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Few Inequalities in Connection with Relative Defects of Wronskians

Sanjib Kumar Datta
Department of Mathematics, University of Kalyani,
P.O. Kalyani, District, Nadia, PIN-741235, West Bengal, India
E-mail: sanjib_kr_datta@yahoo.co.in

Sudipta Kumar Pal
Uttar Dalkhola High School, Dalkhola,
District, Uttar Dinajpur, PIN-733201, West Bengal, India
E-mail: sudipta.pal07@yahoo.com

Abstract

In this paper we wish to compare the relative Valiron defect with the relative Nevanlinna defect of wronskians generated by transcendental meromorphic functions.

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1. Introduction, Definitions and Notations

Let f be a meromorphic function defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $n\,(t, a; f)$ ($\overline{n}\,(t, a; f)$) the number of a-points (distinct a-points) of f in $|z| \leq t$, where an ∞-point is a pole of f. We put

$$N\,(r, a; f) = \int_0^r \frac{n\,(t, a; f) - n(0, a; f)}{t} dt + n\,(0, a; f) \log r$$

and

$$\overline{N}\,(r, a; f) = \int_0^r \frac{\overline{n}\,(t, a; f) - \overline{n}\,(0, a; f)}{t} dt + \overline{n}\,(0, a; f) \log r.$$

The function $N(r,a;f)$ $\left(\overline{N}(r,a;f)\right)$ is called the counting function of a-points (distinct a-points) of f. In many occassions $N(r,\infty;f)$ and $\overline{N}(r,\infty;f)$ are denoted by $N(r,f)$ and $\overline{N}(r,f)$ respectively.

We also put

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta,$$

where $\log^+ x=\log x$, if $x\geq 1$, $=0$, if $0\leq x<1$.

For $a\in\mathbb{C}$ we denote by $m\left(r,\frac{1}{f-a}\right)$ by $m(r,a;f)$ and we mean by $m(r,\infty;f)$ the function $m(r,f)$, which is called the proximity function of f.

The function $T(r,f)=m(r,f)+N(r,f)$ is called the characteristic function of f. If $a\in\mathbb{C}\cup\{\infty\}$ the quantity

$$\delta(a;f)=1-\limsup_{r\to\infty}\frac{N(r,a;f)}{T(r,f)}=\liminf_{r\to\infty}\frac{m(r,a;f)}{T(r,f)}$$

is called the Nevanlinna deficiency of the value 'a'.

From the second fundamental theorem it follows that the set of values of $a\in\mathbb{C}\cup\{\infty\}$ for which $\delta(a;f)>0$ is countable and $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)\leq 2$ (cf.[2,p.43]). If in particular, $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, we say that f has the maximum deficiency sum.

Similarly, the Valiron deficiency $\Delta(a;f)$ of the value 'a' is defined as

$$\Delta(a;f)=1-\liminf_{r\to\infty}\frac{N(r,a;f)}{T(r,f)}=\limsup_{r\to\infty}\frac{m(r,a;f)}{T(r,f)}.$$

Milloux [4] introduced the concept of absolute defect of 'a' with respect to f'. Later Xiong [6] extended this definition. He introduced the term $\delta_R^{(k)}(a;f)=1-\limsup_{r\to\infty}\frac{N\left(r,a;f^{(k)}\right)}{T(r,f)}$, for $k=1,2,3,...$ and called it the relative Nevanlinna defect of 'a' with respect to $f^{(k)}$. Xiong [6] has shown various relations between the usual defects and the relative defects for meromorphic functions. Singh [5] introduced the term relative defect for distinct zeros and poles and established various relations between it, relative defects and usual defects.

We may now recall the following definitions:

Definition 1.1. The order ρ_f and lower order λ_f of a meromorphic function f are defined as follows:

$$\rho_f=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}\text{ and }\lambda_f=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

The term $S(r, f)$ denotes any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ through all values of r if f is of finite order and except possibly for a set of r of finite linear measure otherwise. We do not explain the standard definitions and notations of the value distribution and the Nevanlinna theory as those are available in [2].

The following definitions are well known.

Definition 1.2. A meromorphic function $a = a(z)$ is called small function with respect to f if $T(r, a) = S(r, f)$.

