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Bounded linear operators in Intuitionistic G-fuzzy normed linear spaces

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Abstract

In this paper, concept of several types of intuitionistic G-fuzzy continuous operator and intuitionistic G-fuzzy bounded linear operator are introduced and relations among them are studied. Finally it is shown that the set of all strongly intuitionistic G-fuzzy bounded linear operators is a linear space and introduce the idea of intuionistic G-fuzzy operator norm in such spaces.

Keywords: Strongly intuitionistic G-fuzzy bounded linear operator, Weakly intuitionistic G-fuzzy bounded linear operator, Strongly intuitionistic G-fuzzy continuous linear operator, Weakly intuitionistic G-fuzzy continuous linear operator, Intuitionistic G-fuzzy operator norm

Mathematics Subject Classification: 46S40,03E72

1. INTRODUCTION

In 1965, L. A. Zadeh [1] introduced the concept of fuzzy set theory which has different applications in different fields of Mathematics. Several developments have been done in functional analysis to construct fuzzy functional analysis by employing the idea of fuzzy sets. Fuzzy metrics and fuzzy norms are two important tools to study different results of fuzzy functional analysis. At first Kramosil & Michalek [2] initiated the notion of

fuzzy metric and thereafter many authors came up with different types of fuzzy metrics by generalizing the existing concepts and studied various results (for reference please see [7–13, 15]). A. K. Katsaras [16] was the first author who introduced the concept of fuzzy norm in 1984. In later time many types of fuzzy norms were presented by different Mathematicians to establish several celebrated results of functional analysis in fuzzy settings (for reference please see [5, 6, 14, 17–25, 32]).

K. T. Atanassov [3, 4] extended the concept of fuzzy set theory by proposing the idea of intuitionistic fuzzy sets. J. H. Park [13] has introduced the notion of intuitionistic fuzzy metric spaces and studied some basic properties. After that R. Saadati & J. H. Park [14] have contributed by introducing the idea of the intuitionistic fuzzy topological spaces. They have also introduced the notion of intuitionistic fuzzy normed linear space to investigate some basic results in such spaces. T. Bag & S. K. Samanta [24] modified the definition of intuitionistic fuzzy normed linear space introduced by R. Saadati et al. [14] and studied some elementary results in finite dimensional intuitionistic fuzzy normed linear space.

Mustafa & Sims [26] brought up the concept of G-metric space and studied some topological properties. On the other hand, K. A. Khan [27] initiated the idea of G-norm and studied some basic results. Sun & Yang [7] proposed the definition of G-fuzzy metric space and established some common fixed point theorems. Alternatively S. Chatterjee et al. [22] introduced the notion of G-fuzzy normed linear space and studied its topological structures. The same authors [28] developed some fixed point theorems in G-fuzzy normed linear spaces. In 2023, S. Chatterjee & T. Bag [31] introduced the concept of G-fuzzy operator norm on G-fuzzy normed linear space. In 2013, S. A. Mohiuddine & A. Alotaibi [15] presented the concept of intuitionistic generalized fuzzy metric spaces.

Following the definition of G-fuzzy normed linear space introduced by S. Chatterjee et al. [22], we [32] give the definition of intuitionistic G-fuzzy normed linear space and established various basic results. Our main motivation of this paper are to study various types of bounded linear operators and continuous linear operators in intuitionistic G-fuzzy normed linear spaces and finally introduce the idea of intuitionistic G-fuzzy oprator norm on intuitionistic G-fuzzy normed linear spaces. This paper is organized in the following manner: Section 2 cconsists of some preliminary results which are used to study different results in this paper. In Section 3, various types of fuzzy continuous and fuzzy bounded linear operators are defined to develop some fundamental theorems in intuitionistic G-fuzzy normed linear spaces. In Section 4, intuitionisytic G-fuzzy operator norm is defined for intuitionistic G-fuzzy bounded linear operators.

2. PRELIMINARIES

In this Section some preliminary results are given to study the subsequent results of this paper.

