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On $\varepsilon-$ Farthest Points and their Invariance

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Abstract

For a given bounded subset K of a metric space (X,d), $x \in X$ and $\varepsilon>0$, an element $k_0 \in K$ is called an $\varepsilon-$ farthest point to x if $d(x,k_0) \geq \sup\{d(x,k): k \in K\} - \varepsilon$. In this paper we discuss some results on $\varepsilon-$ farthest points and their invariance under isometric mappings. The underlying spaces are metric spaces.

For a non-empty subset G of a metric space (X,d), $x \in X$ and $\varepsilon>0$, we say that $g_0 \in G$ is an $\varepsilon-$ approximation or good approximation to x if $d(x,g_0) \leq d(x,g)+\varepsilon$ for all $g \in G$ (for $\varepsilon = 0$, such a g_0 is a best approximation to x in G). In the theory of nearest points, R.C. Buck [2] introduced the notion of $\varepsilon-$ approximation. Subsequently, this concept was studied by many researchers (see e.g. [10], [12], [13], [15] and references cited therein). Analogously, the notion of $\varepsilon-$ farthest points was introduced and discussed by Narang [8] in the theory of farthest points. This study was subsequently taken up in [4] and [9]. The notion of invariance of best approximation in normed linear spaces was initiated and discussed by Meinardus [7] and then considered by Brosowski [2], Khan and Khan [5], Mazaheri and Zadeh [6], and in metric spaces by Narang [11]. Mazaheri and Zadeh [6] also discussed the invariance of best approximation, and $\varepsilon-$ approximation in normed linear spaces whereas Narang [12] discussed the invariance of best approximation, and Sharma and Narang [13] of $\varepsilon-$ approximation in metric linear spaces.

In this paper, we discuss some results on $\varepsilon-$ farthest points and their invariance in metric spaces. Let K be a bounded subset of a metric space (X,d), $x \in X$ and $\varepsilon>0$. An element $k_0 \in K$ is said to be

i. a farthest point to x if $d(x,k_0)=\delta(x,K)\equiv\sup\{d(x,y):y\in K\}$,

ii. $\varepsilon-$farthest point or nearly farthest point to x if $d(x,k_0)\geq\delta(x,K)-\varepsilon$.

The set of all farthest points to x in K is denoted by $F_K(x)$ and $\varepsilon-$ farthest points to x in K by $F_{K,\varepsilon}(x)$ i.e.

$$F_K(x)=\{k_0\in K:d(x,k_0)=\delta(x,K)\}\ ,\ F_{K,\varepsilon}(x)=\{k_0\in K:d(x,k_0)\geq\delta(x,K)-\varepsilon\}\ .$$

Clearly, $F_{K,\varepsilon}(x)=K\cap B[x,\delta(x,K)-\varepsilon]^c$, where $B[x,r]^c$ denotes complement of the closed ball $B[x,r]$ with centre x and radius r, $F_{k,\varepsilon}(x)\supseteq F_k(x)=\bigcap_{\varepsilon>0}F_{k,\varepsilon}(x)$.

It can be easily seen that the sets $F_{K,\varepsilon}(x)$ and $F_K(x)$ are bounded sets, are closed sets if K is a closed set, are compact sets if K is a compact set.

Example: On the real line $\mathbb{R}$ with usual metric, let $K=[1,5]$, $x\in\mathbb{R}$, $\varepsilon>0$. Then

$$F_{K,\varepsilon}(x)=\begin{cases}[1,1+\varepsilon[& \text{if } x>3\\ ]5-\varepsilon,5[& \text{if } x<3\\ [1,1+\varepsilon[\cup]5-\varepsilon,5] & \text{if } x=3\end{cases}$$

Concerning the elements of $F_{K,\varepsilon}(x)$, we have:

Proposition 1. If K is a bounded subset of a metrid space (X,d) , $x\in X$ and $\varepsilon>0$, then $k_0\in F_{K,\varepsilon}(x)$ if and only if $k\in B[x,d(x,k_0)+\varepsilon]$ for each $k\in K$.

Proof: Let $k_0\in F_{K,\varepsilon}(x)$. Suppose there is an element $k_1\in K$ such that $k_1\notin B[x,d(x,k_0)+\varepsilon]$, then $d(x,k_0)+\varepsilon<d(x,k_1)\leq\delta(x,K)$ *i.e.* $d(x,k_0)<\delta(x,K)-\varepsilon$ which is a contradiction.

Conversely, suppose $k\in B[x,d(x,k_0)+\varepsilon]$ for all $k\in K$. If $k_0\notin F_{K,\varepsilon}(x)$, then $d(x,k_0)<\delta(x,K)-\varepsilon$, and so there is some $k_1\in K$ such that $d(x,k_0)<d(x,k_1)-\varepsilon$ *i.e.* $d(x,k_1)>d(x,k_0)$. But then $k_1\notin B[x,d(x,k_0)+\varepsilon]$, which is a contradiction.