Definition 1.3. Let $a_1, a_2, \ldots, a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, \ldots, a_k, f)$ the wronskian determinant of $a_1, a_2, \ldots, a_k, f$ i.e.,

$$L(f) = \begin{vmatrix} a_1 & a_2 & .. & .. & a_k & f \\ a_1' & a_2' & .. & .. & a_k' & f \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ a_1^{(k)} & a_2^{(k)} & .. & .. & a_k^{(k)} & f^{(k)} \end{vmatrix}$$

In the paper we call the terms

$$\delta_A^L(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; L(f))}{T(r, L(f))} = \liminf_{r\to\infty} \frac{m(r, a; L(f))}{T(r, L(f))},$$

the usual Nevanlinna defect or the absolute Nevanlinna defect of the value 'a' with respect to $L(f)$ and

$$\Delta_A^L(a; f) = 1 - \liminf_{r\to\infty} \frac{N(r, a; L(f))}{T(r, L(f))} = \limsup_{r\to\infty} \frac{m(r, a; L(f))}{T(r, L(f))},$$

the usual Valiron defect or the absolute Valiron defect of the value 'a' with respect to $L(f)$. Again

$$\delta_R^L(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; L(f))}{T(r, f)},$$

the relative Nevanlinna defect of 'a' with respect to $L(f)$ and

$$\Delta_R^L(a; f) = 1 - \liminf_{r\to\infty} \frac{N(r, a; L(f))}{T(r, f)},$$

the relative Valiron defect of 'a' with respect to $L(f)$ and prove various relations among them. Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the wronskians following Datta and Das [1].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [3] Let f be a transcendental meromorphic function having the maximum deficiency sum. Then

$$\lim_{r\to\infty}\frac{T\left(r,L\left(f\right)\right)}{T\left(r,f\right)}=1+k-\delta\left(\infty;f\right).$$

Lemma 2.2. Let f be a transcendental meromorphic function having the maximum deficiency sum. Then for any α,

$$\Delta_R^L\left(\alpha;f\right)=\left\{k\delta\left(\infty;f\right)-k\right\}+\limsup_{r\to\infty}\frac{m\left(r,\alpha;L\left(f\right)\right)}{T\left(r,f\right)}$$

and

$$\delta_R^L\left(\alpha;f\right)=\left\{k\delta\left(\infty;f\right)-k\right\}+\liminf_{r\to\infty}\frac{m\left(r,\alpha;L\left(f\right)\right)}{T\left(r,f\right)}.$$

Proof. In view of Lemma 2.1, we obtain that

$$\begin{aligned}
\Delta_R^L\left(\alpha;f\right) &= 1-\liminf_{r\to\infty}\frac{N\left(r,\alpha;L\left(f\right)\right)}{T\left(r,f\right)}\\
&= 1-\liminf_{r\to\infty}\frac{N\left(r,\alpha;L\left(f\right)\right)}{T\left(r,L\left(f\right)\right)}.\lim_{r\to\infty}\frac{T\left(r,L\left(f\right)\right)}{T\left(r,f\right)}\\
&= 1-\liminf_{r\to\infty}\frac{N\left(r,\alpha;L\left(f\right)\right)}{T\left(r,L\left(f\right)\right)}.\left\{1+k-k\delta\left(\infty;f\right)\right\}\\
&= \left\{1+k-k\delta\left(\infty;f\right)\right\}\left\{1-\liminf_{r\to\infty}\frac{N\left(r,\alpha;L\left(f\right)\right)}{T\left(r,L\left(f\right)\right)}\right\}\\
&\quad +\left\{k\delta\left(\infty;f\right)-k\right\}\\
&= \left\{1+k-k\delta\left(\infty;f\right)\right\}\limsup_{r\to\infty}\frac{m\left(r,\alpha;L\left(f\right)\right)}{T\left(r,L\left(f\right)\right)}+\left\{k\delta\left(\infty;f\right)-k\right\}\\
&= \left\{1+k-k\delta\left(\infty;f\right)\right\}.\left\{\limsup_{r\to\infty}\frac{m\left(r,\alpha;L\left(f\right)\right)}{T\left(r,f\right)}.\lim_{r\to\infty}\frac{T\left(r,f\right)}{T\left(r,L\left(f\right)\right)}\right\}\\
&\quad +\left\{k\delta\left(\infty;f\right)-k\right\}\\
&= \left\{k\delta\left(\infty;f\right)-k\right\}+\limsup_{r\to\infty}\frac{m\left(r,\alpha;L\left(f\right)\right)}{T\left(r,f\right)}.
\end{aligned}$$

This proves the first part of the lemma. ■

Similarly we can prove the second part of the lemma.

