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On Fuzzy 2-Inner Product Spaces

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Abstract

In this paper, the concept of fuzzy 2-inner product space is introduced. By virtue of this definition α-2-norm is defined and the parallelogram law is proved. Again the relative fuzzy 2-norm with respect to the fuzzy 2-inner product space is defined. Some theorems and polarization identity is proved.

Keywords: Fuzzy 2-inner product space, crisp 2- norm, fuzzy 2-Hilbert space, parallelogram law, polarization identity.

INTRODUCTION:

The concept of fuzzy set was introduced by Zadeh [11] in 1965. A satisfactory theory of 2-norm on a linear space has been introduced and developed by Gähler in [12]. Katsaras [10] in 1984, first introduced the notion of fuzzy norm on a linear space.

In 1992, Felbin [8] introduced an idea of fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear space so that the corresponding metric associated this fuzzy norm is of Kaleva type [9] fuzzy metric.

Cheng and Modenson [7], Bag and Samanta [4-6] have given different definition of fuzzy normed spaces. In recent past lots of work have been done in this topic of fuzzy function analysis but only a few works have been done fuzzy inner product spaces.

Pinaki Majumdar and S.K.Samanta [2] have defined fuzzy inner product in a linear space and some properties of fuzzy inner product function. They also proved parallelogram law and polarization identity.

In this paper we introduce the concept of fuzzy 2-inner product space and some theorems on fuzzy 2- inner product space are established. We have also generalized parallelogram law and polarization identity in fuzzy 2-inner product space.

PRELIMINARIES:

In this section some definition and preliminaries results are given which will be used in this paper.

Definition 2.1 [1]: Let X be a linear space over a field F. A fuzzy subset N of $X \times X \times R$ (R is the set of real numbers) is called a fuzzy 2-norms on X if and only if.

1. for all $t \in R$, with $t \leq 0$, $N(x_1, x_2, t) = 0$
2. for all $t \in R$, with $t>0$, $N(x_1, x_2, t) = 1$ if and only if x_1 and x_2 are linearly dependent.
3. $N(x_1, x_2, t)$ is invarient under any permutation of x_1, x_2.
4. for all $t \in R$, with $t>0$

   $N(x_1, cx_2, t) = N(x_1, x_2, \frac{t}{|c|})$ if $c \neq 0$, $c \in F$.

5. for all $s, t \in R$

   $N(x_1, x_2 + x'_2, s+t) \geq \min\{N(x_1, x_2, s), N(x_1, x'_2, t)\}$

6. $N(x_1, x_2, \bullet)$ is non-decreasing function of R and

   $\lim_{t \to \infty} N(x_1, x_2, t) = 1$

then (X, N) is called a fuzzy 2-normed linear space.

Note (1.1) : N is fuzzy 2- norm on X in the sense that associated to $x \in X$ and $t \in R$, $N(x_1, x_2, t)$ indicates the truth value of the statement. The real number t is greater or equal to norm of x and which belongs to [0, 1].

Example 2.1: Let $(X, \|.,.\|)$ be 2-normed linear space define

$$N(x_1, x_2, t) = \frac{t}{t + \|x_{1,} x_2\|} \quad \text{when } t > 0, t \in R, x_1, x_2 \in X$$

Then (X, N) fuzzy 2-normed linear space.

Theorem 2.1[1]: Let (X, N) be a fuzzy 2- normed linear space. Assume that

(N_7) $N(x_1, x_2, t) > 0$

For all $t > 0$ implies x_1 and x_2 are linearly dependent, define $\|x_{1,} x_2\|_\alpha = \inf\{ t: N(x_1, x_2, t) \geq \alpha \in (0, 1) \}$. Then $\{\|.,.\|_\alpha : \alpha \in [0, 1) \}$ is an ascending family of 2-norms on X. These 2-norms are called α-2-norms on X corresponding to the fuzzy 2-norms.

Definition 2.2 [3]:

Let X be a linear space of dimension greater than one over the field K (either R or C).

The function $\langle ., . | . \rangle : X \times X \times X \to K$ is called a 2-inner product if the following conditions holds:

A1 : $\langle x, x | z \rangle \geq 0$ and $\langle x, x | z \rangle = 0$ iff x and z are linearly dependent.

