^{-1}(\mathrm{cl}(\lambda)) \leq \mathrm{cl}(f^{-1}(\lambda))$. Then $f^{-1}(1) \leq \mathrm{cl}(f^{-1}(\lambda))$, which implies that $1 \leq \mathrm{cl}(f^{-1}(\lambda))$.That is, $\mathrm{cl}(f^{-1}(\lambda)) = 1$. Hence$f^{-1}(\lambda)$ is a fuzzy dense set in (X, T).

PROPOSITION 4.3: If thefunction f: (X, T) → (Y, S) from a fuzzy topological space (X, T) onto another fuzzy topological space (Y, S) is fuzzy continuous, 1-1 and fuzzy open function and if (X, T) is a fuzzy Volterra space, then (Y, S) is a fuzzy Volterra space.

PROOF: Let (X, T) be a fuzzy Volterra space and λ_i' s (i = 1 to N) be fuzzy dense andfuzzy G_δ sets in (Y, S). Thencl $(\lambda_i) = 1$ and$\lambda_i = \wedge_{j=1}^{\infty}(\delta_{ij})$, where $\delta_{ij}'s$ are fuzzy open sets in (Y, S). Now $f^{-1}(\lambda_i) = f^{-1}(\wedge_{j=1}^{\infty}(\delta_{ij})) = \wedge_{j=1}^{\infty} f^{-1}(\delta_{ij})$.

Since f is fuzzy continuous and $\delta_{ij}'s$ are fuzzy open sets in (Y, S), $f^{-1}(\delta_{ij})$'s are fuzzy open sets in (X, T). Hence $\wedge_{j=1}^{\infty} f^{-1}(\delta_{ij})$ is a fuzzy G_δ set in (X, T).

Then $f^{-1}(\lambda_i)$ is a fuzzy G_δ set in (X, T)(i = 1 to N).

Now $\lambda_i$is a fuzzy dense set in (Y, S). Since f is a fuzzy open function from (X, T) onto (Y, S), by proposition 4.2, $f^{-1}(\lambda_i)$ is a fuzzy dense set in (X, T). Since (X, T) is a fuzzy Volterra space, we have cl $(\wedge_{i=1}^{N} f^{-1}(\lambda_i)) = 1$, where $f^{-1}(\lambda_i)$ 'sare fuzzy denseandfuzzy $G_\delta$sets in (X, T).Then, cl $(f^{-1}(\wedge_{i=1}^{N}(\lambda_i)) = 1$.

That is, $f^{-1}(\wedge_{i=1}^{N}(\lambda_i))$ is fuzzy dense in (X, T) . Since f is fuzzy continuous and 1-1, by proposition4.1, f $(f^{-1}(\wedge_{i=1}^{N}(\lambda_i))$ is a fuzzy dense set in (Y, S). That is, cl $(f(f^{-1} \wedge_{i=1}^{N}(\lambda_i))) = 1$……….(A). Since f is onto, $f(f^{-1} \wedge_{i=1}^{N}(\lambda_i))) = \wedge_{i=1}^{N}(\lambda_i)$.Then, from (A), we have cl $(\wedge_{i=1}^{N}(\lambda_i)) = 1$. Therefore (Y, S) is a fuzzy Volterra space.

THEOREM 4.2 [9] : Suppose (X, T) and (Y, S) be fuzzy topological spaces. Let f: (X, T) → (Y, S) be an onto function. Then the following conditions are equivalent.

1. f is somewhat fuzzy open.
2. If$\lambda$is a fuzzy dense set in (Y, S), then $f^{-1}(\lambda)$ is a fuzzy dense set in (X, T).

PROPOSITION 4.4: Ifthefunction f: (X, T) → (Y, S) from a fuzzy topological space (X, T) ontoanother fuzzytopological space (Y, S) is fuzzy continuous, 1-1 and somewhat fuzzy open function, then (X, T) is a fuzzy Volterra space if and only if(Y, S) is a fuzzy Volterra space.

PROOF: Let (X, T) be a fuzzy Volterra space and λ_i' s (i = 1 to N) be fuzzy dense andfuzzy G_δ sets in (Y, S).Then cl $(\lambda_i) = 1$ and$\lambda_i = \wedge_{j=1}^{\infty}(\delta_{ij})$, where $\delta_{ij}'s$ are fuzzy open sets in (Y, S). Hence $f^{-1}(\lambda_i) = f^{-1}(\wedge_{j=1}^{\infty}(\delta_{ij})) = \wedge_{j=1}^{\infty} f^{-1}(\delta_{ij})$

Since f is fuzzy continuous and $\delta_{ij}'s$ are fuzzy open sets in (Y, S), $f^{-1}(\delta_{ij})$'s are fuzzy opensets in (X, T). Hence $\wedge_{j=1}^{\infty} f^{-1}(\delta_{ij})$ is a fuzzy G_δ set in (X, T).

