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Asymmetric Weibull and Complete Asymmetric Weighted Weibull Modelling of the Synthetic stock Return Data in the Nigerian Exchange Market

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Abstract

This study investigated the symmetric Weibull distribution as well as developed its weighted extension for modelling the distributional behaviour of stock returns in the Nigerian Exchange Market, using a simulated dataset constructed to reproduce important characteristics of financial returns. The analysis was motivated by the limitations of the conventional Gaussian assumption. The asymmetric Weibull formulation offers a flexible way to capture differences between positive and negative returns. This provides a useful representation of downside asymmetry that would be difficult to obtain from a conventional symmetric distribution. We examine the cumulative distribution of empirical returns to test for power-law scaling in the tails and estimate the corresponding tail exponent α. Ultimately, this framework offers a clearer picture of return dynamics and provides a more reliable way to model extreme market fluctuations The main contribution of this study is the introduction and application of the complete asymmetric weighted Weibull model. By adding weighting parameters, it provided the model with another layer of flexibility to capture key differences between positive and negative returns. The model comparisons results provide strong empirical support for this extension.

Keywords: Asymmetric Weibull, Weighted Weibull, Asset Returns, Power Law

1. Introduction

Financial markets are characterized by uncertainty, randomness, driven by nonlinear interactions, and unpredictable participant behaviour. A central challenge in financial

economics is accurately modelling the statistical behaviour of asset returns—a requirement for effective risk management, portfolio construction, option pricing, and forecasting.

Historically, standard models assumed asset returns follow a Gaussian (normal) distribution. This assumption was appealing because of its mathematical convenience and ties to the Central Limit Theorem. Under a normal distribution, extreme market swings are treated as virtually impossible outliers. Over the years, several probability distributions have been employed for modelling real-life datasets. However, empirical evidence suggests that many of these classical distributions are inadequate for capturing the behaviour of highly skewed data, thereby posing challenges in both statistical theory and applied modelling. (Areebah *et al*, 2026).

The problem of predicting changes in dynamical behaviour of asset price return was considered by (Osu, 2009). In practice, data across stock, foreign exchange, and commodity markets tell a different story. Empirical return distributions most times diverge from standard Gaussian models, exhibiting pronounced asymmetry, sharp peakedness, and heavy tails. Market crashes and sharp rallies occur far more frequently than normal distribution predict which means that models that ignore tail behaviour systematically underestimate extreme financial risk (Ivana *et al*, 2018)

To address these shortcomings, researchers have turned to alternative distributions, including Lévy stable, Student-t, generalized hyperbolic, generalized Gaussian, and Weibull (stretched exponential) frameworks. The generalized Weibull model stands out for its versatility: its adjustable shape and scale parameters comfortably capture both symmetric and skewed behavior (Chen and Gerlach, 2013). By applying distinct parameters to positive and negative returns, it can also account for market asymmetry—reflecting the real-world reality that gains and losses rarely behave identically.

Beyond the shape of the core distribution, the behaviour of cumulative return tails is equally revealing. Empirical studies consistently shows that return tails decay following a power law: $P(|\ R\ |> x) \propto x^{-\gamma}$, where γ represents the tail exponent. The size of α directly indicates how susceptible a market is to catastrophic shocks, making it a key metric for systemic risk. Considering tail risks as predictors for returns is premised on its characteristic incorporation of tail thickness information, which circumvents plausible misspecification of distributions or higher-order moment problems. (Ogbonna and Olubusoye, 2021)

This paper analyzes the statistical behaviour of asymmetric returns using a generalized Weibull probability density function. We examine the cumulative distribution of empirical returns to test for power-law scaling in the tails and estimate the corresponding tail exponent α. Ultimately, this framework offers a clearer picture of return dynamics and provides a more reliable way to model extreme market fluctuations.

For the Asset Price Returns, let P_t denote the price of an asset at time t. The simple return over one period is defined as $R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$, while the continuously compounded (logarithmic) return is $r_t = \ln\left(\frac{P_t}{P_{t-1}}\right)$. Logarithmic returns are widely used in financial analysis because they possess desirable statistical properties,

including time additivity and mathematical convenience. Figures 1-6 show the synthetic stock dataset, the daily stock returns, the Simulated conditional volatility, the Absolute daily returns and volatility clustering, the frequency of low- and high-volatility states and Empirical probabilities of positive and negative returns.

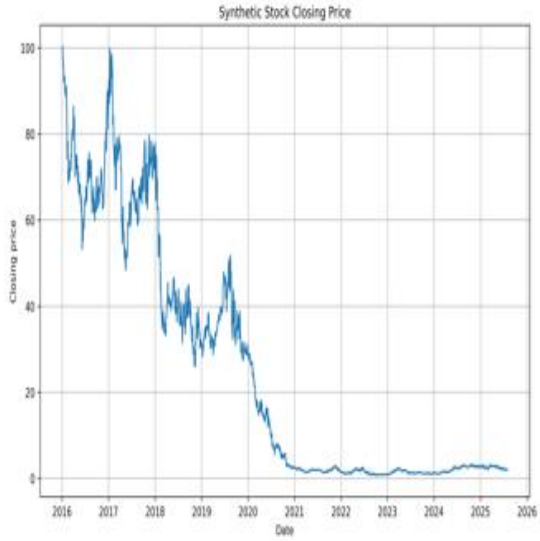


Figure 1. Synthetic stock closing-price trajectory.

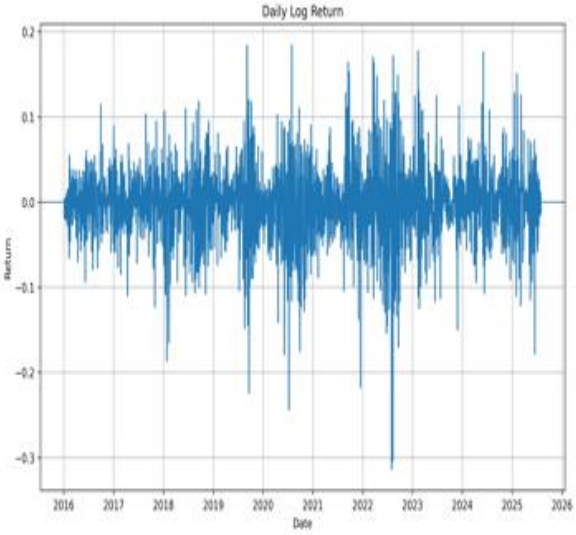


Figure 2. Daily log-return series.

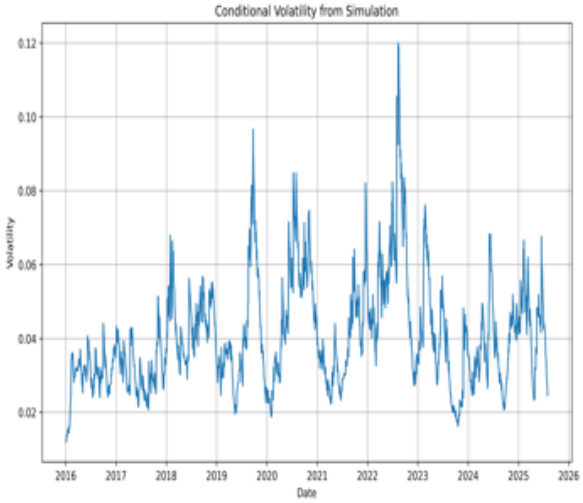


Figure 3. Simulated conditional volatility.