A2 : $\langle x, x | z \rangle = \langle z, z | x \rangle$.

A3 : $\langle x, y | z \rangle = \langle y, x | z \rangle$.

A4 : $\langle \alpha x, y | z \rangle = \alpha \langle x, y | z \rangle$, for all scalars $\alpha \in K$.

A5 : $\langle x_1 + x_2, y | z \rangle = \langle x_1, y | z \rangle + \langle x_2, y | z \rangle$.

Therefore, the pair $(X, \langle ., . | . \rangle)$ is called a 2-inner product space.

3. Definition of fuzzy 2-inner product space and its properties :

In this section we introduce the definition of fuzzy 2-inner product on real or complex linear space and give an example of it. A fuzzy 2-norm is derived from the fuzzy 2-inner product and it is shown that the α-2-norms of the induced fuzzy 2- norm obey parallelogram law.

Definition (3.1): Let X be a linear space over the field C of complex numbers. Let $\eta : X \times X \times X \times C \to [0, 1]$ be mapping such that

1. For s, t $\in C$, $\eta(x_1 + x_1', x_2|x_3, |t| + |s|) \geq \min \{ \eta(x_1, x_2|x_3, |t|), \eta(x_1', x_2|x_3, |s|) \}$
2. For s, t $\in C$, $\eta(x_1, x_2|x_3, |st|) \geq \min \{ \eta(x_1, x_2|x_3, |s|^2), \eta(x_2, x_2|x_3, |t|^2) \}$
3. For t $\in C$, $\eta(x_1, x_2|x_3, t) = \eta(x_2, x_2|x_3, t)$
4. $(\alpha x_1, x_2|x_3, t) = \eta\left(x_1, x_2\middle|x_3, \frac{t}{|\alpha|}\right), \alpha (\neq 0) \in C, t \in C$
5. $(x_1, x_2|x_3, t) = 0, \forall t \in C \setminus R^+$
6. $(x_1, x_2|x_3, t) = 1, \forall t > 0$ iff, $f = 0$
7. $(x_1, x_2|x_3, t) : R \to [0, 1]$ is a monotonic non-decreasing function of R and $\lim_{t \to \infty} \eta(x_1, x_2|x_3, t) = 1$ as $t \to \infty$

Then η is said to be fuzzy 2-inner product space on X and the pair (X, η) is called a fuzzy 2-inner product space.

Example (3.1): Let (X, η) be an 2- inner product space. Define

$$\eta(x_1, x_2|x_3, t) = \begin{cases} \dfrac{t}{t + \langle x_1, x_2 | x_3 \rangle} & when \quad t > 0, t \in R \\ 0, & when \ t \leq 0 \end{cases}$$

Definition (3.2): Let (X, η) be a fuzzy 2-inner product space satisfying the condition

$\{\eta(x_1, x_2|x_3, t^2) > 0, t > 0\}$ implies that x = 0. Then $\forall\, \alpha(0,1)$, define $\|x,x\|_\alpha = \inf\{t: \eta(x_1,x_2|x_3,t^2) \geq \alpha\}$ is a crisp 2- norm on X, called the α-2-norm on X generated from η.

In the sequel we can consider the following condition, For $x_1, x_2, x_3 \in X$ and $s, t \in R$,

$\eta(x_1 + x_2, x_1 + x_2|x_3, 2s^2) \wedge \eta(x_1 - x_2, x_1 - x_2|x_3, 2t^2)$
$\geq \eta(x_1, x_1|x_3, t^2) \wedge \eta(x_2, x_2|x_3, s^2)$.

Theorem (3.2): Let η be a fuzzy 2-inner product space on X. Then N : X × X × R →[0, 1] defined by N(x, y, t) = $\begin{cases} \eta(x,x|y,t^2) when\ t > 0, t \in R \\ 0, \quad when\, t \leq 0 \end{cases}$

is a fuzzy 2-norm on X.