The set K is said to be remotal (uniquely remotal) if $F_K(x)$ is non-empty (singleton) for each $x\in X$.

The set K is said to be $\varepsilon-$remotal ($\varepsilon-$uniquely remotal) if $F_{K,\varepsilon}(x)$ is non-empty (singelton) for each $x\in X$.

A sequence $\langle k_n\rangle$ in K is said to be $\varepsilon-$maximizing sequence for x if $\lim_{n\to\infty}d(x,k_n)\geq\delta(x,K)-\varepsilon$. The set K is said to be $\varepsilon-$nearly compact if for each

$x \in X$, each $\varepsilon-$ maximizing sequence has a subsequence converging to an element of K.

It is known (see [1]) that bounded nearly compact sets are remotal. The following proposition shows that $\varepsilon-$ nearly compact sets are $\varepsilon-$ remotal.

Proposition 2. If K is a boundeded, closed and $\varepsilon-$ nearly compacy subset of a metric space (X,d) then $F_{K,\varepsilon}(x)$ is a non-empty compact set.

Proof: Let $x \in X$ be artitrary. By the definition of $\delta(x,K)$, we can find $k_0 \in K$ such that $d(x,k_0) \geq \delta(x,K)-\varepsilon$ and so $F_{K,\varepsilon}(x)$ is non-empty.

Let $\langle k_n \rangle$ be a sequence in $F_{K,\varepsilon}(x)$ *i.e.* $d(x,k_n) \geq \delta(x,K)-\varepsilon$ for all $n = 1,2,3,...$ and so

$$\lim_{n\to\infty} d(x,k_n) \geq \delta(x,K)-\varepsilon \tag{1}$$

i.e. $\langle k_n \rangle$ is an $\varepsilon-$ maximizing sequence in K. Since K is $\varepsilon-$ nearly compact closed set, $\langle k_n \rangle$ has a subseqnece $\langle k_{n_i} \rangle \to k_0 \in K$. So, (1) implies

$$d(x,k_0) \geq \delta(x,K)-\varepsilon$$

i.e. $k_0 \in F_{K,\varepsilon}(x)$ and so $F_{K,\varepsilon}(x)$ is compact.

If K is a bounded subset of a metric space (X,d) and $\varepsilon>0$, we define

$$F^{-1}_{K,\varepsilon}(k_0) = \left\{x \in X : k_0 \in F_{K,\varepsilon}(x)\right\}$$

$\equiv$ set of all those points of X for which $k_0 \in K$ is $\varepsilon-$ farthest point.

Proposition 3. The set $F^{-1}_{K,\varepsilon}(k_0)$ is a closed subset of X.

Proof: Let x be a limit point of $F^{-1}_{K,\varepsilon}(k_0)$. Then there exists a sequence $\langle x_n \rangle$ in $F^{-1}_{K,\varepsilon}(k_0)$ such that $x_n \to x$.

Since $k_0 \in F_{K,\varepsilon}(x_n)$ for all n, $d(x_n,k_0) \geq \delta(x_n,K)-\varepsilon$.

This gives

$$\lim d(x_n,k_0) \geq \lim \delta(x_n,K)-\varepsilon$$

i.e. $d(x,k_0) \geq \delta(x,K)-\varepsilon$

and so $k_0 \in F_{K,\varepsilon}(x)$ *i.e.* $x \in F^{-1}_{K,\varepsilon}(k_0)$.

Therefore $F^{-1}_{K,\varepsilon}(k_0)$ is closed.

We next discuss the invariance of $\varepsilon-$ farthest points.

Theorem 1. Let T be an isometery on a metric space (X,d) *i.e.* $d(Tx,Ty)=d(x,y)$ for all $x,y\in X, K$ a bounded subset of $X, x\in X$ and $\varepsilon>0$. We have

(i) $T[F_{K,\varepsilon}(x)]=F_{K,\varepsilon}(Tx)$ if $T(K)=K$,

(ii) If x is $T-$ invariant *i.e.* $Tx=x$, then $T[F_{K,\varepsilon}(x)]\subseteq F_{K,\varepsilon}(x)$,

(iii) If x is $T-$ invariance and if $F_{K,\varepsilon}(x)=\left\{k_0\right\}$, then $Tk_0=k_0$,

(iv) If x is $T-$ invariance and if $\left\{k\in K: Tk=k\right\}\cap F_{K,\varepsilon}(x)=\phi$, then either $F_{K,\varepsilon}(x)=\phi$ or $F_{K,\varepsilon}(x)$ has more than one point.