Definition 2.1. [5] Let X be a linear space over $\mathbb{K}$(the field of scalars), a fuzzy subset N of $X \times \mathbb{R}$ is called a fuzzy norm on X iff $\forall x, u \in X$ and $c \in \mathbb{K}$,

$(N1)$ $\forall t \in \mathbb{R}$ with $t \leq 0, N(x,t) = 0$;

$(N2)$ $\forall t \in \mathbb{R}, t > 0, N(x,t) = 1)$ iff $x = 0$;

$(N3)$ $t \in \mathbb{R}, t > 0, N(cx,t) = N(x, \frac{t}{|c|}$ for $c(\neq 0) \in \mathbb{K}$;

$(N4)$ $\forall s, t \in \mathbb{R}, x, y \in X; N(x+y, s+t) \geq \min N(x,s), N(y,t)$;

$(N5)$ $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim\limits_{n\to\infty} N(x,t) = 1$.

The pair (X, N) is referred to as a fuzzy normed linear space.

Example 2.2. [5] Let $(X, \|\cdot\|)$ be a normed linear space. Define $N : X \times \mathbb{R} \to [0,1]$ as follows:

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|}, & \text{if } t > 0; \\ 0, & \text{if } t \leq 0. \end{cases}$$

Then (X, N) is a fuzzy normed linear space.

Definition 2.3. [5] A mapping T from (X, N_1) to (Y, N_2) is said to be fuzzy continuous at $x_0 \in X$, if for given $\epsilon > 0, \alpha \in (0,1), \exists \delta = \delta(\alpha, \epsilon) > 0, \beta = \beta(\alpha, \epsilon) \in (0,1)$ such that $\forall x \in X, N_1(x - x_0, \delta) > \beta \implies N_2(Tx - Tx_0, \epsilon) > \alpha$. If T is fuzzy continuous at each point of X, then T is said to be fuzzy continuous on X.

Definition 2.4. [5] A mapping T from (X, N_1) to (Y, N_2) is said to be strongly fuzzy continuous at $x_0 \in X$, if for each $\epsilon > 0, \exists \delta > 0$ such that $\forall x \in X, N_2(Tx - Tx_0, \epsilon) \geq N_1(x - x_0, \delta)$. T is said to be strongly fuzzy continuous on X if T is strongly fuzzy continuous at each point of X.

Definition 2.5. [5] A mapping T from (X, N_1) to (Y, N_2) is said to be weakly fuzzy continuous at $x_0 \in X$, if for a given $\epsilon > 0, \alpha \in (0,1), \exists \delta = \delta(\alpha, \epsilon) > 0$ such that $\forall x \in X, N_1(x - x_0, \delta) \geq \alpha \implies N_2(Tx - Tx_0, \epsilon) \geq \alpha$. If T is weakly fuzzy continuous at each point of X then T is said to be weakly fuzzy continuous on X.

Definition 2.6. [5] Let $T : (X, N_1) \to (Y, N_2)$ be a mapping where (X, N_1) and (Y, N_2) are two fuzzy normed linear spaces. T is said to be sequentially fuzzy continuous at $x_0 \in X$, if for any sequence $\{x_n\}$ in X with $x_n \to x_0$ implies $Tx_n \to Tx_0$. If T is sequentially continuous at every point of X then T is said to be sequentially continuous on X.

Definition 2.7. [5] Let $T : (X, N_1) \to (Y, N_2)$ be a mapping where (X, N_1) and (Y, N_2) are two fuzzy normed linear spaces. Then T is said to be strongly fuzzy bounded iff $\exists$ a positive real number M such that

$$\forall x \in X, \forall s \in \mathbb{R}, N_2(Tx, s) \geq N_1(x, \frac{s}{M}).$$

Definition 2.8. [5] Let $T : (X, N_1) \to (Y, N_2)$ be a linear operator where (X, N_1) and (Y, N_2) are fuzzy normed linear spaces. T is said to be weakly fuzzy bounded on X if for any $\alpha \in (0, 1), \exists M_\alpha > 0$ such that $\forall x \in X, \forall t \in \mathbb{R}$,

$$N_1(x, \frac{t}{M_\alpha}) \geq \alpha \implies N_2(Tx, t) \geq \alpha.$$

Theorem 2.9. *[5] Let $T : (X, N_1) \to (Y, N_2)$ be a linear operator where (X, N_1) and (Y, N_2) are normed linear spaces. If T is strongly fuzzy bounded then it is weakly fuzzy bounded but not conversely.*

***Remark* 2.10.** [5] The converse of the above theorem can be justified by the following example:

Example 2.11. Let $(X, \|\cdot\|)$ be a normed linear space. We define two functions N_1 and N_2 by

$$N_1(x, t) = \begin{cases} \frac{t^2 - \|x\|^2}{t^2 + \|x\|^2}; & \text{if } t > \|x\| \\ 0; & \text{if } t \leq \|x\|, \end{cases}$$

$$N_2(x, t) = \begin{cases} \frac{t}{t + \|x\|}; & \text{if } t > 0 \\ 0; & \text{if } t \leq 0. \end{cases}$$

Then N_1, N_2 are two fuzzy norms on X. Let us take the linear operator $T : X \to X$ defined by $Tx = x \; \forall x \in X$. Then T is weakly fuzzy bounded linear operator but not stromgly fuzzy bounded linear operator.

