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Exact Values of Dynamical Quantities in Planetary Motion

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Abstract

Exact values of dynamical quantities in planetary motion, such as position, velocity and acceleration, in terms of constants of motion, have appeared in the literature for special locations of the planet, such as the endpoints of the major and minor axes and the latera-recta. In this study, exact values of the jerk vector, in addition to those of the position, velocity and accelerations of the planet and their Cartesian components have been calculated at every 15^o intervals of the true anomaly from the perihelion.

1. INTRODUCTION

Mathematicians cherish exact numbers for the very reason that they do not contain any approximations. For example, exact values of trigonometric functions for 0^o, 30^o, 45^o, 60^o and 90^o (in the first quadrant) and similar values in all other quadrants (for that matter) are widely used by scientists of all disciplines, as they have simple forms and are easy to recall. However, exact values of trigonometric functions are also found for 15^o and 75^o, as well as for 18^o, 36^o, 54^o and 72^o (in the first quadrant) (cf. [1, 2]). In fact, exact values of trigonometric functions are known at every 3^o intervals [3, 4], as well as for many other special angles [1, 2]. Owing to the above, exact values of some physical quantities can be found in terms of some physical constants, if they can be obtained as exact solutions of a problem in terms of trigonometric functions. For example, exact values for the velocities of a planet at specific locations have appeared in the literature [5 – 8]. In this paper, we obtain exact values of dynamical quantities (viz., position, velocity, acceleration and jerk vectors) of a planet and their Cartesian components in terms of known physical constants for every 15^o of its angular positions around the Sun.

2. THEORY OF PLANETARY MOTION

The first scientific laws of planetary motion were enunciated by Johannes Kepler in the early seventeenth century. According to Kepler's first law of planetary motion, the orbit of a planet is an ellipse, with the Sun at one focus. According to Kepler's second law, the areal velocity of the planet about the Sun and equivalently its orbital angular momentum is a constant of motion. As consequence of this law, the orbit of the planet lies in a plane. The equation of the orbital ellipse with major axis lying on the abscissa, the latus-rectum on the ordinate and the perihelion on the positive *x*-axis is expressed in polar coordinates (r, θ) as

$$r = \frac{p}{1 + e\cos\theta} \tag{1}$$

where p is the semi latus-rectum and e the eccentricity of the orbit. The position vector is thus

$$\vec{r} = r\hat{r} = \frac{p}{1 + e\cos\theta}\hat{r} \tag{2}$$

and the angular momentum of the planet is:

$$\ell = mr^2\frac{d\theta}{dt} \tag{3}$$

where m is the mass of the planet.

The rectangular or three-dimensional Cartesian coordinates in this representation have special significance. First, the angular momentum vector $\vec{\ell}$ is always directed along the perpendicular *z*-axis. Second, there exists a constant *Runge-Lenz* vector $\vec{R}$ which is always directed along the positive *x*-axis [9, 10]. And third, there exists yet another constant vector defined by $\vec{S} = \vec{\ell} \times \vec{R}$ which is directed along the positive *y*-axis [10]. In view of this, we calculate the dynamical variables of the planet and their Cartesian components and obtain their exact values at regular intervals of 15° of the angular coordinate θ called the *true anomaly* in orbital mechanics.

3. DYNAMICAL VARIABLES IN PLANETARY MOTION

The dynamical variables of velocity, acceleration and the *jerk* vector (cf. [11]) are defined by the first, second and third derivatives of the position vector with respect to time. By successive differentiation of Eq. (2) and application of the conservation of angular momentum (3), one obtains the velocity, acceleration and jerk vectors, respectively, as follows:

$$\vec{v} = \frac{\ell}{mp}e\sin\theta\hat{r} + \frac{\ell}{mp}(1 + e\cos\theta)\hat{\theta} \tag{4}$$

$$\vec{a} = -\frac{\ell^2}{m^2p^3}(1 + e\cos\theta)^2\hat{r} \tag{5}$$

and

$$\vec{j} = \frac{\ell^3}{m^3 p^5} 2e\sin\theta(1 + e\cos\theta)^3 \hat{r} - \frac{\ell^3}{m^3 p^5}(1 + e\cos\theta)^4 \hat{\theta} \tag{6}$$