Proof :

1. By definition N(x, y, t) = 0 $\forall\, t \in R$ and $t \leq 0$.
2. Again from eq. (6) for t>0, $\eta(x, x|y, t^2) = 1$ iff, f=0 therefore it is follows that N(x, y, t) = 1 iff, x = 0.
3. For all t > 0 and c $\neq 0$, N(cx, cy, t) = $\eta(cx, cx|y, t^2)$

$= \eta\left(x, cx|y, \frac{t^2}{|c|}\right)$

$= \eta\left(x, x|y, \frac{t^2}{|c|^2}\right)$

$= N(x, y, \frac{t}{|c|})$

4. To prove that, $N(x_1 + x_1', x_2, s + t) \geq \min\{N(x_1, x_2, s), N(x_1', x_2, t)\}$ for every

$s, t \in R, x_1, x_1', x_2 \in X$.

Let us consider the following cases:

s+t < 0, (b) s = t = 0, s > 0, t < 0 or s < 0, t > 0, (c) s+t > 0, s, t ≥ 0.

Let us prove (c)

$N(x_1 + x_1', x_2, s + t) = \eta\left(x_1 + x_1', x_1 + x_1'|x_2, (s + t)^2\right)$

$= \eta\left(x_1 + x_1', x_1 + x_1'|x_2, s^2 + st + st + t^2\right)$

$\geq \eta(x_1, x_1|x_2, s^2) \wedge \eta\left(x_1', x_1'|x_2, t^2\right) \wedge \eta\left(x_1, x_1'|x_2, st\right)$

$\geq \eta(x_1, x_1|x_2, t^2) \wedge \eta\left(x_1', x_1'|x_2, t^2\right)$

$= \min\{N(x_1, x_2, s^2), N\left(x_1', x_1'|x_2, t^2\right)\}$

and (b) follows immediately.

5. From equation (7), $\eta(x_1, x_1|x_2, \bullet)$ is a monotonic non-decreasing function and tends to 1 as t → ∞. Thus $N(x_1, x_2, \bullet)$ is a monotonic non-decreasing function and tends to 1 as t → ∞.

Theorem (3.3) (Parallelogram Law) :
Let be a fuzzy 2-inner product space on X, $\alpha \in (0,1)$ and $\|.,.\|_\alpha$ be the α-2-norm generated from fuzzy 2-inner product η on X. Then,

$$\|x - y, z\|_\alpha^2 + \|x + y, z\|_\alpha^2 = 2(\|x, z\|_\alpha^2 + \|y, z\|_\alpha^2).$$

Proof:
$\|x - y, z\|_\alpha^2 + \|x + y, z\|_\alpha^2 = \inf\{ t^2 : t \in R^+$ and N(x-y, z, t) $\geq \alpha\}$ + $\inf\{ s^2 : s \in R^+$ and N(x + y, z, t) $\geq \alpha\}$, where N is the fuzzy 2- norm induced from η.

$= \inf\{ t^2 + s^2 : t, s \in R^+$ and N(x-y, z, t) $\geq \alpha$, N(x + y, z, t) $\geq \alpha\}$

$= \inf\{ t^2 + s^2 : t, s \in R^+$ and N(x-y, z, t) ∧ N(x + y, z, t) $\geq \alpha\}$

Also,
$2(\|x, z\|_\alpha^2 + \|y, z\|_\alpha^2) = 2 \inf\{ p^2 : p \in R^+$ and N(x, z, p) $\geq \alpha\}$ + $2 \inf\{ q^2 : q \in R^+$ and N(y, z, q) $\geq \alpha\}$

$= 2 \inf\{ p^2 + q^2 : p, q \in R^+$ and N(x, z, p) ∧ N(y, z, q) $\geq \alpha\}$

Again, N(x-y, z, $\sqrt{2}p$) ∧ N(x + y, z, $\sqrt{2}q$) □ N(x, z, p) ∧ N(y, z, q)

Hence we get,

$$\|x - y, z\|_\alpha^2 + \|x + y, z\|_\alpha^2 \leq 2(\|x, z\|_\alpha^2 + \|y, z\|_\alpha^2)$$