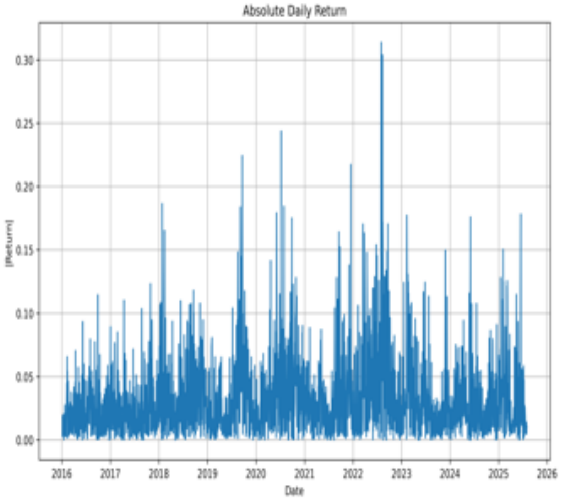


Figure 4. Absolute daily returns and volatility clustering.

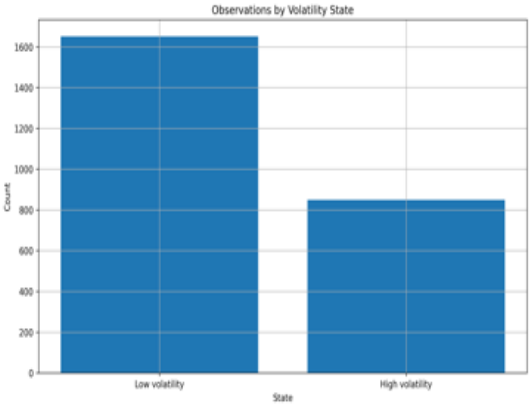


Figure 5. Frequency of low- and high-volatility states.

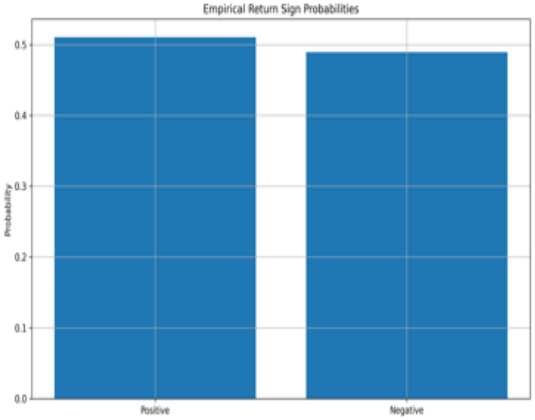


Figure 6. Empirical probabilities of positive and negative returns.

Now let δx represent the normalized asset return. The probability density function (PDF) describes the likelihood that the return assumes a value within an infinitesimal interval. For a continuous random variable, the probability distribution of returns is given as;

$$P(a \leq \delta x \leq b) = \int_a^b f\,(\delta x)\, d(\delta x), \qquad (1)$$

where $f(\delta x)$ denotes the probability density function satisfying $f(\delta x) \geq 0$, and $\int_{-\infty}^{\infty} f(\delta x) d(\delta x) = 1$.

For the asymmetric nature of financial returns, the generalized Weibull distribution is adopted in the form (Andersen *et al*, 1999).

$$P(\delta x) = \begin{cases} \frac{B}{\sqrt{\pi}} \frac{\gamma^+}{Z_+^{\gamma^+/2}} \mid \delta x \mid^{\gamma^+/2-1} \exp\left[-\left(\frac{|\delta x|}{Z_+}\right)^{\gamma^+}\right], & \delta x > 0, \\ \frac{1-B}{\sqrt{\pi}} \frac{\gamma^-}{Z_-^{\gamma^-/2}} \mid \delta x \mid^{\gamma^-/2-1} \exp\left[-\left(\frac{|\delta x|}{Z_-}\right)^{\gamma^-}\right], & \delta x < 0. \end{cases} \qquad (2)$$

Here,

- Z_+ and Z_- are scale parameters controlling the spread of positive and negative returns, respectively;
- γ^+ and γ^- are shape parameters governing the rate of tail decay;
- B represents the probability weight assigned to positive returns, while $1 - B$ corresponds to negative returns.

The use of separate parameters for positive and negative returns enables the model to capture market asymmetry arising from different statistical characteristics of gains and losses. For the normalization of the Weibull PDF, one has that

$$\int_{-\infty}^{\infty} P(\delta x) d(\delta x) = 1. \qquad (3)$$

Thus; Split equation (3) into two part to get

$$\int_0^{\infty} P(\delta x)d\delta x + \int_{-\infty}^{0} P(\delta x)d\delta x = 1 \qquad (4).$$

Now substituting equation (2) into equation (4) and letting $u = \left(\frac{\delta x}{Z_+}\right)^{\gamma^+}$ gives

$$\int_0^{\infty} (\delta x)^{\frac{\gamma^+}{2}-1}\ e^{-\left(\frac{\delta x}{Z_+}\right)^{\gamma^+}} d\delta x.$$

This evaluates to a Gamma function: $\alpha Z_+^{\gamma^+/2}\,\Gamma\left(\frac{1}{2}\right)$. But $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, so the normalization simplifies nicely
$\frac{\mathcal{B}}{\sqrt{\pi}} \times \sqrt{\pi} = \mathcal{B}$. Similarly, $\frac{1-\mathcal{B}}{\sqrt{\pi}} \times \sqrt{\pi} = 1 - \mathcal{B}.$ so that $\mathcal{B} + (1 - \mathcal{B}) = 1$.This shows that the PDF is **normalized.**
In what follows, we generate the moments for any PDF of equation (2). Let $E[X^n] = \int_{-\infty}^{\infty} (\delta x)^n P(\delta x) d(\delta x)$

Because our PDF is piecewise, splitting into two regions gives:
$E[X^n] = \int_0^{\infty} (\delta x)^n P_+(\delta x)d\delta x + \int_{-\infty}^{0} (\delta x)^n P_-(\delta x)d\delta x$. For our distribution (equation 2), integrals reduce to:

$$\int_0^{\infty} x^{a-1}\, e^{-\left(\frac{x}{b}\right)^w} dx = \frac{b^a}{w}\Gamma\left(\frac{a}{w}\right) \qquad (5)$$

Mean $\boldsymbol{\mu = E[\delta x]}$
For the Positive side and using the identity:

$$E_+[\delta x] = \frac{\mathcal{B}}{\sqrt{\pi}} \cdot \frac{\gamma^+}{Z_+^{\gamma^+/2}} \int_0^{\infty} x^{\gamma^+/2}\, e^{-(x/Z_+)^{\gamma^+}} = \frac{\mathcal{B}}{\sqrt{\pi}} \cdot Z_+ \Gamma\left(\frac{1}{2} + \frac{1}{\gamma^+}\right). \qquad (6)$$

For the Negative side: Let $x = -\delta x > 0$, then

$$E_-[\delta x] = -\frac{1-\mathcal{B}}{\sqrt{\pi}} \cdot Z_- \Gamma\left(\frac{1}{2} + \frac{1}{\gamma^-}\right). \qquad (7)$$

Equations (6) and (7) together gives the final mean as.

$$\mu = \frac{\mathcal{B} Z_+}{\sqrt{\pi}}\ \Gamma\left(\frac{1}{2} + \frac{1}{\gamma^+}\right) - \frac{(1-\mathcal{B})Z_-}{\sqrt{\pi}}\ \Gamma\left(\frac{1}{2} + \frac{1}{\gamma^-}\right). \qquad (8)$$

Second Moment $\boldsymbol{E[\delta x^2]}$
The positive side gives:

$$E_+[\delta x^2] = \frac{\mathcal{B}}{\sqrt{\pi}}\ Z_+^2 \Gamma\left(\frac{1}{2} + \frac{2}{\gamma^+}\right). \qquad (9)$$

And the negative side is given as:

$$E_{-}[\delta x^2] = \frac{1-\mathcal{B}}{\sqrt{\pi}}\ Z_{-}^2\Gamma\left(\frac{1}{2}+\frac{2}{\gamma^-}\right). \tag{10}$$