A more traditional treatment of the planetary problem employed Cartesian coordinates [12-15]. The Cartesian components of the velocity, acceleration and jerk vectors are given by [11-15]:

$$v_x = -\frac{\ell}{mp}\frac{y}{r} \tag{7}$$

$$v_y = \frac{\ell}{mp}\left(\frac{x}{r} + e\right) \tag{8}$$

$$a_x = -\frac{\ell^2}{m^2 p}\frac{x}{r^3} \tag{9}$$

$$a_y = -\frac{\ell^2}{m^2 p}\frac{y}{r^3} \tag{10}$$

$$j_x = \frac{\ell^3}{m^3 p^2}\frac{yr + 3xye}{r^5} \tag{11}$$

and

$$j_y = -\frac{\ell^3}{m^3 p^2}\frac{xr + (x^2 - 2y^2)e}{r^5} \tag{12}$$

Expressed as functions of the angular coordinate θ, we have the Cartesian components of the coordinates and the dynamical variables of velocity, acceleration and jerk as follows [11]:

$$x = p\frac{\cos\theta}{1 + e\cos\theta} \tag{13}$$

$$y = p\frac{\sin\theta}{1 + e\cos\theta} \tag{14}$$

$$v_x = -\frac{\ell}{mp}\sin\theta \tag{15}$$

$$v_y = \frac{\ell}{mp}(\cos\theta + e) \tag{16}$$

$$a_x = -\frac{\ell^2}{m^2 p}\cos\theta(1 + e\cos\theta)^2 \tag{17}$$

$$a_y = -\frac{\ell^2}{m^2 p}\sin\theta(1 + e\cos\theta)^2 \tag{18}$$

$$j_x = \frac{\ell^3}{m^3 p^5}\sin\theta(1 + 3e\cos\theta)(1 + e\cos\theta)^3 \tag{19}$$

and

$$j_y = -\frac{\ell^3}{m^3p^5}(3ecos^2\theta + cos\theta - 2e)(1 + ecos\theta)^3 \tag{20}$$

Also, the magnitudes of the velocity, acceleration and jerk vectors are calculated to be respectively the following:

$$v = \frac{\ell}{mp}\sqrt{1 + e^2 + 2ecos\theta} \tag{21}$$

$$a = -\frac{\ell^2}{m^2p^3}(1 + ecos\theta)^2 \tag{22}$$

and

$$j = \frac{\ell^3}{m^3p^5}(1 + ecos\theta)^3\sqrt{1 + 4e^2 + 2ecos\theta - 3e^2cos^2\theta} \tag{23}$$

Equations (13) through (23) express the dynamical variables in planetary motion and their Cartesian components sought for in this study. All the quantities are functions of the single independent variable θ. Succinctly, only sines and cosines of θ appear in these equations. Thus if exact values of sines and cosines exist for any angle, then exact values of the dynamical variables and their components can be found in principle.

4. RESULTS

In the absence of perturbing forces, the motion of a planet repeats itself at regular intervals or cycles. Each cycle consists of two phases: (1) the *Ascending phase* during which the its radial distance from the Sun *r* increases while the planet traverses from the *Perihelion* (the nearest approach of the planet to the Sun) to the *Aphelion* (the farthest retreat of the planet from the Sun); and (2) The *Descending phase* during which the radial distance *r* decreases as the planet traverses from the Aphelion to the Perihelion. The true anomaly of the planet θ increases from 0° to 180° during the first phase and from 180° to 360° during the second.

We now compute the exact values of the dynamical variables for the ascending phase at intervals of 15° from the perihelion using the Equations (13) through (23). Table I shows the exact values of sines and cosines for the various values of θ. Substituting these values in the respective equations and simplifying, we obtain the exact values of the dynamical variables of position, velocity, acceleration and jerk vectors and their Cartesian components. These exact values are entered in Tables II, III, IV and V. Missing are values for the jerk vectors for four angles (15°, 75°, 105° and 165°, Table V) for which no compact forms were attainable.