Also,

$$2(\|x, z\|_\alpha^2 + \|y, z\|_\alpha^2) = 2\left(\left\|\frac{x+y}{2} + \frac{x-y}{2}, z\right\|_\alpha^2 + \left\|\frac{x+y}{2} - \frac{x-y}{2}, z\right\|_\alpha^2\right)$$

$$= \frac{1}{2}(\|(x + y) + (x - y), z\|_\alpha^2 + \|(x + y) - (x - y), z\|_\alpha^2)$$

$$\leq \frac{1}{2}[\|x + y, z\|_\alpha^2 + \|x - y, z\|_\alpha^2 + 2\|x + y, z\|_\alpha \|x - y, z\|_\alpha + \|y + x, z\|_\alpha^2 + \|x - y, z\|_\alpha^2 - 2\|x + y, z\|_\alpha \|x - y, z\|_\alpha]$$

$$\leq \frac{1}{2} \times 2[\|x + y, z\|_\alpha^2 + \|x - y, z\|_\alpha^2]$$

$$= \|x + y, z\|_\alpha^2 + \|x - y, z\|_\alpha^2$$

Therefore,

$$\|x - y, z\|_\alpha^2 + \|x + y, z\|_\alpha^2 = 2(\|x, z\|_\alpha^2 + \|y, z\|_\alpha^2).$$

Hence Proved.

Theorem(3.4): If a fuzzy 2-inner product space (X, η) is strictly convex and if $\eta(x, y|z, t) = \|x, z\|_\alpha \|y, z\|_\alpha$ then x and y are linearly dependent.

Proof : Suppose (X, η) is strictly convex and $\eta(x, y|z, t) = \|x, z\|_\alpha \|y, z\|_\alpha$, then

$$(\|x, z\|_\alpha + \|y, z\|_\alpha)(\|y, z\|_\alpha) \geq \|x + y, z\|_\alpha \|y, z\|_\alpha$$

$\eta\ \eta(x + y, y|z, t)$
$\geq \min\{\eta(x, y|z, t), \eta(y, y|z, t)\}$
$= \|x, z\|_\alpha \|y, z\|_\alpha + \|y, z\|_\alpha \|y, z\|_\alpha$
$= (\|x, z\|_\alpha + \|y, z\|_\alpha)\ \|y, z\|_\alpha$

Therefore $\|x + y, z\|_\alpha = \|x, z\|_\alpha + \|y, z\|_\alpha$ and since (X, η) is strictly convex so x and y are linearly dependent.

Theorem (3.5): Let (X, η) be a fuzzy 2-inner product space. If $\eta(x, y|z, t) = \eta\left(x', y|z, t\right)$ for all $y \in X$ then x and x' are dependent.

Proof: By definition,

$\eta\left(x - x', y|z, t\right) \geq \min\{\eta(x, y|z, t), \eta\left(-x', y|z, t\right)\}$
$= \min\{\eta(x, y|z, t), \eta\left(x', y|z, \frac{t}{|-1|}\right)\}$
$= \min\{\eta(x, y|z, t), \eta\left(x', y|z, t\right)\}$
$= \min\{\ \eta(x, y|z, t)\}$

So x and x' are linearly dependent.

Definition (3.2): A sequence $\{x_n, x'_n\}$ in a fuzzy 2- normed linear space (X, N) is called a cauchy sequence with respect to α -2-norm if

$$\lim\left(\left\|(x_n, x'_n) - (x_m, x'_m)\right\|_\alpha\right) = 0 \quad \text{as} \quad m, n \to \infty,$$

$$\Rightarrow \lim\ \left(\left\|(x_n - x_m), (x'_n - x'_m)\right\|_\alpha\right) = 0 \quad \text{as} \quad m,\ n \to \infty.$$

Definition (3.3): A sequence $\{x_n, x'_n\}$ in a fuzzy 2- normed linear space (X, N) is called convergent sequence with respect to α -2-norm if there exist $(x, x') \in A \times B$, where A, B are subspace of (X, N) such that

$$\lim\left(\left\|(x_n, x'_n) - (x, x')\right\|_\alpha\right) = 0 \quad \text{as} \quad n \to \infty,$$

$$\lim\left(\left\|(x_n - x) - (x'_n - x')\right\|_\alpha\right) = 0 \quad \text{as} \quad n \to \infty.$$

Definition (3.4): A fuzzy 2-normed linear space (X, N) is said to be complete if every cauchy sequence conveges.

Definition(3.5): A complete fuzzy 2-normed linear space (X, N) is called fuzzy 2-Banach space.

Definition (3.6): A complex fuzzy 2-Banach space (X, N) is said to be fuzzy 2-Hilbert space. If its α -2-norm is induced by the fuzzy 2-inner product.