Equations (9) and (10) total the second moment as;

$$E[\delta x^2] = \frac{\mathcal{B}Z_{+}^2}{\sqrt{\pi}}\ \Gamma\left(\frac{1}{2}+\frac{2}{\gamma^+}\right) + \frac{(1-\mathcal{B})Z^2}{\sqrt{\pi}}\ \Gamma\left(\frac{1}{2}+\frac{2}{\gamma^-}\right). \tag{11}$$

Thus, one has the Variance as;

$$Var(\delta x) = \left[\frac{\mathcal{B}Z_{+}^2}{\sqrt{\pi}}\ \Gamma\left(\frac{1}{2}+\frac{2}{\gamma^+}\right) + \frac{(1-\mathcal{B})Z^2}{\sqrt{\pi}}\ \Gamma\left(\frac{1}{2}+\frac{2}{\gamma^-}\right)\right] - \mu^2. \tag{12}$$

Third Moment (for Skewness)
For the positive side,

$$E[\delta x^3] = \frac{\mathcal{B}Z_{+}^3}{\sqrt{\pi}}\ \Gamma\left(\frac{1}{2}+\frac{3}{\gamma^+}\right). \tag{13}$$

The negative gives;

$$E[\delta x^3] = -\frac{(1-\mathcal{B})Z^3}{\sqrt{\pi}}\ \Gamma\left(\frac{1}{2}+\frac{3}{\gamma^-}\right). \tag{14}$$

Equations (13) and (14) total to;

$$E[\delta x^3] = \frac{\mathcal{B}Z_{+}^3}{\sqrt{\pi}}\ \Gamma\left(\frac{1}{2}+\frac{3}{\gamma^+}\right) - \frac{(1-\mathcal{B})Z^3}{\sqrt{\pi}}\ \Gamma\left(\frac{1}{2}+\frac{3}{\gamma^-}\right). \tag{15}$$

One then obtain the Skewness thus;

$$Skewness = \frac{E[(X-\mu)^3]}{\sigma^3} = \frac{E[\delta x^3]-3\mu E[\delta x^2]+2\mu^3}{(Var)^{\frac{3}{2}}}. \tag{16}$$

Note:

- If $\mathcal{B} = 0.5,\ Z_{+} = Z_{-},\ \gamma^{+} = \gamma^{-}$, the distribution is symmetric, implies $skewness\ =\ 0$
- If $\mathcal{B} \neq 0.5$ or parameters differ; implies skewed heavy-tail distribution.

The accumulated (or cumulative tail) probability distribution measures the probability that the magnitude of a return exceeds a specified threshold. For positive returns; $P(\delta x > x) = \int_x^{\infty} P\ (t)\ dt$, while for negative returns,$P(\delta x < -x) = \int_{-\infty}^{-x} P\ (t)\ dt$.
Thus for the positive Weibull density, let

$$S_{+}(x) = \frac{B}{\sqrt{\pi}}\frac{\gamma^{+}}{Z_{+}^{\gamma^+/2}}\int_x^{\infty} t^{\gamma^+/2-1}\exp\ \left[-\left(\frac{t}{Z_{+}}\right)^{\gamma^+}\right]dt. \tag{17}$$

Now set $u = \left(\frac{t}{Z_+}\right)^{\gamma^+}$ so that $t = Z_+ u^{1/\gamma^+}$ and $dt = \frac{Z_+}{\gamma^+} u^{1/\gamma^+ - 1}\, du$ then $t^{\gamma^+/2-1} = Z_+^{\frac{\gamma^+}{2}-1} u^{\left(\frac{\gamma^+}{2}-1\right)/\gamma^+}$.

Therefore,

$$S_+(x) = \frac{B}{\sqrt{\pi}} \int_{\left(\frac{x}{Z_+}\right)^{\gamma^+}}^{\infty} u^{-1/2}\, e^{-u}\, du. \tag{18}$$

Notice that $\left(\frac{\gamma^+/2-1}{\gamma^+}\right) + \left(\frac{1}{\gamma^+} - 1\right) = -\frac{1}{2}$. Hence the exponent simplifies to $u^{-1/2}$. Recall that the **upper incomplete Gamma function** is defined by

$$\Gamma(a, w) = \int_z^{\infty} t^{a-1}\, e^{-t}\, dt \tag{19}$$

Comparing equations (18) and (19) shows that $a = \frac{1}{2},\ and\ \ w = \left(\frac{x}{Z_+}\right)^{\gamma^+}$.
Therefore,

$$S_+(x) = \frac{B}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, \left(\frac{x}{Z_+}\right)^{\gamma^+}\right),\ x > 0. \tag{20}$$

Similarly (using the same substitution $u = \left(\frac{|t|}{Z_-}\right)^{\gamma^-}$)for the negative accumulated probability

$$S_-(x) = P(\delta x < -x) = \frac{1-B}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, \left(\frac{x}{Z_-}\right)^{\gamma^-}\right),\ x < 0. \tag{21}$$

Let the Reliability Function be $R(x) = 1 - F(x) = \int_x^{\infty} P(t)dt$. Then Equations (20) and (21) are respectively the positive and negative reliability functions, $R_+(x)$, $R_-(x)$, say. The accumulated probability of observing a return whose magnitude exceeds x is therefore

$$R(x) = S(x) = S_+(x) + S_-(x) = \frac{B}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, \left(\frac{x}{Z_+}\right)^{\gamma^+}\right) + \frac{1-B}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, \left(\frac{x}{Z_-}\right)^{\gamma^-}\right). \tag{22}$$

This is the accumulated (survival) probability distribution corresponding to our generalized asymmetric Weibull PDF which naturally arises because of the exponential term contained in the Weibull distribution.

Theorem 1 (General Moment):

Let $[X^n] = \int_0^{\infty} (\delta x)^n P_+(\delta x) d\delta x + \int_{-\infty}^{0} (\delta x)^n P_-(\delta x) d\delta x$. For our distribution (equation 2), whose integrals reduce as in equation (5), The Total Moment gives

$$E[\mathrm{X}^n] = \frac{B}{\sqrt{\pi}} \mathrm{Z}_+^n \Gamma\left(\frac{1}{2} + \frac{n}{\gamma^+}\right) + (-1)^n \frac{1-B}{\sqrt{\pi}} \mathrm{Z}_-^n \Gamma\left(\frac{1}{2} + \frac{n}{\gamma^-}\right). \tag{23}$$

Proof:

For the positive side

$$M_n^+ = \int_0^\infty x^n P(x)dx = \frac{B}{\sqrt{\pi}} \frac{\gamma^+}{Z_+^{\frac{\gamma^+}{2}}} \int_0^\infty x^{n+\frac{\gamma^+}{2}-1} \exp\left[-\left(\frac{x}{Z_+}\right)^{\gamma^+}\right] dx \, , m = n + \frac{\gamma^+}{2} - 1 \quad (24)$$

evaluating the integral in (24) yields

$$M_n^+ = \frac{B}{\sqrt{\pi}} Z_+^n \Gamma\left(\frac{1}{2} + \frac{n}{\gamma^+}\right). \quad (25)$$

Similarly the negative side yields

$$M_n^- = \frac{1-B}{\sqrt{\pi}} Z_-^n \Gamma\left(\frac{1}{2} + \frac{n}{\gamma^-}\right). \quad (26)$$

So that the general Moment by combining (25) and (26) gives as required (23).
NB: The odd moments subtract the negative contribution while the even moment gets a positive contributions from both tails, hence capturing the asymmetry between gains and losses (ref). We deduce the moments like Mean, Variance, Skewness and Kurtosis as described by figure 7 below.

