

Beauty of Integers created by Nature

Pradip Kumar Pal

Former Outstanding Scientist, Space Applications Centre, ISRO

Pythagorus Theorem is well known for a long time for estimation of hypotenuse length of a right angle triangle. If A and B are the lengths of the sides with right angle, then the length of the hypotenuse, C is defined by the formula

$$A^2 + B^2 = C^2 \quad \dots(1)$$

Some literature claims that in the ancient scripts, even before Pythagorus, there was mention about estimating the hypotenuse by a linear combination of A and B, such as

$$C = \frac{7}{8} A + \frac{1}{2} B \quad \dots(2)$$

which is valid for some Pythagorean triples (A,B,C) under certain conditions.

Here we are not looking at the trueness of above claims, but we shall analyse the linear methods to estimate the hypotenuse length. Obviously the linear methods are easier for estimation than finding the square root from equation (1). We shall be looking at various categories of triples for which linear method may be applied and try to find the coefficients for linear combination. The coefficients (7/8) and (1/2) are valid for a set of triples like (4n, 3n, 5n) for any integer n. Here validity for any integer n is obvious, as both equations will not be affected by multiplication by n. Actually, there may be infinite number of pair of coefficients to form a linear equation similar to equation (2) for any given value of C in a particular triple. The fact is that the above two coefficients are also valid for another set of triples like (12n, 5n, 13n) for any integer n. If we carefully see, these two coefficients will not be valid for any other Pythagorean triple.

Let us see, how these two coefficients can be found. If we solve the following two equations

$$4x + 3y = 5$$

$$12x + 5y = 13$$

we get the unique pair of coefficients $x = (7/8)$, $y = (1/2)$

Now following the same principle, we shall try to generalize the concept for several other sets of triples.

Generation of Pythagorean triples

In general, Pythagorean triples (A, B, C) can be generated from Euclid's formula

$A = m^2 - n^2$, $B = 2mn$, $C = m^2 + n^2$, for $m > n > 0$ m, n are coprime (no common divisor larger than 1) and not both odd numbers. We shall look here two special categories of triples as described as follows.

First category (I) of Pythagorean triples :

Category (I) is a particular set of triples, generated from Euclid's formula for $m = 2k$ and $n = 1$

For any integer k , a set of triples (A, B, C) can be generated as follows

$$A = 4k^2 - 1, \quad B = 4k, \quad C = 4k^2 + 1 \quad \dots(I)$$

The above triple (A, B, C) satisfies the equation (1), and so it is a Pythagorean triple. The first triple in this category will be (3, 4, 5), second triple (15, 8, 17), third triple (35, 12, 37) and so on. It may be noticed that for this category (I) of triples, $C = A + 2$.

Second category (II) of Pythagorean triples :

For any odd integer $(2n + 1)$, another set of triples (A, B, C) can be generated as follows

$$B = 2n + 1, \quad A = \frac{1}{2}(B^2 - 1), \quad C = A + 1 \quad \dots(II)$$

Here also triple (A, B, C) satisfies the equation (1). The first triple in this category will be (4, 3, 5), second triple (12, 5, 13), third triple (24, 7, 25) and so on.

An alternative way of generating the category (II) triples is that first take the square of any odd integer, then divide it by 2, and then A will be the truncation of that number from left, and C will be the truncation of that number from right.

Generation of coefficients for Linear method

Now we shall generate the pair of coefficients for each set of the above two categories.

Coefficients for Category (I) triples

For category (I), let us solve the following equations for first two triples of this category.

$$3x + 4y = 5$$

$$15x + 8y = 17$$

By solving above two equation we get, $x = 7/9$, $y = 2/3$. By using these coefficients, we can form the equation similar to equation (2) for linear formula for the first two triples of category (I). Similarly solving similar equations for second element (15, 8, 17) and third element (35,12,37) in this category, one can get the coefficients as $x = 23/25$, $y = 2/5$, which will be valid for both of these triples. Similarly, for third and fourth triples, coefficients will be $x = 47/49$, $y = 2/7$. In this way, by solving for each consecutive pair of triples, we get different coefficients for each triple. If one looks carefully, there is a pattern coming out for these coefficients. The general form of the coefficients for any element (A, B, C) of this category (I) will be as follows.

