Regular properties of fuzzy finite state automata with unique membership transition on an input symbol

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Abstract

A fuzzy finite state automaton with unique membership transition (uffsa) on an input symbol is an ordered quintuple $\mathcal{A} = (Q, \Sigma, \mu, i, f)$, where $Q$ is a finite non-empty set of states; $\Sigma$ is a finite non-empty set of input symbols; the fuzzy subset $\mu : Q \times \Sigma \times Q \rightarrow [0, 1]$ is a fuzzy function of $Q \times \Sigma \times [0, 1]$ into $Q$; $i$ is a fuzzy subset of $Q$, i.e., $i : Q \rightarrow [0, 1]$, called the fuzzy subset of initial states; $f$ is a fuzzy subset of $Q$, i.e., $f : Q \rightarrow [0, 1]$, called the fuzzy subset of final states. In this paper we consider the closure properties of the operations such as product, reversal and images of a morphism on an uffsa.

AMS subject classification:

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1. Introduction

In a fuzzy finite state automaton (ffsa), there may be more than one fuzzy transition from a state on an input symbol with a given membership value [3, 5]. This development was followed by the postulation called deterministic fuzzy finite state automaton (dffsa) [3], that is, from a state for an input symbol there will be at most one transition. However, it only acts as a deterministic fuzzy recognizer, that is, for any fuzzy recognizer $M$ there is a deterministic fuzzy recognizer $M_1$ with the same behaviour in the sense, for any string $x$, the membership value of $x$ in the language generated by $M$ and is that of in $M_1$ need not be the same. In [4] an uffsa is introduced by incorporating a condition that the membership function has a unique membership transition on an input symbol, that is, from a state for an input symbol with a given membership value there will be at most one transition, here there may be another transition from the state for the same input symbol with different membership value, so uffsa is much simpler than ffsa.
any string $x$, the membership value of $x$ in the language generated by ffsa and is that of in the corresponding uffsa will be the same.

In this contribution, the authors are mainly interested on analyzing some of the closure properties of fuzzy behaviors, if $\mathcal{A}$ and $\mathcal{B}$ are two uffsa’s with behaviour (regular languages) $L_\mathcal{A}$ and $L_\mathcal{B}$ respectively. $\mathcal{A} \times \mathcal{B}$ is defined and proved that $L_{\mathcal{A} \times \mathcal{B}} = L_\mathcal{A} \land L_\mathcal{B}$, similarly for the shuffle product. The reversal of the automaton $\mathcal{A}$ is defined as $\mathcal{A}^\omega$.

We give inverse image and direct image on fuzzy automata, related to a morphism $F : \Gamma^* \rightarrow \Sigma^*$ where $\Gamma$ and $\Sigma$ are two alphabets, we prove that there exists an $\Sigma-$automaton $\mathcal{A}'$ for the $\Gamma-$automaton $\mathcal{A}$ satisfying that $L_{\mathcal{A}'} = LF(A)$.

2. Basic definitions

For the definition of standard notions, we refer to [2,3,4,8].

**Definition 2.1.** A fuzzy finite state automaton with unique membership transition on an input symbol is denoted by uffsa and is defined by $\mathcal{A} = (Q, \Sigma, \mu, i, f)$, where

(i) $Q$ is a finite non-empty set of states.

(ii) $\Sigma$ is a finite non-empty set of input symbols.

(iii) The fuzzy subset $\mu : Q \times \Sigma \times Q \rightarrow [0, 1]$ is a fuzzy function of $Q \times \Sigma \times [0, 1]$ into $Q$. i.e., if $\forall p \in Q, a \in \Sigma, m \in [0, 1], \mu(p, a, q) = \mu(p, a, q')$ for some $q, q' \in Q$ then $q = q'$.

(iv) $i$ is a fuzzy subset of $Q$, i.e., $i : Q \rightarrow [0, 1]$, called the fuzzy subset of initial states.

(v) $f$ is a fuzzy subset of $Q$, i.e., $f : Q \rightarrow [0, 1]$, called the fuzzy subset of final states.

