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Linearisation of Diophantine Equation for Pythagorean Triples

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Pythagorean triples are the triplets of integers (A,B,C) which satisfies the Diophantine equation

$$A^2 + B^2 = C^2 \qquad \ldots(1)$$

and are generally used for the estimation of hypotenuse length, C of a right angle triangle with sides of length A and B. In an earlier paper by Pal (2022), it was shown that for two different categories of Pythagorean triples, the value of C can be estimated by linear equation like

$$C = x\,A + \,yB \qquad \ldots(2)$$

Here similar linearisation method is extended for general category of Pythagorean triples.

Construction of general Category of Pythagorean triples

To just recollect the procedure of generation of Pythagorean triples from Euclid's formula as mentioned in the earlier paper by Pal (2022), here it is once again repeated that in general, these triples can be generated as follows.

$$A = m^2 - n^2, \quad B = 2mn\,, \quad C = m^2 + n^2$$

where $m > n > 0$ and m, n are coprime and not both odd. These triples (A,B,C) satisfy the equation (1). In the earlier paper, Category (I) of triples has been formed by taking $m = 2k$ and $n = 1$. Here a more general Category (G) of Pythagorean triples is generated by taking $n = 2j$, for $j = 1,2,3,\ldots$ and $m = 2k + 1$, for $k = j$, $j+1, j+2,...$ .In this case, m is odd integer , n even integer and also $m > n$. From this general category (G), first, for example, look at a third Category (III) of triples generated by taking $j = 1$. Then $n = 2$ and $m = 2k + 1$ for $k = 1,2,3,\ldots$. The first triple (for $k = 1$) in this category (III) will be (5,12,13), second triple (for $k = 2$)

(21,20,29), third triple (for k = 3) (45,28,53) and so on. For general category, for any j > 0

$A = (2k+1)^2 - 4j^2, \quad B = 4(2k+1)j\ , \quad C = (2k+1)^2 + 4j^2$, k = j, j+1, j+2, …

Generation of Coefficients for Linear method

Looking at the pattern of coefficients for category (I) and category (II), as described in Pal (2022) and following similar pattern of the coefficients, for category (III) coefficients are constructed as follows

$$x_{III} = \frac{A+B}{A+B+8} \quad , y_{III} = \frac{16}{B+8}$$

so that

$$C = x_{III}A + y_{III}B \qquad \ldots(3)$$

According to the above construction, coefficients (x , y) for the first triple of category (III) will be (17/25 , 4/5), for second triple (41/49 , 4/7) and so on. The equation (3) can be verified for any triple of the Category (III). Here also, it can be mentioned that equation (3) will be valid for any of the category (III) triples multiplied by any integer n like (nA, nB, nC).

In a similar way, the coefficients for general category (G) for any j > 0 may be constructed as follows.

$$x_G = \frac{A+B}{A+B+8j^2} \quad , y_G = \frac{16j^2}{B+8j^2}$$

and the linear form of the equation (1) for the general category of Pythagorean triples will be

$$C = x_G A + y_G B \qquad \ldots(4)$$

This also can be verified for any triple from the general category and it will be valid for triples multiplied by any integer n like (nA, nB, nC).

Discussion

The procedure for earlier two categories, as established in the earlier paper by Pal (2022), has been extended for more general category of Pythagorean triples. One common factor among all these categories of triples may be noticed that the coefficient x is dependent on A, B and j, but coefficient y is dependent on B and j only. Another pattern in these triples may be noticed that in category (G), $C = A + 8j^2$.

One more point worth mentioning is that though the coefficients for linearisation are generated for integers, these will be valid for many real number triples. If the integer triples (A, B, C) are divided by some integer k , like (A/k, B/k, C/k), they will be real number triples and same coefficients will be valid for them as for integer triples (A, B, C).

Here again it may be mentioned that though the linear method is easier for estimation of C, but it is not very useful for practical purpose of estimating the hypotenuse of the triangle, because once it is known to which category the triple belongs, then the value of C is known from the triple. Here it is more important that there is some general pattern in the coefficients for all the categories.

CONCLUSION

The categories of triples mentioned here and in the earlier paper might not have covered all Pythagorean triples. The earlier paper by Pal (2022) has given a clue for the pattern of coefficients of the linear equation for particular categories of triples. Here following the similar procedure, coefficients have been constructed for more general category of Pythagorean triples. Though a general category of Pythagorean triples has been generated here, still there is scope for finding the patterns for other categories of triples. It may be possible to find some other beautiful patterns for linearisation of Diophantine equation for remaining Pythagorean triples. There is no end in finding new and new beauties of integers created by nature.

REFERENCE

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