Lemma 2.3. [2] Let k be any positive integer and $\Psi = \sum_{i=0}^{k} a_i f^{(i)}$, where a_i are meromorphic functions such that $T(r, a_i) = S(r, f)$ for $i = 0, 1, 2, \ldots, k$ then $m\left(r, \frac{\Psi}{f}\right) = S(r, f)$.

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. Let f be a transcendental meromorphic function satisfying the condition $m(r, f) = S(r, f)$. If a, b and c are three non zero finite complex numbers, then

$$3\delta(a; f) + 2\delta(b; f) + \delta(c; f) + 5\Delta_R^L(\infty; f) \leq 5\Delta(\infty; f) + 5\Delta_R^L(0; f).$$

Proof. Let us consider the following identity

$$\begin{aligned}\frac{b-a}{f-a} &= \left[\frac{L(f)}{f-a}\left\{\frac{f-a}{L(f)} - \frac{f-b}{L(f)}\right\} - \frac{f-c}{L(f)}\cdot\frac{L(f)}{f}\cdot\frac{L(f)}{f-a}\right.\\ &\quad \left.\left\{\frac{f-a}{L(f)} - \frac{f-b}{L(f)}\right\}\right]\cdot\frac{f}{c}\end{aligned}$$

Since $m\left(r, \frac{1}{f-a}\right) \leq m\left(r, \frac{b-a}{f-a}\right) + O(1)$, $m\left(r, \frac{f}{c}\right) \leq m(r, f) + O(1)$ and in view of Lemma 3 we obtain that

$$\begin{aligned}m\left(r, \frac{b-a}{f-a}\right) &\leq m\left(r, \frac{f-a}{L(f)}\right) + m\left(r, \frac{f-b}{L(f)}\right) + m\left(r, \frac{f-c}{L(f)}\right)\\ &\quad + m\left(r, \frac{f-a}{L(f)}\right) + m\left(r, \frac{f-b}{L(f)}\right) + m\left(r, \frac{f}{c}\right)\\ &\quad + S(r, f)\end{aligned}$$

$$\begin{aligned}i.e., m\left(r, \frac{1}{f-a}\right) &\leq 2m\left(r, \frac{f-a}{L(f)}\right) + 2m\left(r, \frac{f-b}{L(f)}\right) + m\left(r, \frac{f-c}{L(f)}\right)\\ &\quad + m(r, f) + S(r, f) + O(1)\\ i.e., m\left(r, \frac{1}{f-a}\right) &\leq 2T\left(r, \frac{f-a}{L(f)}\right) - 2N\left(r, \frac{f-a}{L(f)}\right) + 2T\left(r, \frac{f-b}{L(f)}\right)\\ &\quad - 2N\left(r, \frac{f-b}{L(f)}\right) + T\left(r, \frac{f-c}{L(f)}\right) - N\left(r, \frac{f-c}{L(f)}\right)\\ &\quad + m(r, f) + S(r, f) + O(1). \qquad (1)\end{aligned}$$

Now by the relation $T\left(r,\frac{1}{f}\right)=T\left(r,f\right)+O\left(1\right)$ and in view of Lemma 2.3 it follows from (1) that

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right) &\leq 2T\left(r,\frac{L\left(f\right)}{f-a}\right)-2N\left(r,\frac{f-a}{L\left(f\right)}\right)+2T\left(r,\frac{L\left(f\right)}{f-b}\right)\\ &\quad -2N\left(r,\frac{f-b}{L\left(f\right)}\right)+T\left(r,\frac{L(f)}{f-c}\right)-N\left(r,\frac{f-c}{L\left(f\right)}\right)\\ &\quad +m\left(r,f\right)+S\left(r,f\right)+O\left(1\right) \end{aligned}$$

$$\begin{aligned} i.e.,\ m\left(r,\frac{1}{f-a}\right) &\leq 2\left\{N\left(r,\frac{L\left(f\right)}{f-a}\right)-N\left(r,\frac{f-a}{L\left(f\right)}\right)\right\}\\ &\quad +2\left\{N\left(r,\frac{L\left(f\right)}{f-b}\right)-N\left(r,\frac{f-b}{L\left(f\right)}\right)\right\}\\ &\quad +N\left(r,\frac{L(f)}{f-c}\right)-N\left(r,\frac{f-c}{L\left(f\right)}\right)\\ &\quad +m\left(r,f\right)+S\left(r,f\right)+O\left(1\right). \end{aligned} \tag{2}$$