**Definition 2.2.** Let $\mathcal{A} = (Q, \Sigma, \mu, i, f)$ be an uffsa, the extended fuzzy transition for $\mathcal{A}$ is the fuzzy subset $\mu^* : Q \times \Sigma^* \times Q \rightarrow [0, 1]$ has been defined as follows: for all $p, q \in Q, a \in \Sigma, x \in \Sigma^*$,

$$
\mu^*(p, \lambda, q) = \begin{cases}
1, & \text{if } p = q \\
0, & \text{if } p \neq q
\end{cases}
$$

$$
\mu^*(p, xa, q) = \bigvee \{\mu^*(p, x, r) \land \mu(r, a, q) | r \in Q\}
$$

**Definition 2.3.** The fuzzy behaviour of an uffsa $\mathcal{A} = (Q, \Sigma, \mu, i, f)$ is a fuzzy subset of $\Sigma^*$ and denoted by $L_\mathcal{A}, L_{\mathcal{A}} : \Sigma^* \rightarrow [0, 1]$ is defined by $L_\mathcal{A}(x) = \bigvee \{|i(p) \land \mu^*(p, x, q) \land f(q) | q \in Q\} | p \in Q\}$. 
Definition 2.4. A fuzzy subset $L : \Sigma^* \to [0, 1]$ is called a fuzzy behaviour over $\Sigma$. A fuzzy behaviour $L$ over $\Sigma$ is called regular language if there exists an uffsa $\mathcal{A}$ with the fuzzy behavior same as $L$. i.e., $L_{\mathcal{A}} = L$.

3. Product

We consider two unique fuzzy finite state automata $\mathcal{A}$ and $\mathcal{B}$ on which we review a number of basic operations on the Product, Reversal automata and analyze their behaviors.

Definition 3.1. Let $\mathcal{A} = (Q_A, \Sigma, \mu_A, i_A, f_A)$ and $\mathcal{B} = (Q_B, \Sigma, \mu_B, i_B, f_B)$ be two uffsa’s, $Q_A \cap Q_B = \phi$, the product of $\mathcal{A}$ and $\mathcal{B}$ is the uffsa $\mathcal{A} \times \mathcal{B}$ and is defined by $\mathcal{C} = (Q_C, \Sigma, \mu_C, i_C, f_C)$ where $Q_C = Q_A \times Q_B$ and

(i) $\mu_C : (Q_A \times Q_B) \times \Sigma \times (Q_A \times Q_B) \to [0, 1]$ is defined by

$$
\mu_C((p', p''), a, (q', q'')) = \mu_A(p', a, q') \wedge \mu_B(p'', a, q''),
$$

$\forall p', q' \in Q_A, p'', q'' \in Q_B$.

(ii) $i_C : Q_A \times Q_B \to [0, 1]$ is defined by

$$
i_C(p', p'') = \begin{cases} 
i_A(p') \wedge i_B(p''), & \text{if } p' \in Q_A, p'' \in Q_B \\ 0, & \text{Otherwise} \end{cases}
$$

(iii) $f_C : Q_A \times Q_B \to [0, 1]$ is defined by

$$
f_C(q', q'') = \begin{cases} f_A(q') \wedge f_B(q''), & \text{if } q' \in Q_A, q'' \in Q_B \\ 0, & \text{Otherwise} \end{cases}
$$

Lemma 3.2. Let $\mathcal{A} = (Q_A, \Sigma, \mu_A, i_A, f_A)$ and $\mathcal{B} = (Q_B, \Sigma, \mu_B, i_B, f_B)$ be two uffsa’s. Then

$$
\mu^*_C((p', p''), x, (q', q'')) = \mu_A^*(p', x, q') \wedge \mu_B^*(p'', x, q''), \forall x \in \Sigma^*
$$

Theorem 3.3. Let $\mathcal{A} = (Q_A, \Sigma, \mu_A, i_A, f_A)$ and $\mathcal{B} = (Q_B, \Sigma, \mu_B, i_B, f_B)$ be two uffsa’s with $L_{\mathcal{A}}$ and $L_{\mathcal{B}}$ as fuzzy behaviors respectively. Then $L_{\mathcal{C}}$ of $\mathcal{A} \times \mathcal{B}$ is the fuzzy behavior of an uffsa $\mathcal{C}$ such that $L_{\mathcal{C}} = L_{\mathcal{A}} \wedge L_{\mathcal{B}}$.