Exact values for the dynamical variables for the descending phase of the planet can be determined from those for the ascending phase by simple substitutions. Due to the fact that the trigonometric functions of sine and cosine have odd and even parities, respectively, the dynamical variables and their components, too, have either odd or even parities about the perihelion (θ = 0^o). The working formulas are given as follows:

$$sin(2\pi - \theta) = -sin\theta \quad (24)$$
$$cos(2\pi - \theta) = cos\theta \quad (25)$$
$$x(2\pi - \theta) = x(\theta) \quad (26)$$
$$y(2\pi - \theta) = -y(\theta) \quad (27)$$
$$r(2\pi - \theta) = r(\theta) \quad (28)$$
$$v_x(2\pi - \theta) = -v_x(\theta) \quad (29)$$
$$v_y(2\pi - \theta) = v_y(\theta) \quad (30)$$
$$v(2\pi - \theta) = v(\theta) \quad (31)$$
$$a_x(2\pi - \theta) = a_x(\theta) \quad (32)$$
$$a_x(2\pi - \theta) = -a_x(\theta) \quad (33)$$
$$a(2\pi - \theta) = a(\theta) \quad (34)$$
$$j_x(2\pi - \theta) = -j_x(\theta) \quad (35)$$
$$j_y(2\pi - \theta) = j_y(\theta) \quad (36)$$
$$j(2\pi - \theta) = j(\theta) \quad (37)$$

The results are left to the reader as exercise.

Table I. Exact Values of Sines and Cosines at 15^o Intervals

***θ*, deg**	***θ*, rad**	**sin*θ***	**Cos*θ***
0	0	0	1
15	$\frac{\pi}{12}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
75	$\frac{5\pi}{12}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
90	$\frac{\pi}{2}$	1	0
105	$\frac{7\pi}{12}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$-\frac{\sqrt{3}-1}{2\sqrt{2}}$
120	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
135	$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$

150	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
165	$\frac{11\pi}{12}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$-\frac{\sqrt{3}+1}{2\sqrt{2}}$
180	π	0	-1

Table II. Exact Values of Coordinates of Planet at 15° Intervals

θ	***x, p***	***y, p***	***r, p***
0°	$\frac{1}{1+e}$	0	$\frac{1}{1+e}$
15°	$\frac{\sqrt{3}+1}{2\sqrt{2}+\left(\sqrt{3}+1\right)e}$	$\frac{\sqrt{3}-1}{2\sqrt{2}+\left(\sqrt{3}+1\right)e}$	$\frac{2\sqrt{2}}{2\sqrt{2}+\left(\sqrt{3}+1\right)e}$
30°	$\frac{\sqrt{3}}{2+\sqrt{3}e}$	$\frac{1}{2+\sqrt{3}e}$	$\frac{2}{2+\sqrt{3}e}$
45°	$\frac{1}{\sqrt{2}+e}$	$\frac{1}{\sqrt{2}+e}$	$\frac{\sqrt{2}}{\sqrt{2}+e}$
60°	$\frac{1}{2+e}$	$\frac{\sqrt{3}}{2+e}$	$\frac{2}{2+e}$
75°	$\frac{\sqrt{3}-1}{2\sqrt{2}+\left(\sqrt{3}-1\right)e}$	$\frac{\sqrt{3}+1}{2\sqrt{2}+\left(\sqrt{3}-1\right)e}$	$\frac{2\sqrt{2}}{2\sqrt{2}+\left(\sqrt{3}-1\right)e}$
90°	0	1	1
105°	$-\frac{\sqrt{3}-1}{2\sqrt{2}-\left(\sqrt{3}-1\right)e}$	$\frac{\sqrt{3}+1}{2\sqrt{2}-\left(\sqrt{3}-1\right)e}$	$\frac{2\sqrt{2}}{2\sqrt{2}-\left(\sqrt{3}-1\right)e}$
120°	$-\frac{1}{2-e}$	$\frac{\sqrt{3}}{2-e}$	$\frac{2}{2-e}$
135°	$-\frac{1}{\sqrt{2}-e}$	$\frac{1}{\sqrt{2}-e}$	$\frac{\sqrt{2}}{\sqrt{2}-e}$
150°	$-\frac{\sqrt{3}}{2-\sqrt{3}e}$	$\frac{1}{2-\sqrt{3}e}$	$\frac{2}{2-\sqrt{3}e}$
165°	$-\frac{\sqrt{3}+1}{2\sqrt{2}-\left(\sqrt{3}+1\right)e}$	$\frac{\sqrt{3}-1}{2\sqrt{2}-\left(\sqrt{3}+1\right)e}$	$\frac{2\sqrt{2}}{2\sqrt{2}-\left(\sqrt{3}+1\right)e}$
180°	$-\frac{1}{1-e}$	0	$\frac{1}{1-e}$