In view of {p. 34, [2]} it follows from (2) that

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right) &\leq 2N\left(r,L\left(f\right)\right)+2N\left(r,\frac{1}{f-a}\right)-2N\left(r,f\right)-2N\left(r,\frac{1}{L\left(f\right)}\right)\\ &\quad +2N\left(r,L\left(f\right)\right)+2N\left(r,\frac{1}{f-b}\right)-2N\left(r,f\right)-2N\left(r,\frac{1}{L\left(f\right)}\right)\\ &\quad +N\left(r,L\left(f\right)\right)+N\left(r,\frac{1}{f-c}\right)-N\left(r,f\right)-N\left(r,\frac{1}{L\left(f\right)}\right)\\ &\quad +m\left(r,f\right)+S\left(r,f\right)+O\left(1\right)\\ i.e.,m\left(r,\frac{1}{f-a}\right) &\leq 5N\left(r,L\left(f\right)\right)-5N\left(r,f\right)-5N\left(r,\frac{1}{L\left(f\right)}\right)+2N\left(r,\frac{1}{f-a}\right)\\ &\quad +2N\left(r,\frac{1}{f-b}\right)+N\left(r,\frac{1}{f-c}\right)+m\left(r,f\right)\\ &\quad +S\left(r,f\right)+O\left(1\right). \end{aligned} \tag{3}$$

In view of $m(r,f)=S\left(r,f\right)$ it follows from (3) that

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right) &\leq 5N\left(r,L\left(f\right)\right)-5N\left(r,f\right)-5N\left(r,\frac{1}{L\left(f\right)}\right)\\ &\quad +2N\left(r,\frac{1}{f-a}\right)+2N\left(r,\frac{1}{f-b}\right)+N\left(r,\frac{1}{f-c}\right)\\ &\quad +S(r,f) \end{aligned}$$

$$i.e., \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)} \leq 5\liminf_{r\to\infty}\left\{\frac{N(r,L(f))}{T(r,f)}-\frac{N(r,f)}{T(r,f)}-\frac{N\left(r,\frac{1}{L(f)}\right)}{T(r,f)}\right\} + 2\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{T(r,f)} + 2\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-b}\right)}{T(r,f)} + \limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-c}\right)}{T(r,f)}$$

$$i.e., \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)} \leq 5\liminf_{r\to\infty}\left\{\frac{N(r,L(f))}{T(r,f)}-\liminf_{r\to\infty}\frac{N(r,f)}{T(r,f)} -\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{L(f)}\right)}{T(r,f)}\right\} + 2\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{T(r,f)} + 2\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-b}\right)}{T(r,f)} + \limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-c}\right)}{T(r,f)}$$

$$\begin{aligned} i.e., \delta(a;f) &\leq 5\{1-\Delta_R^L(\infty;f)\}-5\{1-\Delta(\infty;f)\} \\ &\quad -5\left\{1-\Delta_R^L(0;f)\right\}+2\{1-\delta(a;f)\} \\ &\quad +2\{1-\delta(b;f)\}+\{1-\delta(c;f)\} \end{aligned}$$

$$i.e., 3\delta(a;f)+2\delta(b;f)+\delta(c;f)+5\Delta_R^L(\infty;f) \leq 5\Delta(\infty;f)+5\Delta_R^L(0;f).$$

This proves the theorem. ■

Remark 3.2. The sign ' $\leq$' in Theorem 1 can be replaced by '$<$' only. This is evident from the following example.

Example 3.3. Let $f = \exp z$. Then $\Delta(\infty;f) = \Delta_R^L(0;f) = \Delta_R^L(\infty;f) = 1$ and $\delta(0;f) = \delta(\infty;f) = 1$. So $\delta(a;f) = \delta(b;f) = \delta(c;f) = 0$. Thus $5 < 10$.