Proof. Let $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ and $L_{\mathcal{C}}$ be the fuzzy behavior accepted by $\mathcal{C} = (Q_C, \Sigma, \mu_C, i_C, f_C)$. Let $((p', p''), x, (q', q'')) \in (Q_A \times Q_B) \times \Sigma \times (Q_A \times Q_B)$ and let $x \in \Sigma^*$
Now,

\[ L_{\mathcal{E}}(x) = \bigvee \left\{ \left( i_A(p', p'') \land \mu_C^*(p', p'', x, (q', q'')) \right) \land \left( f_C(q', q'') | (q', q'') \in Q_A \times Q_B \right) \right\} \]

\[ \land \left( (p', p'') \in Q_A \times Q_B \right) \]

\[ = \bigvee \left\{ \left( i_A(p') \land i_B(p'') \land \mu_A^*(p', x, q') \land (\mu_B^*(p'', x, q')) \right) \land \right. \]

\[ \left. \left( (f_A(q') \land f_B(q'')) | (q', q'') \in Q_A \times Q_B \right) | (p', p'') \in Q_A \times Q_B \right] \text{ (by definition 5)} \]

\[ = \left( \bigvee \left\{ i_A(p') \land \mu_A^*(p', x, q') \land f_A(q') \right| q' \in Q_A \right) \land \right. \]

\[ \left. \left( \bigvee \left\{ i_B(p'') \land \mu_B^*(p'', x, q'') \land f_B(q'') \right| q'' \in Q_B \right) \right) \quad \left( p' \in Q_A \right) \land \]

\[ \left. \left( p'' \in Q_B \right) \right) \]

\[ L_{\mathcal{E}}(x) = L_{\mathcal{A}}(x) \land L_{\mathcal{B}}(x) \]

\[ \blacksquare \]

**Example 3.4.** Consider an uffsa \( \mathcal{A} = (Q_A, \Sigma, \mu_A, i_A, f_A) \) where \( Q_A = \{q_1, q_2, q_3\} \), \( \Sigma = \{a, b\} \), \( \mu_A : Q_A \times \Sigma \times Q_A \rightarrow [0, 1] \) is defined as follows:

\[
\begin{align*}
\mu_A(q_1, a, q_1) &= 0.8 \\
\mu_A(q_1, b, q_1) &= 1.0 \\
\mu_A(q_3, b, q_2) &= 0.3 \\
\mu_A(q_2, a, q_3) &= 0.7 \\
\mu_A(q_3, a, q_3) &= 0.4 \\
\mu_A(q_1, b, q_2) &= 0.6
\end{align*}
\]

\( i_A : Q_A \rightarrow [0, 1] \) is defined by \( i_A(q_1) = 1.0, i_A(q_2) = 0.5 \).

\( f_A : Q_A \rightarrow [0, 1] \) is defined by \( f_A(q_3) = 1.0 \).

The fuzzy behavior of \( \mathcal{A} \) is the fuzzy subset \( L_{\mathcal{A}} : \Sigma^* \rightarrow [0, 1] \) such that

\[
L_{\mathcal{A}}(x) = \begin{cases} 
0.3, & \text{if } x \in \{a, b\}^*ba\{a, b\}^* \\
0.3, & \text{if } x \in \{a, b\}^* \\
0, & \text{Otherwise}
\end{cases}
\]

Consider another uffsa \( \mathcal{B} = (Q_B, \Sigma, \mu_B, i_B, f_B) \) where \( Q_B = \{q_1', q_2', q_3'\} \), \( \Sigma = \{a, b\} \), \( \mu_B : Q_B \times \Sigma \times Q_B \rightarrow [0, 1] \) is defined as follows:

\[
\begin{align*}
\mu_B(q_1', a, q_1') &= 0.9 \\
\mu_B(q_1', b, q_1') &= 0.6 \\
\mu_B(q_3', b, q_3') &= 0.5 \\
\mu_B(q_2', a, q_3') &= 0.8 \\
\mu_B(q_2', a, q_3') &= 0.8 \\
\mu_B(q_3', b, q_3') &= 0.9
\end{align*}
\]

\( i_B : Q_B \rightarrow [0, 1] \) is defined by \( i_B(q_1') = 0.8, i_B(q_2') = 0.1 \).

\( f_B : Q_B \rightarrow [0, 1] \) is defined by \( f_B(q_3') = 0.3 \).