Theorem (3.6): Polarisation identity : If x, y, z are the elements of the fuzzy 2-inner product space (X, η), then

$$4\eta(x, y|z, st) = \|x+y, z\|_\alpha^2 - \|x-y, z\|_\alpha^2 + i\|x+iy, z\|_\alpha^2 - i\|x-iy, z\|_\alpha^2.$$

Proof : We know that

$$\|x+y, z\|_\alpha^2 - \|x-y, z\|_\alpha^2 + i\|x+iy, z\|_\alpha^2 - i\|x-iy, z\|_\alpha^2$$

$$=\eta((x+y),(x+y)|z,(s+t)^2) - \eta((x-y),(x-y)|z,(s-t)^2)$$
$$+ i\eta((x+iy),(x+iy)|z,(s+t)^2) - i\eta((x-iy),(x-iy)|z,(s-t)^2)$$
$$= \eta((x+y),(x+y)|z,s^2+st+st+t^2) - \eta((x-y),(x-y)|z,s^2-st-st+t^2)$$
$$+ i\eta((x+iy),(x+iy)|z,s^2+st+st+t^2) - i\eta((x-iy),(x-iy)|z,s^2-st-st+t^2)$$
$$= \eta(x,x|z,s^2)+\eta(x,y|z,st)+\eta(y,x|z,st)+\eta(y,y|z,t^2)-\eta(x,x|z,s^2)-\eta(x-y|z,-st)$$
$$-\eta(-y,x|z,-st)-\eta(-y,-y|z,t^2)+i\eta(x,x|z,s^2)+i\eta(x,iy|z,st)+i\eta(iy,x|z,st)$$
$$+i\eta(iy,iy|z,t^2)-i\eta(x,x|z,s^2)-i\eta(x,-iy|z,-st)-i\eta(-iy,x|z,-st)-i\eta(-iy,-iy|z,t^2)$$
$$= \eta(x,x|z,s^2)+\eta(x,y|z,st)+\eta(y,x|z,st)+\eta(y,y|z,t^2)-\eta(x,x|z,s^2)-\eta\left(x,y|z,\frac{-st}{|-1|}\right)$$
$$-\eta\left(y,x|z,\frac{st}{|-1|}\right)-\eta(y,y|z,t^2)+i\eta(x,x|z,s^2)+i\eta(x,iy|z,st)+i\eta(iy,x|z,st)$$
$$+i\eta(iy,iy|z,t^2)-i\eta(x,x|z,s^2)-i\eta\left(x,iy|z,\frac{-st}{|-1|}\right)-i\eta\left(iy,x|z,\frac{-st}{|-1|}\right)-i\eta(iy,iy|z,t^2)$$
$$= \eta(x,y|z,st)+\eta(y,x|z,st)-\eta(x,y|z,-st)-\eta(y,x|z,-st)+i\eta(x,iy|z,st)+i\eta(iy,x|z,st)$$
$$+i\eta(x,iy|z,st)+i\eta(iy,x|z,st)$$
$$= \eta(x,y|z,st)+\eta(y,x|z,st)+\eta(x,y|z,st)+\eta(y,x|z,st)+i\eta(x,iy|z,st)+i\eta(iy,x|z,st)$$
$$+i\eta(x,iy|z,st)+i\eta(iy,x|z,st)$$
$$= \eta(x,y|z,st)+\eta(y,x|z,st)+\eta(x,y|z,st)+\eta(y,x|z,st)+i(-i)\eta(x,y|z,st)+i(i)\eta(y,x|z,st)$$
$$+i(-i)\eta(x,y|z,st)+i(i)\eta(y,x|z,st)$$
$$= \eta(x,y|z,st)+\eta(y,x|z,st)+\eta(x,y|z,st)+\eta(y,x|z,st)+\eta(x,y|z,st)-\eta(y,x|z,st)$$
$$+\eta(x,y|z,st)-\eta(y,x|z,st)$$
$$= 4\eta(x,y|z,st)$$

Hence

$$4\eta(x, y|z, st) = \|x+y, z\|_\alpha^2 - \|x-y, z\|_\alpha^2 + i\|x+iy, z\|_\alpha^2 - i\|x-iy, z\|_\alpha^2.$$

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