Definition 2.12. [32] Let X be a linear space over $\mathbb{K}$(the field of scalars), $*, \diamond$ be continuous t-norm and continuous t-conorm respectively and G_N, G_M be two fuzzy sets on $X^3 \times [0, \infty)$ satisfying the following conditions:

$(\mathcal{I}GFN1)$ $G_N(x,y,z,t)+G_M(x,y,z,t)\leq 1$;

$(\mathcal{I}GFN2)$ $G_N(x,y,z,0)=0$;

$(\mathcal{I}GFN3)$ $(G_N(x,y,z,t)=1\ \forall t>0)\iff x=y=z=\theta$;

$(\mathcal{I}GFN4)$ $G_N(\alpha x,\alpha y,\alpha z,t)=G_N(x,y,z,\frac{t}{|\alpha|})$ for $\alpha(\neq 0)\in\mathbb{K}$;

$(\mathcal{I}GFN5)$ $G_N(x,y,z,t)=G_N(p(x,y,z),t)$ where $p(x,y,z)$ is a permutation function of x,y,z;

$(\mathcal{I}GFN6)$ $G_N(x+x',y+y',z+z',s+t)\geq G_N(x,y,z,s)*G_N(x',y',z',t)$;

$(\mathcal{I}GFN7)$ $G_N(x+y,\theta,z,t)\geq G_N(x,y,z,t)$;

$(\mathcal{I}GFN8)$ $\lim\limits_{t\to\infty}G_N(x,y,z,t)=1$;

$(\mathcal{I}GFN9)$ $G_M(x,y,z,0)=1$;

$(\mathcal{I}GFN10)$ $(G_M(x,y,z,t)=0\ \forall t>0)\iff x=y=z=\theta$;

$(\mathcal{I}GFN11)$ $G_M(\alpha x,\alpha y,\alpha z,t)=G_M(x,y,z,\frac{t}{|\alpha|})$ for $\alpha(\neq 0)\in\mathbb{K}$;

$(\mathcal{I}GFN12)$ $G_M(x,y,z,t)=G_N(p(x,y,z),t)$ where $p(x,y,z)$ is a permutation function of x,y,z;

$(\mathcal{I}GFN13)$ $G_M(x+x',y+y',z+z',s+t)\leq G_M(x,y,z,s)\diamond G_M(x',y',z',t)$;

$(\mathcal{I}GFN14)$ $G_M(x+y,\theta,z,t)\leq G_M(x,y,z,t)$;

$(\mathcal{I}GFN15)$ $\lim\limits_{t\to\infty}G_M(x,y,z,t)=0$.

Then $(X,G_N,G_M,*,\diamond)$ is said to be an intuitionistic G-fuzzy normed linear space.

***Remark* 2.13.** [32] Let $(X,G_N,G_M,*,\diamond)$ be an intuitionistic G-fuzzy normed linear space. Then $G_N(x,y,z,\cdot)$ is a non-decreasing function of $[0,\infty)$ and $G_M(x,y,z,\cdot)$ is a non-increasing function of $[0,\infty)$.

Example 2.14. [32] Let us consider the linear space $X=\mathbb{R}$ over the field $\mathbb{R}$, the t-norm $*$ defined by $a*b=\min\{a,b\}$ and the t-conorm $\diamond$ defined by $a\diamond b=\max\{a,b\}$. We define $G_N,G_M:X^3\times[0,\infty)\to[0,1]$ as follows

$$G_N(x,y,z,t)=\begin{cases}\frac{t}{t+|x|+|y|+|z|}, & \text{if } t>0\\ 0, & \text{if } t=0\end{cases}$$

$$G_M(x,y,z,t) = \begin{cases} \frac{|x|+|y|+|z|}{t+|x|+|y|+|z|}, & \text{if } t>0 \\ 1, & \text{if } t=0 \end{cases}.$$

Then $(X, G_N, G_M, *, \diamond)$ is an intuitionistic G-fuzzy normed linear space.