The fuzzy behavior of \( \mathcal{B} \) is the fuzzy subset \( L_{\mathcal{B}} : \Sigma^* \rightarrow [0, 1] \) such that

\[
L_{\mathcal{B}}(x) = \begin{cases} 
0.3, & \text{if } x \in \{a, b\}^*ba\{a, b\}^* \\
0.1, & \text{if } x \in \{a, b\}^* \\
0, & \text{Otherwise}
\end{cases}
\]
Now, the product of $A$ and $B$ is defined by $C = (Q_C, \Sigma, \mu_C, i_C, f_C)$ where $\mu_C : Q_C \times \Sigma \times Q_C \rightarrow [0, 1]$ is defined as follows:

$$
\begin{align*}
\mu_C((q_1, q'_1), a, (q_1, q'_1)) &= 0.8 \\
\mu_C((q_1, q'_1), b, (q_1, q'_1)) &= 0.6 \\
\mu_C((q_2, q'_2), a, (q_3, q'_3)) &= 0.5 \\
\mu_C((q_2, q'_2), b, (q_3, q'_3)) &= 0.8 \\
\mu_C((q_3, q'_3), a, (q_3, q'_3)) &= 0.4 \\
\mu_C((q_3, q'_3), b, (q_3, q'_3)) &= 0.3
\end{align*}
$$

$i_C : Q_C \rightarrow [0, 1]$ is defined by $i_C(q_1, q'_1) = 0.8$, $i_C(q_2, q'_2) = 0.6$.

$f_C : Q_C \rightarrow [0, 1]$ is defined by $f_C(q_3, q'_3) = 0.3$.

The transition diagram is as shown below:

![Transition Diagram]

The fuzzy behavior of $C$ is the fuzzy subset $L_C : \Sigma^* \rightarrow [0, 1]$ such that

$$
L_C(x) = \begin{cases} 
0.3, & \text{if } x \in \{a, b\}^*ba\{a, b\}^* \\
0.1, & \text{if } x \in \{a, b\}^* \\
0, & \text{Otherwise}
\end{cases}
$$

### 4. Reversal

A path $c$ in $A$ with $|c| = \sigma_1\sigma_2\ldots\sigma_k$ yields a path $c^\rho$ in $A^\rho$ with $|c^\rho| = \sigma_k\sigma_{k-1}\ldots\sigma_1$ where $\rho : \Sigma^* \rightarrow \Sigma^*$ is the reversal function defined by $\rho(1) = 1$, $\rho(\sigma) = \sigma$, $\rho(st) = \rho(t)\rho(s)$, where $s, t \in \Sigma^*$.

**Definition 4.1.** Let $A = (Q, \Sigma, \mu, i, f)$ be an uffsa with membership transition on an input symbol, the reversal uffsa is defined by $A^\rho = (Q, \Sigma, f, i)$ where

(i) $\mu : Q \times \Sigma \times Q \rightarrow [0, 1]$ is defined by

$$
\mu(p, a, q) = \begin{cases} 
\mu(q, a, p), & \text{if } p, q \in Q \\
0, & \text{Otherwise}
\end{cases}
$$

(ii) $i : Q \rightarrow [0, 1]$ is defined by

$$
i(p) = \begin{cases} 
f(p), & \text{if } p \in Q \\
0, & \text{Otherwise}
\end{cases}
$$
(iii) \( f : Q \rightarrow [0, 1] \) is defined by
\[
    f(q) = \begin{cases} 
    i(q) & \text{if } q \in Q \\
    0 & \text{Otherwise}
    \end{cases}
\]

The fuzzy behavior of \( A^\varrho \) is defined by
\[
    L_{A^\varrho} = \lor \{ [i(q) \land \mu^x(q, x, p) \land f(p)] : p \in Q \} | q \in Q \}
\]

5. Images of a morphism

We consider fuzzy finite state automata, related to a morphism \( F : \Gamma^* \rightarrow \Sigma^* \) where \( \Gamma \) and \( \Sigma \) are two alphabets, its concept will manifest on an Inverse Image and Direct Image by incorporating their conditions that the membership function has a unique membership transition on an input symbol.