Table III. Exact Values of Velocities of Planet at 15° Intervals

θ	v_x, ℓ/mp	v_y, ℓ/mp	v, ℓ/mp
0°	0	$1+e$	$1+e$
15°	$-\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{(\sqrt{3}+1)+2\sqrt{2}e}{2\sqrt{2}}$	$\sqrt{1+\frac{\sqrt{3}+1}{\sqrt{2}}e+e^2}$
30°	$-\frac{1}{2}$	$\frac{\sqrt{3}+2e}{\sqrt{2}}$	$\sqrt{1+\sqrt{3}e+e^2}$
45°	$-\frac{1}{\sqrt{2}}$	$\frac{1+\sqrt{2}e}{\sqrt{2}}$	$\sqrt{1+\sqrt{2}e+e^2}$
60°	$-\frac{\sqrt{3}}{2}$	$\frac{1+2e}{2}$	$\sqrt{1+e+e^2}$
75°	$-\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{(\sqrt{3}-1)+2\sqrt{2}e}{2\sqrt{2}}$	$\sqrt{1+\frac{\sqrt{3}-1}{\sqrt{2}}e+e^2}$
90°	-1	e	$\sqrt{1+e^2}$
105°	$-\frac{\sqrt{3}+1}{2\sqrt{2}}$	$-\frac{(\sqrt{3}-1)-2\sqrt{2}e}{2\sqrt{2}}$	$\sqrt{1-\frac{\sqrt{3}-1}{\sqrt{2}}e+e^2}$
120°	$-\frac{\sqrt{3}}{2}$	$-\frac{1-2e}{2}$	$\sqrt{1-e+e^2}$
135°	$-\frac{1}{\sqrt{2}}$	$-\frac{1-\sqrt{2}e}{\sqrt{2}}$	$\sqrt{1-\sqrt{2}e+e^2}$
150°	$-\frac{1}{2}$	$-\frac{\sqrt{3}-2e}{2}$	$\sqrt{1-\sqrt{3}e+e^2}$
165°	$-\frac{\sqrt{3}-1}{2\sqrt{2}}$	$-\frac{(\sqrt{3}+1)-2\sqrt{2}e}{2\sqrt{2}}$	$\sqrt{1-\frac{\sqrt{3}+1}{\sqrt{2}}e+e^2}$
180°	0	$-(1-e)$	$1-e$

Table IV. Exact Values of Accelerations of Planet at 15° Intervals

θ	a_x, ℓ^2/m^2p^3	a_y, ℓ^2/m^2p^3	a, ℓ^2/m^2p^3
0°	$-(1+e)^2$	0	$(1+e)^2$
15°	$-\frac{(\sqrt{3}+1)[2\sqrt{2}+(\sqrt{3}+1)e]^2}{16\sqrt{2}}$	$-\frac{(\sqrt{3}-1)[2\sqrt{2}+(\sqrt{3}+1)e]^2}{16\sqrt{2}}$	$\frac{[2\sqrt{2}+(\sqrt{3}+1)e]^2}{8}$
30°	$-\frac{\sqrt{3}(2+\sqrt{3}e)^2}{8}$	$-\frac{(2+\sqrt{3}e)^2}{8}$	$\frac{(2+\sqrt{3}e)^2}{4}$