Definition 2.15. [32] Let $(X, G_N, G_M, *, \diamond)$ be an intuitionistic G-fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be convergent and converges to $x \in X$ if $\lim_{l,m,n\to\infty} G_N(x_l - x, x_m - x, x_n - x, t) = 1$ and $\lim_{l,m,n\to\infty} G_M(x_l - x, x_m - x, x_n - x, t) = 0\ \forall t > 0$.

Definition 2.16. [32] Let $(X, G_N, G_M, *, \diamond)$ be an intuitionistic G-fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be Cauchy if the following hold:

$$\lim_{l,m,n\to\infty} G_N(x_l - x_m, x_m - x_n, x_n - x_l, t) = 1 \text{ and}$$

$$\lim_{l,m,n\to\infty} G_M(x_l - x_m, x_m - x_n, x_n - x_l, t) = 0\ \forall t > 0.$$

Definition 2.17. [22] Let X be a linear space over the field $\mathcal{F}$ (real or complex) and θ be the origin of X. Let $Q = \{\|\cdot,\cdot,\cdot\|_\alpha : \alpha \in (0,1)\}$ be a family of mappings from X^3 into $[0,\infty)$. Q, is called a quasi G-norm family and (X,Q) is called generating space of quasi G-norm family if the following conditions are satisfied:

$(QN1)$ $\|x,y,z\|_\alpha = 0, \forall \alpha \in (0,1)$ iff $x = y = z = \theta$.

$(QN2)$ $\|cx,cy,cz\|_\alpha = |c|\|x,y,z\|_\alpha, \forall x,y,z \in X, \forall \alpha \in (0,1)$ and $\forall c \in \mathcal{F}$.

$(QN3)$ $\|x,y,z\|_\alpha$ is invariant under permutations of x,y and $z \in X,\ \forall \alpha \in (0,1)$.

$(QN4)$ For any $\alpha \in (0,1) \exists \beta \in (0,\alpha]$ such that

$$\|x+x', y+y', z+z'\|_\alpha \le \|x,y,z\|_\beta + \|x',y',z'\|_\beta.$$

$(QN5)$ $\|x,y,z\|_\alpha \ge \|x+y,\theta,z\|_\alpha, \forall x,y,z \in X, \alpha \in (0,1)$.

$(QN6)$ For any $x,y,z \in X, \|x,y,z\|_\alpha$ is non-increasing for $\alpha \in (0,1)$.

Again (X,Q) is called a generating space of sub-strong quasi G-norm family, strong quasi G-norm family and semi G-norm family respectively, if $(QN4)$ is strengthened to $(QN4u)$, $(QN4t)$ and $(QN4e)$, where $(QN4u)$ For any $\alpha \in (0,1), \beta \in (0,\alpha]$ such that

$$\|\sum_{i=1}^n x_i, \sum_{i=1}^n y_i, \sum_{i=1}^n z_i\|_\alpha \le \sum_{i=1}^n \|x_i, y_i, z_i\|_\beta$$

for any $n \in \mathbb{N}$ and $x_i; y_i; z_i \in X (i = 1, 2, \cdots, n)$. $(QN4t)$ For any $\alpha \in (0, 1), \exists \beta \in (0, \alpha]$ such that

$$\|x + x', y + y', z + z'\|_\alpha \leq \|x, y, z\|_\alpha + \|x', y', z'\|_\beta.$$

$(QN4e)$ For any $\alpha \in (0, 1)$, it holds that

$$\|x + x', y + y', z + z'\|_\alpha \leq \|x, y, z\|_\alpha + \|x', y', z'\|_\alpha.$$

Theorem 2.18. *[22] Let $(X, G_N, *)$ be a G-fuzzy normed linear space such that $*$ is lower-semi-continuous. We define $\|\cdot, \cdot, \cdot\|_\alpha$ as follows:*

$$\|x, y, z\|_\alpha = \wedge\{t > 0 : G_N(x, y, z, t) \geq 1 - \alpha\}$$

for every $\alpha \in (0, 1)$ and for all $x, y, z \in X$. Let us denote $Q_1 = \{\|\cdot, \cdot, \cdot\|_\alpha : \alpha \in (0, 1)\}$. Then (X, Q_1) is a generating spaces of quasi-G-norm family.