Proof:

(i) Let $T(k_0)\in T[F_{K,\varepsilon}(x)]$ *i.e.* $k_0\in F_{K,\varepsilon}(x)$. This gives $d(x,k_0)\geq d(x,k)-\varepsilon$ for all $k\in K$ and so $d(Tx,Tk_0)\geq d(Tx,Tk)-\varepsilon$ for all $Tk\in T(K)=K$ *i.e.* $T(k_0)\in F_{K,\varepsilon}(Tx)$. Therefore $T[F_{K,\varepsilon}(x)]\subseteq F_{K,\varepsilon}(Tx)$. Conversely, suppose $T(k_0)\in F_{K,\varepsilon}(Tx)$ *i.e.* $T(k_0)\in F_{T(K),\varepsilon}(Tx)$. Then $d(Tx,Tk_0)\geq d(Tx,Tk)-\varepsilon$ for all $T(k)\in TK$ *i.e.* $d(x,k_0)\geq d(x,k)-\varepsilon$ for all $k\in K$. Therefore $k_0\in F_{K,\varepsilon}(x)$ and so $Tk_0\in T[F_{K,\varepsilon}(x)]$. Hence $F_{K,\varepsilon}(Tx)\subseteq T[F_{K,\varepsilon}(x)]$.

(ii) Let $T(k_0)\in T[F_{K,\varepsilon}(x)]$ *i.e.* $k_0\in F_{K,\varepsilon}(x)$. Consider $d(Tx,Tk_0)=d(x,k_0)$ $\geq\delta(x,K)-\varepsilon$. This gives $T(k_0)\in F_{K,\varepsilon}(x)$. Hence $T[F_{K,\varepsilon}(x)]\subseteq F_{K,\varepsilon}(x)$.

(iii) By (ii) $T(k_0)\in\left\{k_0\right\}$ and hence $Tk_0=k_0$.

(iv) By (ii) $T(k_0)\in F_{K,\varepsilon}(x)$. But by hypothesis, no invariant point can be an $\varepsilon-$ farthest point to x, therefore $T(k_0)\neq k_0$. So, if $T(k_0)=k_0$ then $\varepsilon-$ farthest point to x does not exist, *i.e.* $F_{K,\varepsilon}(x)=\phi$. If $Tk_0\neq k_0$ then x has at least two $\varepsilon-$ farthest points to x.

Theorem 2. Let T be an isometery on a metric space $(X,d), K$ a bounded subset of $X, x\in X$ and $\varepsilon>0$.

Then $T[F_{K,\varepsilon}(x)]=F_{T(K),\varepsilon}(Tx)$.

Proof: Since T is isometery, $d(Tx,Ty)=d(x,y)$ for all $x,y\in X$. The proof follows from $d(x,k_0)\geq d(x,k)-\varepsilon$ for all $k\in K\Leftrightarrow d(Tx,Tk_0)\geq d(Tx,Tk)-\varepsilon$ for all $Tk\in T(K)$.

Remarks:

1. Maps satisfying $\mathrm{T}\left[F_{K,\varepsilon}(x)\right]=F_{T(k),\varepsilon}(Tx)$ are called $\varepsilon-$ farthest point preserving. Therefor isometery maps preserve $\varepsilon-$ farthest points.
2. Taking $\varepsilon=0$, we obtain $\mathrm{T}\left[F_K(x)\right]=F_{T(k)}(Tx)$. Such maps are called farthest point preserving. So isometery maps are farthest point preserving.

3. If $\mathrm{T}:(X,d)\to(Y,d')$ is an onto isometery and K is a bounded subset of X, then (i) K is remotal (uniquely remotal) in X if and only if $T(K)$ is remotal (uniquely remotal) subset of Y. (ii) K is $\varepsilon-$ remotal ($\varepsilon-$ uniquely remotal) in X if and only if T(K) is $\varepsilon-$ remotal ($\varepsilon-$ uniquely remotal) subset of Y.
4. In th theory of nearest points, the notion of $\varepsilon-$ coapproximation in metric spaces was defined by Narang [10] as under:

For a non-empty subset G of a metric space $(X,\ d)$, $x\in X$ and $\varepsilon>0$, an element $g_0\in G$ is called an $\varepsilon-$ coapproximation to x if $d(g_0,g)\le d(x,g)+\varepsilon$ for all $g\in G$.

For $\varepsilon=0$, such a $g_0\in G$ is called a coapproximation to x. These concepts were discussed by many researchers (see [11] - [14] and references cited therein.).

It is easy to see that analogous notions of $\varepsilon-$ cofarthest and cofarthest points which could be defined as under :

For a bounded subset K of a metric space (X,d) and $\varepsilon>0$, an element $k_0\in K$ is called an $\varepsilon-$ cofarthest point (cofarthest point) to $x\in X$ if $d(k_0,k)\ge\delta(x,K)-\varepsilon$ ($d(k_0,k)\ge\delta(x,K)$) for all $k\in K$ are not possible in the theory of farthest points.

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