**Definition 5.1. Inverse image:** Let \( A = (Q, \Sigma, \mu, i, f) \) be a \( \Sigma \)-automaton and the \( \Gamma \)-automaton is defined by \( A = (Q, \Gamma, \mu^{-1}, i, f) \) and \( F : \Gamma^* \rightarrow \Sigma^* \) is defined as a morphism where \( \Gamma \) and \( \Sigma \) are two alphabets, \( \mu^{-1} : Q \times \Gamma \times Q \rightarrow [0, 1] \) is defined as follows:

(i) \( \mu^{-1}(p, a', q) = \mu(p, a, q) \) if \( F(a') = a, a \in \Sigma, a' \in \Gamma, \forall p, q \in Q \).

(ii) \( \mu^{-1}(q, a', q) = 1 \), if \( F(a') = 1 \in \Sigma^*, \forall q \in Q \).

(iii) \( i : Q \rightarrow [0, 1] \) is defined by
\[
    i(p) = \begin{cases} 
    i(p) & \text{if } p \in Q \\
    0 & \text{Otherwise}
    \end{cases}
\]

(iv) \( f : Q \rightarrow [0, 1] \) is defined by
\[
    f(q) = \begin{cases} 
    f(q) & \text{if } q \in Q \\
    0 & \text{Otherwise}
    \end{cases}
\]

**Example 5.2.** Consider an uffsa \( A = (Q, \Sigma, \mu, i, f) \) where \( Q = \{q_1, q_2, q_3\}, \Sigma = \{a, b\}, \mu : Q \times \Sigma \times Q \rightarrow [0, 1] \) is as shown below in transition diagram:

\( i : Q \rightarrow [0, 1] \) is defined by \( i(q_1) = 0.9, i(q_2) = 0.3 \).

\( f : Q \rightarrow [0, 1] \) is defined by \( f(q_3) = 0.2 \).

The transition diagram is as shown below:
The fuzzy behavior of $\mathcal{A}^i$ is the fuzzy subset $L_{\mathcal{A}^i} : \Sigma^* \rightarrow [0, 1]$ such that

$$
L_{\mathcal{A}^i}(x) = \begin{cases} 
0.2, & \text{if } x \in \{a, b\}^*aa\{a, b\}^* \\
0.2, & \text{if } x \in \{a, b\}^*ba\{a, b\}^* \\
0.2, & \text{if } x \in a\{a, b\}^* \\
0, & \text{Otherwise}
\end{cases}
$$

Consider another ufssa $\mathcal{A}'' = (Q, \Gamma, \mu^{-1}, i, f)$ where $Q = \{q_1, q_2, q_3\}, \Gamma = \{a', b'\}$, $\mu^{-1} : Q \times \Gamma \times Q \rightarrow [0, 1]$ is defined as follows:

- $\mu^{-1}(q_1, a', q_1) = 1.0$
- $\mu^{-1}(q_1, b', q_1) = 1.0$
- $\mu^{-1}(q_1, a', q_2) = 0.5$
- $\mu^{-1}(q_1, b', q_2) = 0.4$
- $\mu^{-1}(q_2, a', q_3) = 0.7$
- $\mu^{-1}(q_3, a', q_3) = 1.0$
- $\mu^{-1}(q_3, b', q_3) = 1.0$

$i : Q \rightarrow [0, 1]$ is defined by $i(q_1) = 0.9, i(q_2) = 0.3$.

$f : Q \rightarrow [0, 1]$ is defined by $f(q_3) = 0.2$.

The fuzzy behavior of $\mathcal{A}''$ is the fuzzy subset $L_{\mathcal{A}''} : \Gamma^* \rightarrow [0, 1]$ such that

$$
L_{\mathcal{A}''}(x) = \begin{cases} 
0.2, & \text{if } x \in \{a', b'\}^*a'\{a', b'\}^* \\
0.2, & \text{if } x \in \{a', b'\}^*b'\{a', b'\}^* \\
0, & \text{Otherwise}
\end{cases}
$$

Therefore, $L_{\mathcal{A}^i}(x) = L_{\mathcal{A}''}(F^{-1}(x))$.