Theorem 2.19. *[22] In the Theorem 2.18, if t-norm $*$ is defined by $a * b = \min\{a, b\}$ $\forall a, b \in [0, 1]$ then (X, Q) is a generating space of semi-G-norm family.*

Theorem 2.20. *[31] Let $(X, G_{N_1}, *_1)$ and $(Y, G_{N_2}, \min)$ be two G-fuzzy normed llinear spaces. Let $T_1, T_2, T_3 \in BGF(X, Y)$ and $\{\|\cdot, \cdot, \cdot\|^2_\alpha : \alpha \in (0, 1)\}$ be a family of semi-G-norms of G_{N_2} and $\{\|\cdot, \cdot, \cdot\|^1_\alpha : \alpha \in (0, 1)\}$ be a family of quasi-G-norms of G_{N_1}.Again suppose that G_{N_1} satisfies $(QN7)$. $((QN7) : \forall \alpha \in (0, 1), \|x, y, z\|_\alpha > 0, \forall (x, y, z) \neq (\theta, \theta, \theta).)$*
We define $\|T_1, T_2, T_3\|_\alpha = \bigvee_{\substack{x,y,z \in X \\ (x,y,z) \neq (\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|Tix, T_iy, T_iz\|^2_\alpha}{\|x,y,z\|^1_{1-\alpha}}$.
Then the mapping $G_N : \{BGF(X, Y)\}^3 \times \mathbb{R} \to [0, 1]$ defined by

$G_N(T_1, T_2, T_3, s) =$

$$\begin{cases} \vee\{\alpha \in (0, 1) : \|T_1, t_2, T_3\|_{1-\alpha} \leq s\}, & \text{if } (T_1, T - 2, T_3, s) \neq (\bigcirc, \bigcirc, \bigcirc, 0) \\ 0, & \text{otherwise} \end{cases}$$

is a G-fuzzy normed linear space with respect to the t-norm '$\min$'.

3. MAIN RESULT

In this Section fuzzy continuous operator, strongly intuitionistic G-fuzzy linear operator, weakly intuitionistic G-fuzzy operator, sequentially intuitioistic G-fuzzy continuous operator as well as strongly intuitionistic G-fuzzy bounded linear opeerator, weakly intuitionistic G-fuzzy bounded linear operator are defined in intuitionistic G-fuzzy normed linear space and some relations among them are established.

Definition 3.1. Let $(X, G_{N_1}, G_{M_1}, *_1, \diamond_1)$ and $(Y, G_{N_2}, G_{M_2}, *_2, \diamond_2)$ be two intuitionistic G-fuzzy normed linear spaces. A function $T : X \to Y$ is said to be intuitionistic G-fuzzy continuous at $x_0 \in X$ if for given $\epsilon > 0$ and $\alpha \in (0,1), \exists \delta = \delta(\alpha, \epsilon) > 0$ and $\beta = \beta(\alpha, \epsilon) \in (0,1)$ such that
$G_{N_1}(x - x_0, y - x_0, z - x_0, \delta) > \beta, G_{M_1}(x - x_0, y - x_0, z - x_0, \delta) < 1 - \beta$ implies
$G_{N_2}(Tx - Tx_0, Ty - Tx_0, Tz - Tx_0, \epsilon) > \alpha, G_{M_2}(Tx - Tx_0, Ty - Tx_0, Tz - Tx_0, \epsilon) < 1 - \alpha$.
If T is intuitionistic G-fuzzy continuous at every point of X then T is said to be intuitionistic G-fuzzy continuous on X.

Definition 3.2. Let $(X, G_{N_1}, G_{M_1}, *_1, \diamond_1)$ and $(Y, G_{N_2}, G_{M_2}, *_2, \diamond_2)$ be two intuitionistic G-fuzzy normed linear spaces. A function $T : X \to Y$ is said to be strongly intuitionistic G-fuzzy continuous at $x_0 \in X$ if for each $\epsilon > 0, \exists \delta > 0$ such that

$G_{N_1}(x - x_0, y - x_0, z - x_0, \delta) \leq G_{N_2}(Tx - Tx_0, Ty - Tx_0, Tz - Tx_0, \epsilon)$ and
$G_{M_1}(x - x_0, y - x_0, z - x_0, \delta) \geq G_{M_2}(Tx - Tx_0, Ty - Tx_0, Tz - Tx_0, \epsilon)\ \forall x, y, z \in X.$

If T is strongly intuitionistic G-fuzzy continuous at every point of X then T is said to be strongly intuitionistic G-fuzzy continuous on X.