$$x_I = \frac{A+B}{A+B+2} \quad , \quad y_I = \frac{4}{B+2}$$

and the equation (2) will look like

$$C = x_I A + y_I B$$

As in this category (I), $A = 4k^2 - 1$, $B = 4k$, the coefficients can also be expressed as

$$x_I = \frac{4k(k+1)-1}{(2k+1)^2} \quad , \quad y_I = \frac{2}{2k+1}$$

Thus, for any triple (A, B, C) of category (I), the hypotenuse C may be estimated by linear combination of A and B as in the following equation

$$C = \left(\frac{A+B}{A+B+2} \right) A + \left(\frac{4}{B+2} \right) B \quad \dots(3)$$

The equation (3) can be verified for any triple of the Category (I).

Coefficients for Category (II) triples

Now for category (II), one can proceed in similar way as for the first category (I), by solving equations for each pair of consecutive triples in this category (II). One can find that for first element in this category (4, 3, 5), coefficients will be $x = 7/8$, $y = 1/2$. Though the same coefficients will be also valid for the second triple (12, 5, 13), the separate coefficients can be generated by solving similar equations for second and third triples of this category (II). The new coefficients will be $x = 17/18$, $y = 1/3$. Repeating the same procedure, the coefficient for third triple will be $x = 31/32$, $y = 1/4$. Here again a different pattern can be noticed for the coefficients. In this category (II), the general form of the coefficients for any triple of category (II) will be as follows.

$$x_{II} = \frac{A+B}{A+B+1} \quad , \quad y_{II} = \frac{2}{B+1}$$

and the linear equation (2) will look like

$$C = x_{II}A + y_{II}B$$

Thus for any triple (A, B, C) of category (II), the hypotenuse may be estimated by linear combination of A and B as in the following equation

$$C = \left(\frac{A+B}{A+B+1} \right) A + \left(\frac{2}{B+1} \right) B \quad \dots(4)$$

The equation (4) can be verified for any triple of category (II).

As it was mentioned in the beginning, the equations (3) and (4) with corresponding coefficients will also be valid for any of these triples multiplied by any integer, like (nA, nB, nC).

Discussion

From the pattern of linear coefficients, we have generated the general form of coefficients for the above two categories (I) and (II) of Pythagorean triples. Here it may be noticed that the first coefficient is a function of A and B, whereas the second coefficient is a function of only B. Except for the first triple in category (II), in all other triples, (A, B, C) are generated in such a way that $A > B$. Though by interchanging A and B, there will be no change in C, if C is estimated by equation (1), but there will be different coefficients, if C is estimated by equations (3) and (4). Actually the coefficients (x,y) will be interchanged to (y,x). The coefficient which is a function of A and B will not change, but in the coefficient which is a function of B

only, B has to be replaced with A.

It may be mentioned, that as this linearization method is only applicable for particular categories of triples, it may not be very useful for practical application. Practically, first one has to find to which category of triples the pair (A, B) belongs, and then the appropriate formula (3) or (4) has to be applied. Once it is known, to which category the pair (A, B) belongs then C is already known as the third element of the triple of that category. Here it is more important to note that there is certain general patterns in the coefficients of the linear formula, which can replace the quadratic formula.

Conclusion

It is interesting to note that the quadratic formulation of original Pythagorus formula (1) can be expressed by a linear expression for a large number of integers, and there is some general pattern in the coefficients for the linear formation for different categories of triples. This is another beauty of Integers created by Nature. It may be difficult to prove, but definite patterns are there. There are many other beautiful patterns in integers, some of which were coming in the mind of Ramanujam without any proof. Compiling many other beauties of integers, Stephen Hawking authored the book “ God created the Integers”, first published in 2005. One thing is clear that the ‘Beauty of Integers is created by Nature’, and it is not yet fully explored. Though the beauty of integers may not look really beautiful to many of us, they are necessary part of our daily life in various ways.