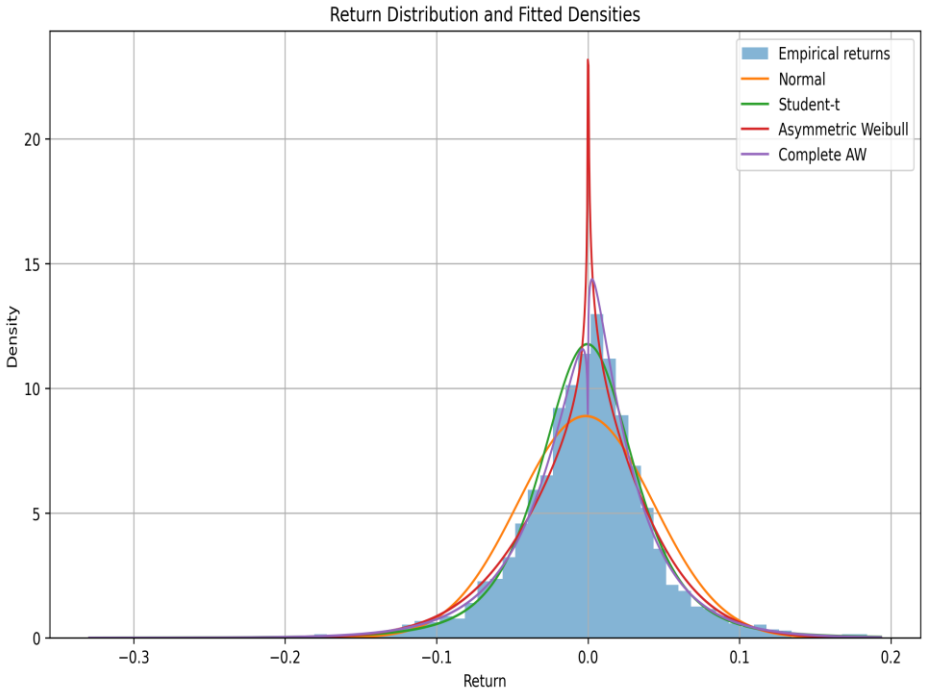


Figure 7. Return histogram with normal, Student-t, asymmetric Weibull and complete AW densities.

Power-Law Tail Behavior
Although the generalized Weibull distribution effectively models the central region of financial returns, empirical studies frequently show that the extreme tails of the accumulated probability distribution approximately satisfy (Ahsanullah et al., 2013)

$$P(|\ \delta x\ |> x)\gamma x^{-\gamma}\ , \tag{27}$$

where γ is called the tail exponent. Taking logarithms gives

$$\log P\ (|\ \delta x\ |> x) = -\gamma \log x + C, \tag{28}$$

where C is a constant. This linear relationship forms the basis for estimating the tail exponent using log-log plots or regression techniques (Clauset *et al*., 2009; Gabaix, 2008). A smaller value of αindicates heavier tails and a higher probability of extreme market movements, whereas larger values correspond to thinner tails and behavior closer to the Gaussian distribution (Cont, 2001). The tail exponent is an important statistical measure in financial risk analysis because it quantifies the frequency of extreme returns (Embrechts *et al*., 1997). Markets with relatively small tail exponents experience more frequent crashes and unusually large gains than markets with larger exponents. Consequently, accurate estimation of the tail exponent is essential for developing realistic models of market risk and improving financial decision-making (Danielsson and de Vries, 2000). Now Let $C = \frac{\mathcal{B}}{\sqrt{\pi}}\frac{\gamma^{+}}{Z^{\gamma/2}}$ and let $\alpha = 1 - \frac{\gamma^{+}}{2}$ .Then the PDF of equation (2) for $\delta x > 0$ becomes;

$$P(\delta x) = C|\delta x|^{-\alpha} \exp\left[-\left(\frac{\delta x}{Z_{+}}\right)^{\gamma^{+}}\right]\ . \tag{29}$$

We say that the prefactor obeys a power law with exponent $\alpha = 1 - \frac{\gamma^{+}}{2}$.

For large arguments, $\Gamma(a, z) \sim z^{a-1}e^{-z}$, $z \to \infty$. Hence equation (20) becomes;

$$S_{+}(x) \sim \frac{B}{\sqrt{\pi}}\left(\frac{x}{Z_{+}}\right)^{-\gamma^{+}/2} \exp\ \left[-\left(\frac{x}{Z_{+}}\right)^{\gamma^{+}}\right], \tag{30}$$

and similarly equation (21) becomes,

$$S_{-}(x) \sim \frac{1-B}{\sqrt{\pi}}\left(\frac{x}{Z_{-}}\right)^{-\gamma^{-}/2} \exp\ \left[-\left(\frac{x}{Z_{-}}\right)^{\gamma^{-}}\right]. \tag{31}$$

Figures 8, 9 and 10 describe this asymptotic behaviour of the model equations (30) and (31). These expressions demonstrate that the survival function has a stretched-exponential tail. The multiplicative factor is algebraic (a power of x) (Laherrere and Sornette, 1998), but the exponential term dominates for very large x. Therefore, the theoretical generalized Weibull model does **not** possess a true asymptotic power-law tail (Sornette, 2006).

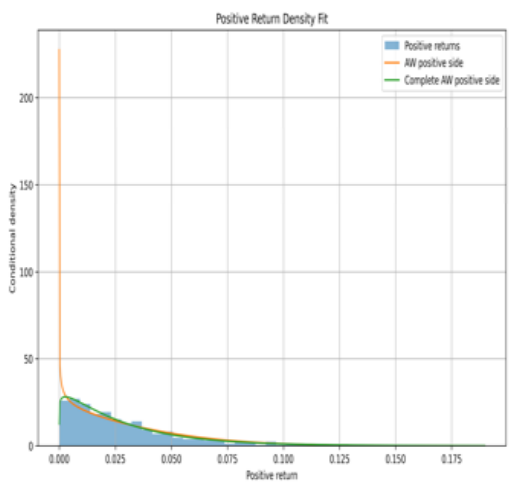


Figure 8. Positive-return conditional density and fitted models.

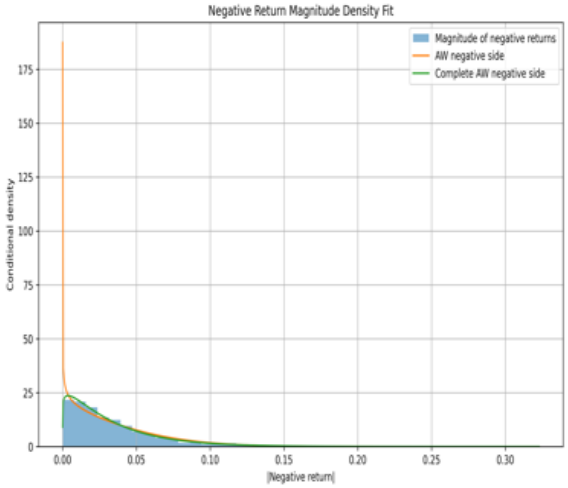


Figure 9. Negative-return magnitude conditional density and fitted models.

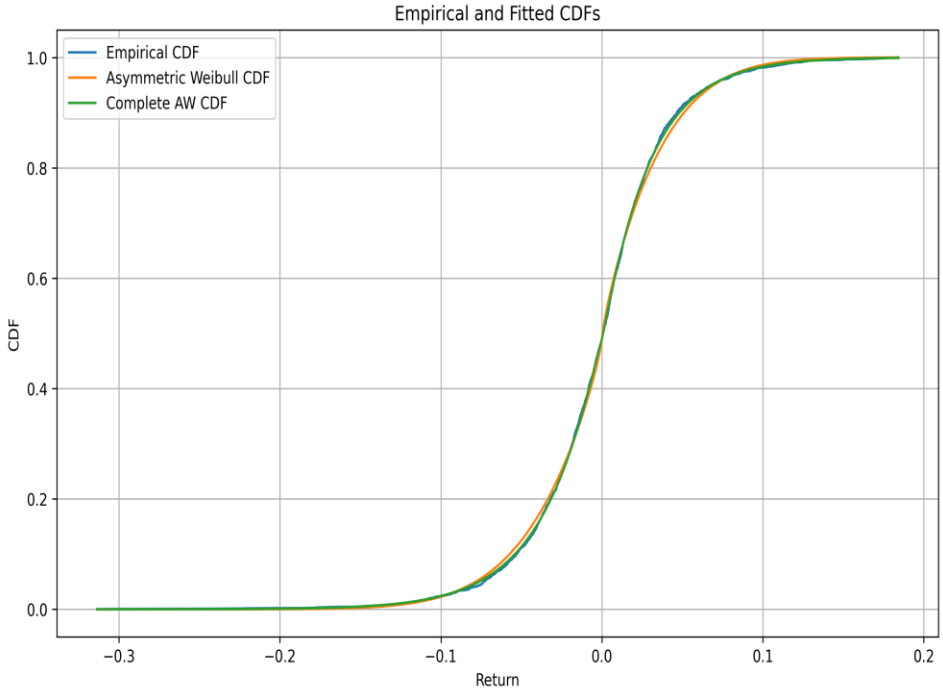


Figure 10. Empirical CDF and fitted asymmetric models.