**Lemma 5.3.** $\mu^*(p, x, r) = \mu^{*-1}(p, x', r)$ where $F(x') = x$.

**Proof.** Let $x \in \Sigma^*$ where $x = a_1a_2a_3 \cdots a_n$

Now,

$$
F(a'_1) = a_1 \Rightarrow a'_1 = F^{-1}(a_1) \\
F(a'_2) = a_2 \Rightarrow a'_2 = F^{-1}(a_2) \\
F(a'_3) = a_3 \Rightarrow a'_3 = F^{-1}(a_3) \\
\cdots \cdots \cdots \cdots \cdots \\
F(a'_{n-1}) = a_{n-1} \Rightarrow a'_{n-1} = F^{-1}(a_{n-1}) \\
F(a'_n) = a_n \Rightarrow a'_n = F^{-1}(a_n)
$$

So,

$$
a_1a_2a_3 \cdots a_n = F(a'_1)F(a'_2)F(a'_3) \cdots F(a'_n) = F(a'_1a'_2a'_3 \cdots a'_n) = F(x') \Rightarrow x = F(x')
$$
Theorem 5.4. Let \( F : \Gamma^* \to \Sigma^* \) be a morphism where \( \Gamma \) and \( \Sigma \) are alphabets. Let \( A = (Q, \Sigma, \mu, i, f) \) be an uffsa, then there exists a \( \Gamma \)-automaton \( A' = (Q, \Gamma, \mu^{-1}, i, f) \) such that the fuzzy behavior of \( A \) and \( A' \) are equal. i.e., \( L_A(x) = L_{A'}(F(x)) \).

Now, \[
\mu^*(p, x, r) = \mu^*(p, a_1a_2a_3 \cdots a_n, r) = \mu^{*-1}(p, F(a_1')F(a_2')F(a_3') \cdots F(a_n'), r) = \mu^{*-1}(p, F^{-1}(a_1'a_2'a_3' \cdots a_n'), r).
\]

Hence \( \mu^*(p, x, r) = \mu^{*-1}(p, x', r). \) \( \blacksquare \)

Theorem 5.5. Direct Image: A function \( F : \Gamma^* \to \Sigma^* \) be a morphism satisfying \( F(a') \neq 1, \forall a' \in \Gamma \). Given a \( \Gamma \)-automaton, \( A = (Q, \Gamma, \mu, i, f) \). Define a \( \Sigma \)-automaton \( A' = (Q', \Sigma, \mu', i', f') \) with \( Q' \supset Q \) where \( \mu' : Q' \times \Sigma \times Q' \to [0, 1] \) is defined by

(i) if \( \mu(p, a', q) > 0 \), then include \( \mu'(p, a', q) = \mu(p, a, q) \)

where \( F(a) = a' \in \Gamma, a \in \Sigma \).

(ii) if \( \mu(p, a', q) > 0, F(a) = a_1'a_2'a_3' \cdots a_n', n > 1 \), then include

\[
\mu'(p_{i-1}', a_i', p_i') = \mu(p, a, q)
\]

where \( i = 1, 2, 3, \cdots, n \)

where \( p_0 = p, p_n' = q \) and \( p_1', p_2', \cdots, p_{n-1}' \) are new states in \( Q' - Q \).

(iii) \( i' : Q' \to [0, 1] \) is defined by

\[
i'(p) = \begin{cases} 
i(p) & \text{if } p \in Q \\
0 & \text{Otherwise} \end{cases}
\]

(iv) \( f' : Q' \to [0, 1] \) is defined by

\[
f'(q) = \begin{cases} 
f(q) & \text{if } q \in Q \\
0 & \text{Otherwise} \end{cases}
\]

Theorem 5.6. A function \( F : \Gamma^* \to \Sigma^* \) be a morphism satisfying \( F(a') \neq 1, \forall a' \in \Gamma \), let \( A = (Q, \Gamma, \mu, i, f) \) be an uffsa, then there exists an \( \Sigma \)-automaton \( A' = (Q', \Sigma, \mu', i, f) \) with \( Q' \supset Q \) such that the fuzzy behavior of \( A \) and \( A' \) are equal. i.e., \( L_A(x) = L_{A'}(F(x)) \).
Proof. Let \( x = a_1'a_2'a_3' \cdots a_n' \) where \( a_i \in \Sigma, i = 1, 2, \cdots, n \)