Then $f^{-1}(\lambda_i)$ is a fuzzy G_δ set in (X, T)(i = 1 to N).

Now $\lambda_i$is a fuzzy dense set in (Y, S). Since f is a somewhat fuzzy open function from (X, T) onto (Y, S), by theorem 4.2, $f^{-1}(\lambda_i)$ is a fuzzy dense set in (X, T). Since (X, T) is a fuzzy Volterra space, we have cl $(\wedge_{i=1}^{N} f^{-1}(\lambda_i)) = 1$, where $f^{-1}(\lambda_i)$'s are fuzzy denseandfuzzy $G_\delta$sets in (X, T). Then, cl $(f^{-1}(\wedge_{i=1}^{N}(\lambda_i)) = 1$.

That is, $f^{-1}(\wedge_{i=1}^{N}(\lambda_i)$ is fuzzy dense in (X, T).Since f is fuzzy continuous and1-1, by proposition 4.1, f $(f^{-1}(\wedge_{i=1}^{N}(\lambda_i)))$ is a fuzzy dense set in (Y, S). That is, cl $(f(f^{-1}\wedge_{i=1}^{N}(\lambda_i))) = 1$…………(A).Since f is onto, $f(f^{-1}(\wedge_{i=1}^{N}(\lambda_i))) = \wedge_{i=1}^{N}(\lambda_i)$. Then, from (A), we have cl $(\wedge_{i=1}^{N}(\lambda_i)) = 1$. Therefore (Y, S) is a fuzzy Volterra space.

Conversely, let (Y, S) be a fuzzy Volterra space and λ_i' s (i = 1 to N) be fuzzy denseandfuzzy G_δ sets in (Y, S).Then cl $(\lambda_i) = 1$ and $\lambda_i = \wedge_{j=1}^{\infty}(\delta_{ij})$, where $\delta'_{ij}s$ are fuzzy open sets in (Y, S). Hence $f^{-1}(\lambda_i) = f^{-1}(\wedge_{j=1}^{\infty}(\delta_{ij})) = \wedge_{j=1}^{\infty} f^{-1}(\delta_{ij})$.Since f is fuzzy continuous, $f^{-1}(\delta_{ij})$' s are fuzzy open sets in (X, T). Hence $f^{-1}(\lambda_i)$is a fuzzy G_δ set in (X, T). Since f is a somewhat fuzzy open function from (X, T) onto (Y, S), by theorem 4.2, $f^{-1}(\lambda_i)$ is a fuzzydense set in (X, T). Therefore$f^{-1}(\lambda_i)$'s are fuzzy dense and fuzzy G_δ sets in(X, T).

Now we claim that cl $(\wedge_{i=1}^{N}(f^{-1}(\lambda_i)) = 1$.Suppose that cl $(\wedge_{i=1}^{N}((f^{-1}(\lambda_i))) \neq 1$. Then $1 -$ cl $(\wedge_{i=1}^{N}((f^{-1}(\lambda_i)) \neq 0$, which implies that int $(\vee_{i=1}^{N}[1 - (f^{-1}(\lambda_i)])$ $= \text{Int}(\vee_{i=1}^{N}[(f^{-1}(1-\lambda_i))]) \neq 0$.Then there will be a non-zero fuzzy open set η_i in (X, T) such that$\eta_i \leq \vee_{i=1}^{N}[f(1-\lambda_i)]$ .Then $f(\eta_i) \leq f\ (\vee_{i=1}^{N}[f^{-1}(1-\lambda_i)]) \leq (\vee_{i=1}^{N}[f f^{-1}(1-\lambda_i)])$. Since f is onto, $ff^{-1}(1-\lambda_i) = (1-\lambda_i)$. Hence $f(\eta_i) \leq (\vee_{i=1}^{N}(1-\lambda_i)) = 1 - (\wedge_{i=1}^{N}(\lambda_i)$. Then, $\text{int}[f(\eta_i)] \leq int[1 - (\wedge_{i=1}^{N}\lambda_i)]$ implies that $\text{int}[f(\eta_i)] \leq 1 - \text{cl}(\wedge_{i=1}^{N}(\lambda_i)) = 1 - 1 = 0$ (since (Y, S) is a fuzzy Volterra spacecl$(\wedge_{i=1}^{N}(\lambda_i)) = 1$).That is, $\text{int}[f(\eta_i)] = 0$. But this is a contradiction to f being a somewhat fuzzy open function for which$\text{int}[f(\eta_i)] \neq 0$. Therefore (X, T) is a fuzzy Volterra space.