Proposition 1: let equation (29) exist. Then for any real number $m \geq 0$, one can transform the Weibull function into the upper incomplete gamma function of the form;

$$\int_0^{\infty} x^m \exp\left[-\left(\frac{x}{Z}\right)^{\gamma}\right] dx = \frac{Z^{m+1}}{\gamma}\Gamma\left(\frac{m+1}{\gamma}\right) \tag{32}$$

Proof:

Let $u = \left(\frac{x}{\mathrm{Z}}\right)^{\gamma}$. Then $x = \mathrm{Z}u^{\frac{1}{\gamma}}$ and $dx = \frac{\mathrm{Z}}{\gamma}u^{\frac{1}{\gamma}-1}du$. Substitution in equation (32) yields

$$\int_0^{\infty} x^m \exp\left[-\left(\frac{x}{Z}\right)^{\gamma}\right]dx = \int_0^{\infty} \mathrm{Z}^m u^{\frac{m}{\gamma}} \exp[-u]\left(\frac{\mathrm{Z}}{\gamma}\right)u^{\frac{1}{\gamma}-1}du$$

$$= \frac{\mathrm{Z}^{m+1}}{\gamma}\int_0^{\infty} u^{\frac{m+1}{\gamma}-1}\exp[-u]du$$

$$= \frac{\mathrm{Z}^{m+1}}{\gamma}\Gamma\left(\frac{m+1}{\gamma}\right) = K\,\Gamma\left(\frac{1}{2},\mathrm{u}\right)$$

This is the upper incomplete Gamma function as required.

In this sequel, we construct the hazard function (which is related to the inverse of Mills Ratio) for the positive side given equation (2) using the relationship; $h_+(x) = \frac{P(x)}{R(x)} = \frac{P(x)}{\int_x^{\infty} P(t)dt}$ as

$$h_+(x) = \frac{\frac{B}{\sqrt{\pi}}\frac{\gamma^+}{\mathrm{Z}_+^{\gamma^+/2}}|x|^{\gamma^+/2-1}\exp\left[-\left(\frac{|x|}{\mathrm{Z}_+}\right)^{\gamma^+}\right]}{\frac{B}{\sqrt{\pi}}\Gamma\left(\frac{1}{2},\left(\frac{x}{\mathrm{Z}_+}\right)^{\gamma^+}\right)} = \frac{\frac{\gamma^+}{\mathrm{Z}_+^{\gamma^+/2}}|x|^{\gamma^+/2-1}\exp\left[-\left(\frac{|x|}{\mathrm{Z}_+}\right)^{\gamma^+}\right]}{\Gamma\left(\frac{1}{2},\left(\frac{x}{\mathrm{Z}_+}\right)^{\gamma^+}\right)}. \quad (33)$$

Similarly, the negative side yields

$$h_-(x) = \frac{\frac{\gamma^-}{\mathrm{Z}_+^{\gamma^-/2}}|x|^{\gamma^-/2-1}\exp\left[-\left(\frac{|x|}{\mathrm{Z}_-}\right)^{\gamma^-}\right]}{\Gamma\left(\frac{1}{2},\left(\frac{x}{\mathrm{Z}_-}\right)^{\gamma^-}\right)}. \quad (34)$$

If $h_-(x)$ exceeds $h_+(x)$ for large x, it implies that extreme losses occur more readily than extreme gains. This is often associated with downside risk and market asymmetry.

Derivation of the Asymmetric Weighted Weibull Distribution

Given the PDF of our asymmetric Weibull of equation (2), Taking the positive side of the above model without the Probability weight B, let

$$P_+(x) = \frac{\gamma^+}{\sqrt{\pi}\mathrm{Z}_+^{\gamma^+/2}}x^{\gamma^+/2-1}\exp\left[-\left(\frac{|x|}{\mathrm{Z}_+}\right)^{\gamma^+}\right], \quad x > 0. \quad (35)$$

Recall that the CDF already given is $G_+(x) = \frac{1}{\sqrt{\pi}}\gamma\left(\frac{1}{2}, \left(\frac{x}{Z_+}\right)^{\gamma^+}\right)$ and that $\gamma\left(\frac{1}{2}, z\right) = \sqrt{\pi}\,\mathrm{erf}\left(\sqrt{z}\right)$. Then the CDF can be rewritten as;

$$G_+(cx) = \mathrm{erf}\left\{\left(\frac{cx}{Z_+}\right)^{\frac{\gamma^+}{2}}\right\}. \tag{36}$$

We now apply the relation for our asymmetric weighted Weibull distribution as (Saman *et al*, 2010);

$$g_{w,+}(x) = \frac{1}{P(cX_1 > X_2)} g_+(x) G_+(cx)\ ;\ c > 0 \tag{37}$$

For our distribution, we define the scaling by $c = a^{1/\gamma^+}$ so that $c^{\gamma^+/2} = a^{1/2}$.
Then the weighted density of (36) yields;

$$G_+\left(a^{1/\gamma^+}x\right) = \mathrm{erf}\left\{\sqrt{a}\left(\frac{x}{Z_+}\right)^{\frac{\gamma^+}{2}}\right\}. \tag{38}$$

By calculating the normalizing probability**,** **l**et $Y_i = \left(\frac{X_i}{Z_+}\right)^{\gamma^+}$, stating that $Y_i \sim \mathrm{Gamma}\left(\frac{1}{2}, 1\right)$.
We need $P\left(a^{\frac{1}{\gamma^+}}X_1 > X_2\right)$. Raising both sides to the power of γ^+, we have $aX_1^{\gamma^+} > X_2^{\gamma^+}$. Therefore $aY_1 > Y_2$. Then $P\left(a^{\frac{1}{\gamma^+}}X_1 > X_2\right) = P(Y_2 < aY_1)$ For two independent $\mathrm{Gamma}\left(\frac{1}{2}, 1\right)$ variables, this probability is

$$P\left(a^{\frac{1}{\gamma^+}}X_1 > X_2\right) = \frac{2}{\pi}\tan^{-1}\left(\sqrt{a}\right). \tag{39}$$

We therefore obtain for our model that

$$D(a) = \frac{2}{\pi}\tan^{-1}\left(\sqrt{a}\right). \tag{40}$$

Putting equations (37). (38) and (40) together, we have that;

$$g_{w,+}(x) = \frac{1}{D(a)} g_+(x) G_+\left(a^{1/\gamma^+}x\right). \tag{41}$$

Which expresses as

$$g_{w,+}(x) = \frac{\pi}{2\tan^{-1}(\sqrt{a})}\frac{\gamma^{+}}{\sqrt{\pi}\mathrm{Z}_{+}^{\gamma^{+}/2}}x^{\gamma^{+}/2-1}\exp\left[-\left(\frac{x}{\mathrm{Z}_{+}}\right)^{\gamma^{+}}\right]\mathrm{erf}\left\{\sqrt{a}\left(\frac{x}{Z_{+}}\right)^{\frac{\gamma^{+}}{2}}\right\},\quad x>0,$$

and simplifies to

$$g_{w,+}(x) = \frac{\sqrt{\pi}\,\gamma^{+}}{2\mathrm{Z}_{+}^{\gamma^{+}/2}\tan^{-1}(\sqrt{a})}\,x^{\gamma^{+}/2-1}\exp\left[-\left(\frac{x}{\mathrm{Z}_{+}}\right)^{\gamma^{+}}\right]\mathrm{erf}\left\{\sqrt{a_{+}}\left(\frac{x}{Z_{+}}\right)^{\frac{\gamma^{+}}{2}}\right\},\quad x>0\,. \tag{42}$$