Now,

\[
L_{s\sigma}'(x) = \bigvee \{ (i(p) \land \mu^*(p, a_1'a_2'a_3' \cdots a_n'), \land f(q) | q \in Q) | p \in Q \}
\]

\[
= \bigvee \{ (i(p) \land \mu^*(p, a_1a_2a_3, \cdots, a_n'), q) \land f(q) | q \in Q) | p \in Q \}
\]

\[
= \bigvee \{ (i(p) \land \mu'(p_0', a_1', a_2', p_1') \land \mu'(p_1', a_1', a_2', p_2') \land \cdots \land \mu'(p_{n-1}', a_{n-1}', a_n', p_n')
\]

\[
\land f'(q) | p_0' = p, p_n' = q \in Q' | p_1', p_2', \ldots, p_n' \in Q' - Q \}
\]

\[
= \bigvee \{ (i(p) \land \mu'(p, a_1'a_2'a_3' \cdots a_n', q)
\]

\[
\land f'(q) | p, q \in Q' | p_1', p_2', \ldots, p_{n-1}' \in Q' - Q \}
\]

\[
= \bigvee \{ (i(p) \land \mu'(p, F(x), q) \land f'(q) | q \in Q' | p \in Q' \}
\]

\[
= L_{s\sigma}'F(x).
\]

Therefore, \( L_{s\sigma}'F(x) = L_{s\sigma}(x) \).

\[
\square
\]

6. Shuffle Product

Given subsets \( A \subset \Sigma^*, B \subset \Gamma^* \) with \( \Sigma \cap \Gamma = \phi \), the Shuffle product

\[ (A \Delta B) \subset (\Sigma \cup \Gamma)^* \]

consists of all words of the form \( s_1g_1s_2g_2 \cdots s_ng_n \in (\Sigma \cup \Gamma)^* \) with \( s_1, s_2, s_3, \ldots, s_n \in A, \ g_1, g_2, g_3, \ldots, g_n \in B \).

Definition 6.1. Let \( \mathcal{A} = (Q_A, \Sigma, \mu_A, i_A, f_A) \) and \( \mathcal{B} = (Q_B, \Gamma, \mu_B, i_B, f_B) \) be two uffsa’s; \( Q_A \cap Q_B = \phi \), the shuffle product of \( \mathcal{A} \) and \( \mathcal{B} \) is the uffsa \( \mathcal{C} = \mathcal{A} \Delta \mathcal{B} \) by setting \( Q_C = Q_A \times Q_B, i_C = i_A \times i_B, f_C = f_A \times f_B \), i.e., \( \mathcal{C} = (Q_C, (\Sigma \cup \Gamma), \mu_C, i_C, f_C) \) where

(i) \( \mu_C : Q_C \times (\Sigma \cup \Gamma) \times Q_C \rightarrow [0, 1] \) is defined as follows

\[
\forall p, q \in Q_A, p', q' \in Q_B, a \in \Sigma.
\]

\[
\mu_C((p, p'), a, (q, q')) = \begin{cases} 
\mu_A(p, a, q), & \text{if } p' = q' \\
\mu_B(p', a, q'), & \text{if } p = q \\
0, & \text{Otherwise}
\end{cases}
\]

(ii) \( i_C : Q_C \rightarrow [0, 1] \) is defined by

\[
i_C(p, p') = \begin{cases} 
i_A(p) \land i_B(p') & \text{, if } p \in Q_A, p' \in Q_B \\
0 & \text{, Otherwise}
\end{cases}
\]

(iii) \( f_C : Q_C \rightarrow [0, 1] \) is defined by

\[
f_C(p, p') = \begin{cases} 
f_A(p) \land f_B(p') & \text{, if } p \in Q_A, p' \in Q_B \\
0 & \text{, Otherwise}
\end{cases}
\]
Lemma 6.2. If \( x = a_1 b_1 a_2 b_2 \cdots a_n b_n, a_i \in \Sigma, b_i \in \Gamma \) then

\[
\mu^*_C((q_i, q'_i), x, (q_j, q'_j)) = \bigvee\{\mu^*_A(q_i, a_1 a_2 \cdots a_n, q_j) \land \mu^*_B(q'_i, b_1 b_2 \cdots b_n, q'_j)\} \\
\forall q_i, q_j \in Q_A, q'_i, q'_j \in Q_B.
\]

Proof. We prove the result by induction on \( n \).