THEOREM 4.3 [9] : Let (X, T) and (Y, S) be any two fuzzy topological spaces. Let f: (X, T) → (Y, S) be a function. Then the following are equivalent.

1. f is somewhat fuzzy continuous .
2. If λ is a fuzzy closed set of (Y, S) such that $f^{-1}(\lambda) \neq 1$, then there exists a properfuzzy closed set μ of (X, T) such that $\mu > f^{-1}(\lambda)$.
3. If $\lambda$is a fuzzy dense set in (Y, S), then $f^{-1}(\lambda)$ is a fuzzy dense set in (X, T).

PROPOSITION 4.5: Ifthefunction f: (X, T) → (Y, S) from a fuzzy topological space (X, T) ontoanother fuzzytopologicalspace (Y, S) is somewhat fuzzy continuous, 1-1 and fuzzy open function, then (X, T) is a fuzzy Volterra space if and only if(Y, S) is a fuzzyVolterra space.

PROOF: Let (X, T) be a fuzzy Volterra space and λ_i' s (i = 1 to N) be fuzzy dense and fuzzy G_δ sets in(X, T). Then $Cl(\lambda_i) = 1$ and $\lambda_i = \wedge_{j=1}^{\infty}(\delta_{ij})$, where $\delta_{ij}'s$ are fuzzy open sets in (X, T). Since f is a fuzzy open function, $\delta_{ij}'s$are fuzzy opensets in (X, T) implies that f (δ_{ij}) 's are fuzzy open sets in (Y, S) . Hence $\wedge_{j=1}^{\infty} f(\delta_{ij})$ is a fuzzy $G_\delta$set in (Y, S). Now f^{-1} $(\wedge_{j=1}^{\infty} f(\delta_{ij})) = \wedge_{j=1}^{\infty} f^{-1}(f(\delta_{ij})) = \wedge_{j=1}^{\infty}(\delta_{ij}) = \lambda_i$ [since f is 1-1, f^{-1} f $(\lambda_i) = \lambda_i$] .Now f (λ_i) = f $(f^{-1}(\wedge_{j=1}^{\infty} f(\delta_{ij})) = \wedge_{j=1}^{\infty} f(\delta_{ij})$. [since f is onto] .Hence f (λ_i) is a fuzzy $G_\delta$set in (Y, S).

Since f is a fuzzy somewhat fuzzy continuous function and λ_i' s (i = 1 to N) are fuzzy dense sets in (X, T), by theorem 4.3, f (λ_i)'s arefuzzy dense sets in (Y, S).

Now we claim that cl $(\wedge_{i=1}^{N}(f(\lambda_i)) = 1$.Suppose thatcl $(\wedge_{i=1}^{N}(f(\lambda_i)) \neq 1$.

Then 1−cl $(\wedge_{i=1}^{N}(f(\lambda_i)) \neq 0$, which implies that int$(1-(\wedge_{i=1}^{N}(f(\lambda_i))) \neq 0$.Then Int $(\vee_{i=1}^{N}[1-f(\lambda_i)]) = \text{int}(\vee_{i=1}^{N}[f(1-\lambda_i)]) \neq 0$ { Since f is both 1-1 and onto, f (1 — λ_i) = 1 — f (λ_i) }.Then there will be a non-zero fuzzy open set $\eta_i$in(Y, S) such that $\eta_i \leq \vee_{i=1}^{N}[f(1-\lambda_i)]$ .Then $f^{-1}(\eta_i) \leq f^{-1}(\vee_{i=1}^{N}[f(1-\lambda_i)])$. Since fis a somewhat fuzzy continuous function and $\eta_i \in S$, int $(f^{-1}(\eta_i)) \neq 0$ implies thatint$(f^{-1}(\vee_{i=1}^{N}[f(1-\lambda_i)])) \neq 0$. Then int $((\vee_{i=1}^{N} f^{-1}[f(1-\lambda_i)])) \neq 0$, implies thatint $((\vee_{i=1}^{N}(1-\lambda_i))) \neq 0$, (Sincef is 1-1, f^{-1} f $(\lambda_i) = \lambda_i$] .Then int $(1-[\wedge_{i=1}^{N}(\lambda_i)]) \neq 0$ implies that $1-$cl $(\wedge_{i=1}^{N}(\lambda_i)) \neq 0$.This implies that cl $(\wedge_{i=1}^{N}(\lambda_i)) \neq 1$, a contradiction to (X, T) being a fuzzy Volterra space. Hence we must have cl $(\wedge_{i=1}^{N}(f(\lambda_i)) = 1$, wheref (λ_i) 's are fuzzy dense and fuzzy G_δ sets in(Y, S). Therefore (Y, S) is a fuzzy Volterra space.