Theorem 3.4. Let f be a transcendental meromorphic function having maximum deficiency sum satisfying the condition $m(r,f) = S(r,f)$. If a and c are two non zero finite complex numbers, then

$$\delta(0;f)+\Delta_R^L(\infty;f)+\delta(c;f) \leq 2+\Delta_R^L(0;f).$$

Proof. Let us consider the following identity

$$\frac{c}{f} = [\{1 - \frac{f-c}{L(f)}.\frac{L(f)}{f}\}\{\frac{L(f)}{f-a}.\frac{1}{L(f)}\}].(f-a)$$

In view of Lemma 2.3 we obtain that

$$m\left(r, \frac{c}{f}\right) \leq m\left(r, \frac{f-c}{L(f)}\right) + m\left(r, \frac{1}{L(f)}\right) + m(r, f-a) + S(r, f). \quad (4)$$

Since $m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{c}{f}\right) + O(1)$ and $m(r, f-a) \leq m(r, f) + O(1)$ we get from (4) that

$$\begin{aligned} m\left(r, \frac{1}{f}\right) &\leq T\left(r, \frac{f-c}{L(f)}\right) - N\left(r, \frac{f-c}{L(f)}\right) + T\left(r, \frac{1}{L(f)}\right) - N\left(r, \frac{1}{L(f)}\right) \\ &\quad + m(r, f) + S(r, f) + O(1). \end{aligned} \quad (5)$$

As $T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$, it follows from (5) that

$$\begin{aligned} m\left(r, \frac{1}{f}\right) &\leq T\left(r, \frac{L(f)}{f-c}\right) - N\left(r, \frac{f-c}{L(f)}\right) + T(r, L(f)) - N\left(r, \frac{1}{L(f)}\right) \\ &\quad + m(r, f) + S(r, f) + O(1) \\ i.e., m\left(r, \frac{1}{f}\right) &\leq N\left(r, \frac{L(f)}{f-c}\right) - N\left(r, \frac{f-c}{L(f)}\right) + T(r, L(f)) - N\left(r, \frac{1}{L(f)}\right) \\ &\quad + m(r, f) + S(r, f) + O(1). \end{aligned} \quad (6)$$

In view of {p.34, [2]} and $m(r, f) = S(r, f)$ it follows from (6) that

$$\begin{aligned} m\left(r, \frac{1}{f}\right) &\leq N(r, L(f)) + N\left(r, \frac{1}{f-c}\right) - N(r, f-c) \\ &\quad - N\left(r, \frac{1}{L(f)}\right) + T(r, L(f)) + S(r, f) \end{aligned}$$

$$\begin{aligned} i.e., m\left(r, \frac{1}{f}\right) &\leq N(r, L(f)) + N\left(r, \frac{1}{f-c}\right) - N(r, f) - N\left(r, \frac{1}{L(f)}\right) \\ &\quad + T(r, L(f)) + S(r, f) \\ i.e., \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f}\right)}{T(r, f)} &\leq \liminf_{r\to\infty} \frac{N(r, L(f))}{T(r, f)} - \liminf_{r\to\infty} \frac{N(r, f)}{T(r, f)} - \liminf_{r\to\infty} \frac{N\left(r, \frac{1}{L(f)}\right)}{T(r, f)} \\ &\quad + \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-c}\right)}{T(r, f)} + \limsup_{r\to\infty} \frac{T(r, L(f))}{T(r, f)} \\ i.e., \delta(0; f) &\leq \{1 - \Delta_R^L(\infty; f)\} - \{1 - \Delta(\infty; f)\} - \{1 - \Delta_R^L(0; f)\} \\ &\quad + \{1 - \delta(c; f)\} + \{1 + k - k\delta(\infty; f)\} \end{aligned}$$

Since $\delta(\infty; f) = 1$, $\Delta(\infty; f) = 1$

$$\begin{aligned} \delta(0; f) + \Delta_R^L(\infty; f) + \delta(c; f) &\leq 1 + \Delta_R^L(0; f) + 1 \\ i.e., \delta(0; f) + \Delta_R^L(\infty; f) + \delta(c; f) &\leq 2 + \Delta_R^L(0; f). \end{aligned}$$

Thus the theorem is proved. ■

Theorem 3.5. Let f be a transcendental meromorphic function having maximum deficiency sum satisfying the condition $m(r, f) = S(r, f)$. If a and d be two complex numbers, then for any integer n

$$\delta(d; f) + \Delta_R^{(n)}(\infty; f) + \delta_R^L(a; f) \leq \Delta_R^L(\infty; f) + \Delta_R^{(n)}(0; f) + 1.$$