Similarly, the negative side yields

$$g_{w,-}(x) = \frac{\sqrt{\pi}\,\gamma^{-}}{2\mathrm{Z}_{-}^{\gamma^{-}/2}\tan^{-1}(\sqrt{a})}\,x^{\gamma^{-}/2-1}\exp\left[-\left(\frac{x}{\mathrm{Z}_{-}}\right)^{\gamma^{-}}\right]\mathrm{erf}\left\{\sqrt{a_{-}}\left(\frac{x}{Z_{-}}\right)^{\frac{\gamma^{-}}{2}}\right\},\ x<0 \tag{43}$$

Now, we have the **complete asymmetric weighted distribution as;**

$$g_{W}(x) = \begin{cases} \frac{B\sqrt{\pi}\,\gamma^{+}}{2\mathrm{Z}_{+}^{\gamma^{+}/2}\tan^{-1}(\sqrt{a})}\,x^{\gamma^{+}/2-1}\exp\left[-\left(\frac{x}{\mathrm{Z}_{+}}\right)^{\gamma^{+}}\right]\mathrm{erf}\left\{\sqrt{a_{+}}\left(\frac{x}{Z_{+}}\right)^{\frac{\gamma^{+}}{2}}\right\}, & x>0 \\ \frac{(1-B)\sqrt{\pi}\,\gamma^{-}}{2\mathrm{Z}_{-}^{\gamma^{-}/2}\tan^{-1}(\sqrt{a})}\,x^{\gamma^{-}/2-1}\exp\left[-\left(\frac{x}{\mathrm{Z}_{-}}\right)^{\gamma^{-}}\right]\mathrm{erf}\left\{\sqrt{a_{-}}\left(\frac{x}{Z_{-}}\right)^{\frac{\gamma^{-}}{2}}\right\}, & x<0 \end{cases}. \tag{44}$$

Note that a special case occurs if $a_{+}=a_{-}=1$. Then $\tan^{-1}(1)=\frac{\pi}{4}$, hence $\frac{\sqrt{\pi}}{2\tan^{-1}(1)}=\frac{2}{\sqrt{\pi}}$.

Then the asymmetric weighted density becomes

$$g_{W}(x) = \begin{cases} \frac{2B\,\gamma^{+}}{\sqrt{\pi}\mathrm{Z}_{+}^{\gamma^{+}/2}}\,x^{\gamma^{+}/2-1}\exp\left[-\left(\frac{x}{\mathrm{Z}_{+}}\right)^{\gamma^{+}}\right]\mathrm{erf}\left\{\left(\frac{x}{Z_{+}}\right)^{\frac{\gamma^{+}}{2}}\right\}, & x>0 \\ \frac{2(1-B)\,\gamma^{-}}{\sqrt{\pi}\mathrm{Z}_{-}^{\gamma^{-}/2}}\,x^{\gamma^{-}/2-1}\exp\left[-\left(\frac{x}{\mathrm{Z}_{+}}\right)^{\gamma^{-}}\right]\mathrm{erf}\left\{\left(\frac{x}{Z_{-}}\right)^{\frac{\gamma^{-}}{2}}\right\}, & x<0 \end{cases}. \tag{45}$$

So $a_{+}=a_{-}$ means only that the weighting mechanism is common to gains and losses not that the distribution becomes symmetric. Figures 11 and 12 show 30-day rolling return volatility and Central return-density fit with equation (45).

Thoerem 2: Let equation (45) exits. Then for a positive random variable x with density $g_{w}(x)$, its moment is Mellin transform by;

$$E[X^n] = \mathcal{B}\frac{\sqrt{a}Z_+^n}{\tan^{-1}(\sqrt{a})}\Gamma\left(1+\frac{n}{\gamma^+}\right)F_1\left(\frac{1}{2},1+\frac{n}{\gamma^+};\frac{3}{2};-a\right)\ (-1)^n\binom{1-}{\mathcal{B}}\frac{\sqrt{a}Z_-^n}{\tan^{-1}(\sqrt{a})}\Gamma\left(1+\frac{n}{\gamma^-}\right)F_1\left(\frac{1}{2},1+\frac{n}{\gamma^-};\frac{3}{2};-a\right). \quad (46)$$

Proof: Define the Mellin transform of (45) by $M_g(k) = \int_0^\infty x^{k-1}g(x)dx$ so that $M_g(k+1) = \int_0^\infty x^k g(x)dx$.
Thus $M_g(k+1) = E[X^n]$. One can easily see that the Mellin transform is the generalization of the moment-generating mechanisms for positive variables. Now the Mellin Transform of the positive branch of the weighted Weibull distribution is $M_+(k) = \int_0^\infty x^{k-1}g_{w+}(x)dx$, which is substituted in the weighted PDF of (45) to give;

$$M_+(k) = \frac{\sqrt{\pi}\,\gamma^+}{2Z_+^{\gamma^+/2}\tan^{-1}(\sqrt{a})}\times$$
$$\int_0^\infty x^{k+\gamma^+/2-2}\exp\left[-\left(\frac{x}{Z_+}\right)^{\gamma^+}\right]\mathrm{erf}\left\{\sqrt{a_+}\left(\frac{x}{Z_+}\right)^{\frac{\gamma^+}{2}}\right\}dx\,. \quad (47)$$

Now let $u = \left(\frac{x}{Z_+}\right)^{\gamma^+}$, so that $x = Z_+u^{1/\gamma^+}$, $dx = \frac{Z_+}{\gamma^+}u^{1/\gamma^+-1}du$ and $x^{k+\gamma^+/2-2}dx = \frac{Z_+^{k+\gamma^+/2-2}}{\gamma^+}\,u^{(k-1)/\gamma^+-1/2}\,du$.
Substituting this into the Mellin integral of (47) yields;

$$M_+(k) = \frac{\sqrt{\pi}\,\gamma^+}{2Z_+^{\gamma^+/2}\tan^{-1}(\sqrt{a})}\times$$
$$\frac{Z_+^{k+\gamma^+/2-2}}{\gamma^+}\int_0^\infty u^{(k-1)/\gamma^+-1/2}\,e^{-u}\mathrm{erf}(\sqrt{au})du. \quad (48)$$

Which simplifies to

$$M_+(k) = \frac{\sqrt{\pi}Z_+^{k-1}}{2\tan^{-1}(\sqrt{a})}\int_0^\infty u^{(k-1)/\gamma^+-1/2}\,e^{-u}\mathrm{erf}(\sqrt{au})du \quad (49)$$

We convert the error function by using $\mathrm{erf}(z) = \frac{2z}{\sqrt{\pi}}F_1\left(\frac{1}{2};\frac{3}{2};-z^2\right)$ and $z = \sqrt{au}$. This gives
$\mathrm{erf}\left(\sqrt{au}\right) = \frac{2\sqrt{au}}{\sqrt{\pi}}F_1\left(\frac{1}{2};\frac{3}{2};-au\right)$ and by letting $\lambda = \frac{k-1}{\gamma^+}$, (49) becomes

$$M_+(k) = \frac{\sqrt{\pi}Z_+^{k-1}}{2\tan^{-1}(\sqrt{a})}\times\frac{2\sqrt{a}}{\sqrt{\pi}}\int_0^\infty u^\lambda\,e^{-u}F_1\left(\frac{1}{2};\frac{3}{2};-au\right)du = \frac{\sqrt{a}Z_+^{k-1}}{\tan^{-1}(\sqrt{a})}I\,, \quad (50)$$

where $I = \int_0^\infty u^\lambda\,e^{-u}F_1\left(\frac{1}{2};\frac{3}{2};-au\right)du$.
To evaluate the integral, we use the Gamma-confluent hypergeometric integral

$$\int_0^{\infty} u^{\lambda} e^{-u} F_1(a; c; -au)du = \Gamma(\lambda + 1)F_1(a; \lambda + 1; c; -a). \quad (51)$$