Conversely, let (Y, S) be a fuzzy Volterra space and λ_i's (i = 1 to N) be fuzzy denseandfuzzy G_δ sets in (X, T). Then Cl (λ_i) = 1 and $\lambda_i = \wedge_{j=1}^{\infty}(\delta_{ij})$, where $\delta_{ij}'s$ are fuzzy open sets in (X, T).

Since f is a somewhat fuzzy continuous function and λ_i' s (i = 1 to N) are fuzzy dense sets in (X, T), by theorem 4.3, f (λ_i)'s are fuzzydense sets in (Y, S).Since f is a fuzzy open function, $\delta_{ij}'s$are fuzzy opensets in (X, T) implies that f (δ_{ij}) 's are fuzzy open sets in (Y, S) .Hence $\wedge_{j=1}^{\infty} f(\delta_{ij})$ is a fuzzy$G_\delta$set in (Y, S). Now f^{-1} $(\wedge_{j=1}^{\infty} f(\delta_{ij})) = \wedge_{j=1}^{\infty} f^{-1}(f(\delta_{ij})) = \wedge_{j=1}^{\infty}(\delta_{ij}) = \lambda_i$ [since f is 1-1, f^{-1} f $((\delta_{ij})=(\delta_{ij})$] . Now f (λ_i) = f $(f^{-1}(\wedge_{j=1}^{\infty} f(\delta_{ij})) = \wedge_{j=1}^{\infty} f(\delta_{ij})$ [since f is onto, ff^{-1} $((\delta_{ij})=(\delta_{ij})$] .Hence f (λ_i) is a fuzzy $G_\delta$set in(Y, S). Therefore f (λ_i) is a fuzzy denseandfuzzy G_δ sets in (Y, S).

Now we claim that cl $(\wedge_{i=1}^{N}((\lambda_i)) = 1$, where λ_i's (i = 1 to N) are fuzzy denseandfuzzy G_δ sets in (X, T). Suppose that cl $(\wedge_{i=1}^{N}((\lambda_i))) \neq 1$.Then, $1-$cl $(\wedge_{i=1}^{N}((\lambda_i)) \neq 0$, which implies that int$(1-(\wedge_{i=1}^{N}((\lambda_i)) \neq 0$. Thenwe haveint$(\vee_{i=1}^{N}[1-(\lambda_i)]) \neq 0$. Then there will be a non-zero fuzzy open set $\eta_i$in(X, T) suchthat$\eta_i \leq \vee_{i=1}^{N}[(1-\lambda_i)]$ .Then $f(\eta_i) \leq f(\vee_{i=1}^{N}[(1-\lambda_i)])$. Then $f(\eta_i) \leq (\vee_{i=1}^{N}[f(1-\lambda_i)])$. Since f is 1-1 and onto, $f(1-\lambda_i) = 1-f(\lambda_i)$. Hence $f(\eta_I) \leq (\vee_{i=1}^{N}[(1-f(\lambda_i)]) = 1-(\wedge_{i=1}^{N} f(\lambda_i))$. Then $\text{int}[f(\eta_i)] \leq$ int $(1-(\wedge_{i=1}^{N} f(\lambda_i))) = 1-\text{cl}(\wedge_{i=1}^{N} f(\lambda_i)) = 1-1 = 0$(since(Y, S) is a fuzzy Volterraspace, we havecl$(\wedge_{i=1}^{N} f(\lambda_i)) = 1$).That is, $\text{int}[f(\eta_i)] = 0$.But thisis a contradiction to f being a fuzzy open function for which$\text{int}[f(\eta_i)] = [f(\eta_i)$. Therefore (X, T) is a fuzzy Volterra space.

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