Let \( n = 1 \).

\[
\mu^*_C((q_i, q'_i), a_1 b_1, (q_j, q'_j)) \\
= \mu_C((q_i, q'_i), a_1, (q_j, q'_j)) \land \mu_C((q_i, q'_i), b_1, (q_j, q'_j)) \\
\forall q_i \in Q_A, q'_i \in Q_B. \\
[ \because l = j \text{ for all non-zero membership values}].
\]

\[
= \bigvee \{\mu_A(q_i, a_1, q_j) \land \mu_B(q'_i, b_1, q'_j)|q_i, q_j \in Q_A, q'_i, q'_j \in Q_B\}
\]

Thus the result is true for \( n = 1 \).

Assume that the result if true for all strings of the length \( \leq n - 1 \).
Now,

\[
\mu^*_C((q_i, q'_i), a_1 b_1 a_2 b_2 \cdots a_n b_n, (q_j, q'_j)) \\
= \mu^*_C((q_i, q'_i), a_1 b_1 \cdots a_{n-1} b_{n-1}, (q_p, q'_p)) \\
\land \mu_C((q_p, q'_p), a_n b_n, (q_j, q'_j)) \\
= \bigvee \{\mu^*_A(q_i, a_1 a_2 \cdots a_{n-1}, q_p) \\
\land \mu^*_B(q'_i, b_1 b_2 \cdots b_{n-1}, q'_p) \land \mu_A(q_p, a_n, q_j) \\
\land \mu_B(q'_p, b_n, q'_j)|q_p \in Q_A, q'_p \in Q_B\}
\]

\[
= \bigvee \{\mu^*_A(q_i, a_1 a_2 \cdots a_n, q_j) \\
\land \mu^*_B(q'_i, b_1 b_2 \cdots b_n, q'_j)|\forall q_i, q_j \in Q_A, q'_i, q'_j \in Q_B\}
\]

Hence the result is true for \( n \).

Theorem 6.3. Let \( \mathcal{A} = (Q_A, \Sigma, \mu_A, i_A, f_A) \) and \( \mathcal{B} = (Q_B, \Gamma, \mu_B, i_B, f_B) \) be two uffsa’s with \( L_{\mathcal{A}} \) and \( L_{\mathcal{B}} \) as the fuzzy behaviors respectively. Then \( L_{\mathcal{C}} \) of \( \mathcal{A} \bigtriangleup \mathcal{B} \) is the fuzzy behavior of an uffsa \( \mathcal{C} \) such that \( L_{\mathcal{C}} = L_{\mathcal{A}} \land L_{\mathcal{B}} \).
Proof. Let \( x = a_1b_1a_2b_2 \cdots a_nb_n \), where \( a_1a_2 \cdots a_n \in \Sigma, \ b_1b_2 \cdots b_n \in \Gamma \).

\[
L_{\varphi} = \bigvee \{ \{ i_C(q_i, q'_i) \land \mu_C^*(c_i, q_i, x, (q_j, q'_j)) \
\land f_C(q_j, q'_j) \in Q_A \times Q_B \}(q_i, q'_i) \in Q_A \times Q_B \} \\
= \bigvee \{ i_A(q_i) \land i_B(q'_i) \land \mu_A^*(c_i, a_i, q_j) \land \mu_B^*(q'_i, b_i, q'_j)) \\
\land f_A(q_j) \land f_B(q'_j) \in Q_A \times Q_B \}(q_i, q'_i) \in Q_A \times Q_B \}[\text{by lemma 3}] \\
= \bigvee \{ i_A(q_i) \land \mu_A^*(c_i, a_i, q_j) \land f_A(q_j) | q_j \in Q_A \}| q_i \in Q_A \} \\
\bigvee \{ i_B(q'_i) \land \mu_B^*(q'_i, b_i, q'_j) \land f_B(q'_j) | q'_j \in Q_B \}| q'_i \in Q_B \}
\]

\[
L_{\varphi} = L_A \land L_B \blacksquare
\]

7. Conclusion

In this paper, we have made an attempt to study more about the regular properties of the unique fuzzy finite state automata (uffsa) on which we have analyzed a number of basic operations on the product, reversal and images of a morphism on a set of strings such as inverse image, direct image and shuffle product by incorporating their conditions that the membership function has a unique transition on an input symbol. We have proved some results with examples in this direction. Further, we have extended and explored the notion of uffsa’s as a background for doing further work in theory of fuzzy automaton to the fuzzy context. Finally, we have made a humble beginning in this direction. However, many concepts are yet to be fuzzyfied in the context of uffsa.

References