Thus $I = \Gamma\left(1 + \frac{k-1}{\gamma^+}\right) F_1\left(\frac{1}{2}, 1 + \frac{k-1}{\gamma^+}; \frac{3}{2}; -a\right)$, so that (50) becomes

$$M_+(k) = \frac{\sqrt{a} Z_+^{k-1}}{\tan^{-1}(\sqrt{a})} \Gamma\left(1 + \frac{k-1}{\gamma^+}\right) F_1\left(\frac{1}{2}, 1 + \frac{k-1}{\gamma^+}; \frac{3}{2}; -a\right). \quad (52)$$

Equation (52) is the Mellin transform of the positive weighted branch. Consequently, the negative branch gives

$$M_-(k) = \frac{\sqrt{a} Z_-^{k-1}}{\tan^{-1}(\sqrt{a})} \Gamma\left(1 + \frac{k-1}{\gamma^-}\right) F_1\left(\frac{1}{2}, 1 + \frac{k-1}{\gamma^-}; \frac{3}{2}; -a\right). \quad (53)$$

The k-th moment is

$$E[X^{k-1}] = \mathcal{B} M_+(k) + (-1)^{k-1}(1 - \mathcal{B}) M_-(k) \text{ , for } k - 1 \text{ an integer.} \quad (54)$$

Therefore, for integer $n \geq 0$

$$E[X^n] = \mathcal{B} M_+(n + 1) + (-1)^n (1 - \mathcal{B}) M_-(n + 1), \quad (55)$$

as required and completes the proof.

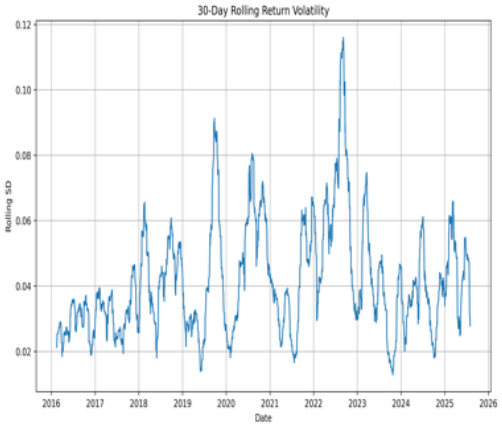


Figure 11. 30-day rolling return volatility fit with (45).

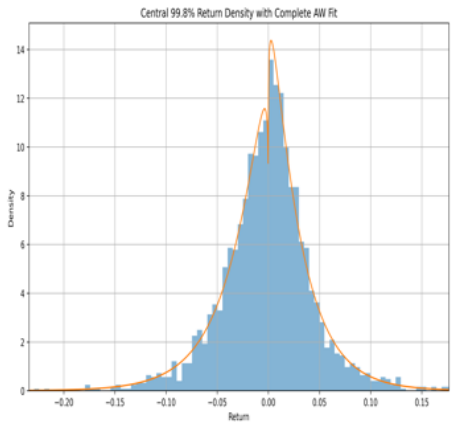


Figure 12. Central return-density fit with (45)

Data and simulation design (*Synthetic data; 2,500 trading-day observations, 2016–2025)*

The analysis uses the synthetic stock dataset generated for the proposed asymmetric weighted Weibull framework. The file contains 2,500 business-day observations with Date, Open, High, Low, Close, Volume, Return, Abs_Return, Volatility and State. The first return is missing because a return requires a preceding closing price;

consequently, 2,499 daily returns are available for return-distribution analysis as seen in table 1. The simulation was deliberately constructed to reproduce stylized financial-return features: volatility regimes, volatility clustering, heavy-tailed innovations, mild serial dependence, and asymmetric positive/negative shock scales. The observations are synthetic and must not be described as observations from a real stock.

Table 1: Data screening and descriptive analysis

Statistic	Value
N	2499
Positive returns	1276
Negative returns	1223
Positive proportion	0.510604
Negative proportion	0.489396
Mean	-0.00156463
Median	0.000778562
Std. deviation	0.0448862
Minimum	-0.313617
Maximum	0.184252
Q1	-0.0241683
Q3	0.0220336
Skewness	-0.40317
Excess kurtosis	3.88383
Shapiro-Wilk W	0.957909
Shapiro-Wilk p	2.65369e-26
Jarque-Bera	1630.04
Jarque-Bera p	0

The empirical sign probability is $\hat{B} = 0.510604$ for positive returns and 0.489396 for negative returns. The mean return is −0.001565, while the median is 0.000779, indicating a modest downward pull from negative observations. The standard deviation is 0.044886. Skewness is −0.4032 and excess kurtosis is 3.8838, indicating negative asymmetry and substantial leptokurtosis relative to a Gaussian distribution. Both Shapiro–Wilk ($p = 2.654\times10^{-26}$) and Jarque–Bera ($p < 0.001$) strongly reject

normality. Thus, a normal-return assumption is inappropriate for this dataset. The Graphical exploratory analysis uses the table 1 for figures 1 to 18.

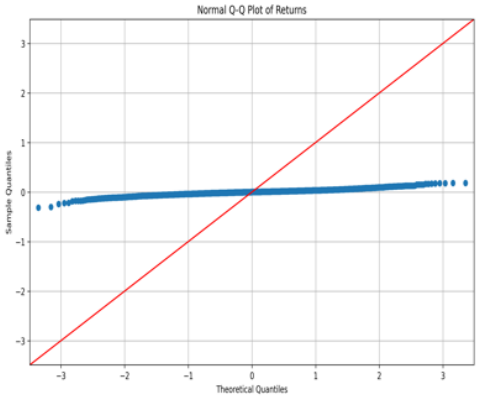


Figure 13. Normal Q-Q plot.

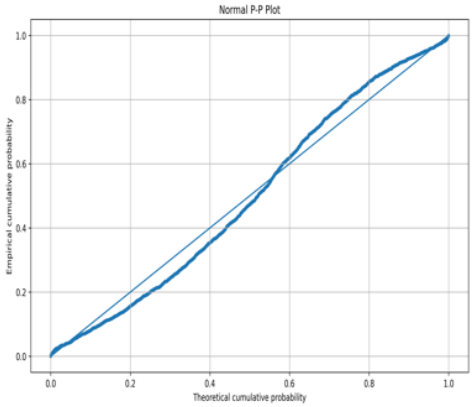


Figure 14. Normal P-P plot.

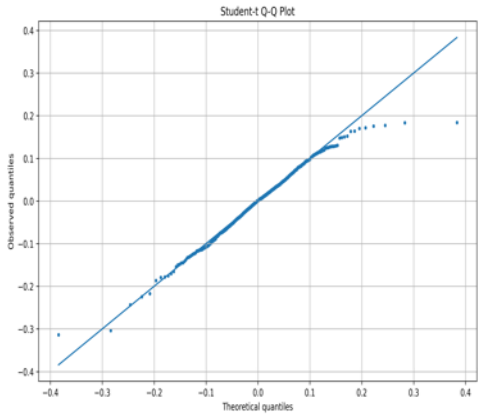


Figure 15. Student-t Q-Q plot.