45°	$-\frac{(\sqrt{2}+e)^2}{2\sqrt{2}}$	$-\frac{(\sqrt{2}+e)^2}{2\sqrt{2}}$	$\frac{(\sqrt{2}+e)^2}{2}$
60°	$-\frac{(2+e)^2}{8}$	$-\frac{\sqrt{3}(2+e)^2}{8}$	$\frac{(2+e)^2}{4}$
75°	$-\frac{(\sqrt{3}-1)[2\sqrt{2}+(\sqrt{3}-1)e]^2}{16\sqrt{2}}$	$-\frac{(\sqrt{3}+1)[2\sqrt{2}+(\sqrt{3}-1)e]^2}{16\sqrt{2}}$	$\frac{[2\sqrt{2}+(\sqrt{3}-1)e]^2}{8}$
90°	0	1	1
105°	$\frac{(\sqrt{3}-1)[2\sqrt{2}-(\sqrt{3}-1)e]^2}{16\sqrt{2}}$	$-\frac{(\sqrt{3}+1)[2\sqrt{2}-(\sqrt{3}-1)e]^2}{16\sqrt{2}}$	$\frac{[2\sqrt{2}-(\sqrt{3}-1)e]^2}{8}$
120°	$\frac{(2-e)^2}{8}$	$-\frac{\sqrt{3}(2-e)^2}{8}$	$\frac{(2-e)^2}{4}$
135°	$\frac{(\sqrt{2}-e)^2}{2\sqrt{2}}$	$-\frac{(\sqrt{2}-e)^2}{2\sqrt{2}}$	$\frac{(\sqrt{2}-e)^2}{2}$
150°	$\frac{\sqrt{3}(2-\sqrt{3}e)^2}{8}$	$-\frac{(2-\sqrt{3}e)^2}{8}$	$\frac{(2-\sqrt{3}e)^2}{4}$
165°	$\frac{(\sqrt{3}+1)[2\sqrt{2}-(\sqrt{3}+1)e]^2}{16\sqrt{2}}$	$-\frac{(\sqrt{3}-1)[2\sqrt{2}-(\sqrt{3}+1)e]^2}{16\sqrt{2}}$	$\frac{[2\sqrt{2}-(\sqrt{3}+1)e]^2}{8}$
180°	$(1-e)^2$	0	$(1-e)^2$

Table V. Exact Values of Jerks of Planet at 15° Intervals

Θ	j_x, ℓ^3/m^3p^5	j_y, ℓ^3/m^3p^5	j, ℓ^3/m^3p^5
0°	0	$-(1+e)^4$	$(1+e)^4$
15°	*	*	*
30°	$\frac{(2+3\sqrt{3}e)(2+\sqrt{3}e)^3}{32}$	$-\frac{(2\sqrt{3}+e)(2+\sqrt{3}e)^3}{32}$	$\frac{\sqrt{4+4\sqrt{3}e+7e^2}(2+\sqrt{3}e)^3}{16}$
45°	$\frac{(\sqrt{2}+3e)(\sqrt{2}+e)^3}{4\sqrt{2}}$	$-\frac{(\sqrt{2}-e)(\sqrt{2}+e)^3}{4\sqrt{2}}$	$\frac{\sqrt{2+2\sqrt{2}e+5e^2}(\sqrt{2}+e)^3}{4}$
60°	$\frac{\sqrt{3}(2+3e)(2+e)^3}{32}$	$-\frac{(2-5e)(2+e)^3}{32}$	$\frac{\sqrt{4+4e+13e^2}(\sqrt{2}+e)^3}{16}$
75°	*	*	*
90°	1	$2e$	$\sqrt{1+4e^2}$
105°	*	*	*
120°	$\frac{\sqrt{3}(2-3e)(2-e)^3}{32}$	$\frac{(2+5e)(2-e)^3}{32}$	$\frac{\sqrt{4-4e+13e^2}(\sqrt{2}-e)^3}{16}$
135°	$\frac{(\sqrt{2}-3e)(\sqrt{2}-e)^3}{4\sqrt{2}}$	$\frac{(\sqrt{2}+e)(\sqrt{2}-e)^3}{4\sqrt{2}}$	$\frac{\sqrt{2-2\sqrt{2}e+5e^2}(\sqrt{2}-e)^3}{4}$

150°	$\frac{(2-3\sqrt{3}e)(2-\sqrt{3}e)^3}{32}$	$\frac{(2\sqrt{3}-e)(2-\sqrt{3}e)^3}{32}$	$\frac{\sqrt{4-4\sqrt{3}e+7e^2}(2-\sqrt{3}e)^3}{16}$
165°	*	*	*
180°	0	$(1+e)^4$	$(1+e)^4$

** Compact form unattainable*

5. DISCUSSION

Because of their flexibility, complicated trigonometric expressions often reduce to drastically simpler forms. The present study emphatically illustrates this point as seemingly complicated forms for the dynamical variables were reduced to elegant compact forms in terms of the constants of motion. It may be well to search for other problems which can similarly betray this elegance.

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