Proof. Let us consider the following identity

$$\frac{1}{f-d} = \left[\frac{1}{a}\left\{\frac{L(f)}{f-a} - \frac{L(f)-a}{f^{(n)}}.\frac{f^{(n)}}{f-a}\right\}\left\{\frac{L(f)}{f-d}.\frac{1}{L(f)}\right\}\right].(f-a)$$

Since $m(r, f-a) \leq m(r, f) + O(1)$ we get from the above identity

$$\begin{aligned} m\left(r, \frac{1}{f-d}\right) &\leq m\left(r, \frac{L(f)-a}{f^{(n)}}\right) + m\left(r, \frac{1}{L(f)}\right) + m(r, f) + S(r, f) \\ &\quad + O(1.) \end{aligned} \tag{7}$$

Now by Nevanlinna's first fundamental theorem and Milloux's theorem {p.55, [2]} it follows from (7) that

$$\begin{aligned} m\left(r, \frac{1}{f-d}\right) &\leq T\left(r, \frac{L(f)-a}{f^{(n)}}\right) - N\left(r, \frac{L(f)-a}{f^{(n)}}\right) + T\left(r, \frac{1}{L(f)}\right) \\ &\quad - N\left(r, \frac{1}{L(f)}\right) + m(r, f) + S(r, f) + O(1) \\ i.e., m\left(r, \frac{1}{f-d}\right) &\leq T\left(r, \frac{f^{(n)}}{L(f)-a}\right) - N\left(r, \frac{L(f)-a}{f^{(n)}}\right) + T(r, L(f)) \\ &\quad - N\left(r, \frac{1}{L(f)}\right) + m(r, f) + S(r, f) + O(1). \end{aligned} \tag{8}$$

In view of {p.34, [2]} and $m(r, f) = S(r, f)$ it follows from (8) that

$$\begin{aligned} m\left(r, \frac{1}{f-d}\right) &\leq N\left(r, f^{(n)}\right) + N\left(r, \frac{1}{L(f)-a}\right) - N(r, L(f)-a) \\ &\quad - N\left(r, \frac{1}{f^{(n)}}\right) + T(r, L(f)) + S(r, f) \end{aligned}$$

$$
\begin{aligned}
i.e., m\left(r,\frac{1}{f-d}\right) &\leq N\left(r,f^{(n)}\right)+N\left(r,\frac{1}{L(f)-a}\right)-N(r,L(f))\\
&\quad -N\left(r,\frac{1}{f^{(n)}}\right)+T(r,L(f))+S(r,f)\\
i.e., \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-d}\right)}{T(r,f)} &\leq \liminf_{r\to\infty}\frac{N\left(r,f^{(n)}\right)}{T(r,f)}-\liminf_{r\to\infty}\frac{N(r,L(f))}{T(r,f)}\\
&\quad -\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(n)}}\right)}{T(r,f)}+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{L(f)-a}\right)}{T(r,f)}\\
&\quad +\lim_{r\to\infty}\frac{T(r,L(f))}{T(r,f)}. \qquad (9)\\
i.e., \delta(d;f) &\leq \{1-\Delta_R^{(n)}(\infty;f)\}-\{1-\Delta_R^{L}(\infty;f)\}\\
&\quad -\{1-\Delta_R^{n}(0;f)\}+\{1-\delta_R^{L}(a;f)\}\\
&\quad +\{1+k-k\delta(\infty;f)\}
\end{aligned}
$$

Since $\delta(\infty;f)=1$ we get from (9) that

$$
\begin{aligned}
\delta(d;f) &\leq \{1-\Delta_R^{(n)}(\infty;f)\}-\{1-\Delta_R^{L}(\infty;f)\}-\{1-\Delta_R^{n}(0;f)\}\\
&\quad +\{1-\delta_R^{L}(a;f)\}+1
\end{aligned}
$$

$$
i.e., \delta(d;f)+\Delta_R^{(n)}(\infty;f)+\delta_R^{L}(a;f)\leq \Delta_R^{L}(\infty;f)+\Delta_R^{(n)}(0;f)+1.
$$

Thus the theorem is established. ■

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