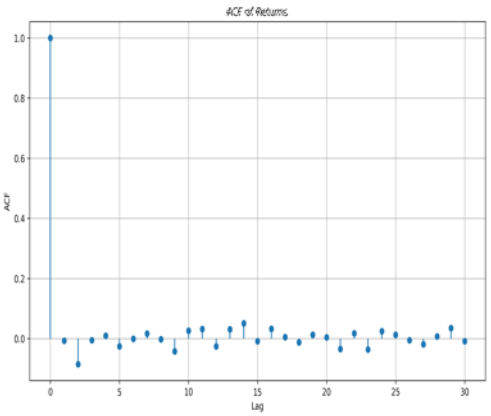


Figure 16. ACF of returns.

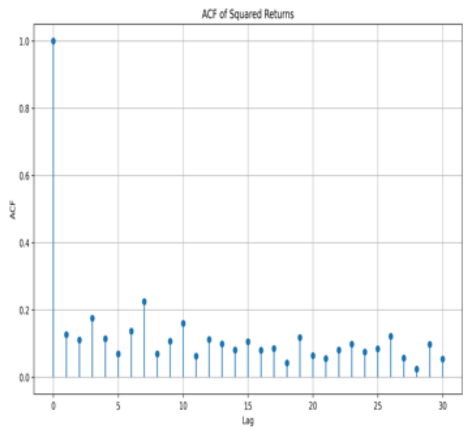


Figure 17. ACF of squared returns.

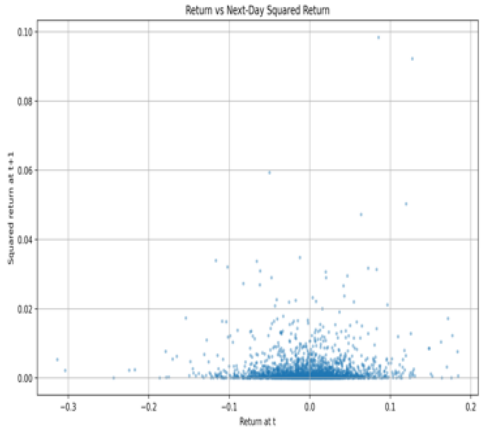


Figure 18. Return versus next-day squared return (leverage proxy).

Serial-dependence diagnostics

The return ACF is generally small, whereas the squared-return ACF is materially more persistent. This is consistent with the intended financial stylization in which raw returns can have weak linear dependence while volatility exhibits clustering (see for example tables 2 and 3)

Table 2:Ljung–Box test: returns

index	lb_stat	lb_pvalue
10	27.274	0.00235656
20	43.9618	0.00152274

Table 3:Ljung–Box test: squared returns

index	lb_stat	lb_pvalue
10	471.783	4.69554e-95
20	666.831	2.28235e-128

Maximum-likelihood estimation

The asymmetric Weibull model of (2) and the asymmetric weighted Weibull model of (45) were estimated by maximum likelihood after separating the positive returns from the magnitudes of negative returns. The complete model was estimated with positivity constraints on scale, shape and weighting parameters. The probability weight B was estimated by the observed fraction of positive returns (as seen in table 4).

Table 4: Maximum-likelihood estimation

Parameter	Asymmetric Weibull	Complete AW
B	0.510604	0.510604
gamma+	1.597547	1.087328
gamma-	1.611285	1.099885
Z+	0.057057	0.039159
Z-	0.065510	0.043887
a+	—	1.536078
a-	—	1.226057

Estimated asymmetric Weibull parameters: B = 0.510604, γ^{+} = 1.597547, γ^{-} = 1.611285, Z^{+} = 0.057057 and Z^{-} = 0.065510. The larger negative-side scale indicates greater dispersion of losses than gains. Estimated complete-model parameters: B = 0.510604, γ^{+} = 1.087328, γ^{-} = 1.099885, Z^{+} = 0.039159, Z^{-} = 0.043887, a^{+} = 1.536078 and a^{-} = 1.226057. The difference between a^{+} and a^{-} provides an additional mechanism for modelling asymmetric weighting across the two sides. Table 5 provides the goodness fit and model comparison of our model equations (2), (45) and other statistical distributions as seen in figures 1 to 18.

Table 5: Goodness-of-fit and model comparison

Model	LogLik	Parameters	AIC	BIC	KS_D	Histogram_ISE
Asymmetric Weibull	4333.502393	5	-8657.004787	-8627.886557	0.023652	0.260492
Complete asymmetric weighted Weibull	4377.211347	7	-8740.422694	-8699.657172	0.009731	0.024696
Normal	4210.528707	2	-8417.057413	-8405.410122	—	—
Student-t	4369.862562	3	-8733.725123	-8716.254185	—	—

The complete asymmetric weighted Weibull model has the largest log-likelihood (4377.211), the lowest AIC (−8740.423) and the lowest BIC among the fitted models considered. The ordinary asymmetric Weibull has log-likelihood 4333.502, AIC −8657.005 and BIC −8627.887. The Student-t model is a strong competitor (log-likelihood 4369.863, AIC −8733.725, BIC −8716.254), but the complete asymmetric model improves the AIC by approximately 6.70 and the log-likelihood by approximately 7.35. The KS distance is 0.009731 for the complete model versus 0.023652 for the simpler asymmetric Weibull. The histogram integrated squared error is also substantially smaller for the complete model (0.02470 versus 0.26049). These results support the additional weighting component, although formal p-values for a KS test after parameter estimation should not be interpreted using the ordinary fixed-parameter KS reference distribution; a parametric bootstrap is preferable.

Conclusion

This study investigated the symmetric Weibull distribution as well as developed its weighted extension for modelling the distributional behaviour of stock returns in the Nigerian Exchange Market, using a simulated dataset constructed to reproduce important characteristics of financial returns. The analysis was motivated by the limitations of the conventional Gaussian assumption, centring on its inability to properly represent asymmetry, excess kurtosis, volatility clustering and the occurrence of extreme observations. It was obvious from the empirical characteristics of the returns that it exhibited negative skewness, substantial excess kurtosis, relatively weak linear dependence in returns but stronger dependence in squared returns, while exhibiting consistent volatility clustering.

The asymmetric Weibull formulation offers a flexible way to capture differences between positive and negative returns. By fitting separate scale and shape parameters for the two sides of the distribution, the model was able to accommodate differences between gains and losses. In our simulation, the negative return side showed a larger scale parameter, pointing to greater dispersion of losses than gains. This provides a useful representation of downside asymmetry that would be difficult to obtain from a conventional symmetric distribution.The model comparisons results provide strong empirical support for this extension.

The main contribution of this study is the introduction and application of the complete asymmetric weighted Weibull model. By adding weighting parameters, it provided the model with another layer of flexibility to capture key differences between positive and negative returns. The estimation results demonstrate that these weighting parameters are not redundant: the estimated values of (a_+) and (a_-) differ significantly, reflecting distinct weighting behaviours for gains compared to that of losses, hence offering a clear upgrade over the standard asymmetric Weibull models.

The complete asymmetric weighted Weibull model achieved the highest log-likelihood alongside the lowest AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion) scores among all primary candidate models. It also significantly reduced both the Kolmogorov–Smirnov distance and the histogram integrated squared error compared to the standard asymmetric Weibull model. While the Student-t distribution performed competitively, the complete asymmetric model still outperformed it in log-likelihood and AIC, confirming that asymmetric weighting significantly improves fit across the return distribution.

Examining the cumulative probability distribution reveals a key distinction between empirical and theoretical tail behaviour. The empirical distribution of extreme returns roughly follows a power-law scale, offering a practical measure for extreme market events. However, the theoretical survival function of the generalized Weibull model takes a stretched-exponential form rather than a true asymptotic power law, as its exponential component eventually dominates. As a result, the Weibull framework works best as a flexible model for intermediate and central return behaviour, while empirical power-law analysis serves as a complementary tool for extreme tails. These findings show that explicitly modelling gain-loss asymmetry alongside a flexible weighting mechanism yields a much stronger fit than standard symmetric models. For the data we analysed, the complete asymmetric weighted Weibull model offered the best fit across all metrics, providing a robust framework for capturing asymmetric risk and return dispersion.

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