Definition 3.3. Let $(X, G_{N_1}, G_{M_1}, *_1, \diamond_1)$ and $(Y, G_{N_2}, G_{M_2}, *_2, \diamond_2)$ be two intuitionistic G-fuzzy normed linear spaces. A function $T : X \to Y$ is said to be weakly intuitionistic G-fuzzy continuous at $x_0 \in X$ if for given $\epsilon > 0, \alpha \in (0,1), \exists \delta = \delta(\alpha, \epsilon) > 0$ such that

$G_{N_1}(x - x_0, y - x_0, z - x_0, \delta) \geq \alpha, G_{M_1}(x - x_0, y - x_0, z - x_0, \delta) \leq 1 - \alpha$ implies
$G_{N_2}(Tx - Tx_0, Ty - Tx_0, Tz - Tx_0, \epsilon)$
$\geq \alpha, G_{M_2}(Tx - Tx_0, Ty - Tx_0, Tz - Tx_0, \epsilon) \leq 1 - \alpha$
$\forall x, y, z \in X.$

If T is weakly intuitionistic G-fuzzy continuous at every point of X then T is said to be weakly intuitionistic G-fuzzy continuous on X.

Definition 3.4. Let $(X, G_{N_1}, G_{M_1}, *_1, \diamond_1)$ and $(Y, G_{N_2}, G_{M_2}, *_2, \diamond_2)$ be two intuitionistic G-fuzzy normed linear spaces. A function $T : X \to Y$ is said to be sequentially intuitionistic G-fuzzy continuous at $x_0 \in X$ if for three sequences

$\{x_n\}, \{y_n\}, \{z_n\}$ of points in X with

$$\lim_{n\to\infty} G_{N_1}(x_n - x_0, y_n - x_0, z_n - x_0, t) = 1,$$
$$\lim_{n\to\infty} G_{M_1}(x_n - x_0, y_n - x_0, z_n - x_0, t) = 0 \; \forall t > 0$$
$$\text{implies } \lim_{n\to\infty} G_{N_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, t) = 1,$$
$$\lim_{n\to\infty} G_{M_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, t) = 0 \; \forall t > 0.$$

If T is sequentially intuitionistic G-fuzzy continuous at every point of X then T is said to be sequntially intuitionistic G-fuzzy continuous on X.

Theorem 3.5. *Let $(X, G_{N_1}, G_{M_1}, *_1, \diamond_1)$ and $(Y, G_{N_2}, G_{M_2}, *_2, \diamond_2)$ be two intuitionistic G-fuzzy normed linear spaces. A function $T : X \to Y$ is intuitionistic G-fuzzy continuous iff T is sequentially intuitionistic G-fuzzy continuous.*

Proof. Suppose that T is intuitionistic G-fuzzy continuous at $x_0 \in X$.
Let $\epsilon > 0$ be given. Choose $\alpha \in (0, 1)$. Since T is intuitionistic G-fuzzy continuous, then $\exists \delta = \delta(\alpha, \epsilon) > 0$ and $\beta = \beta(\alpha, \epsilon) \in (0, 1)$ such that $G_{N_1}(x-x_0, y-x_0, z-x_0, \delta) > \beta$ and $G_{M_1}(x - x_0, y - x_0, z - x_0, \delta) < 1 - \beta$ implies $G_{N_2}(Tx - Tx_0, Ty - Tx_0, Tz - Tx_0, \epsilon) > \alpha$ and $G_{M_2}(Tx - Tx_0, Ty - Tx_0, Tz - Tx_0, \epsilon) < 1 - \alpha$.
Let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences in X such that $\lim_{n\to\infty} G_{N_1}(x_n - x_0, y_n - x_0, z_n - x_0, t) = 1$ and $\lim_{n\to\infty} G_{M_1}(x_n - x_0, y_n - x_0, z_n - x_0, t) = 0 \; \forall t > 0$.
So $\exists n_0 \in \mathbb{N}$ such that $G_{N_1}(x_n - x_0, y_n - x_0, z_n - x_0, \delta) > \beta, G_{M_1}(x_n - x_0, y_n - x_0, z_n - x_0, \delta) < 1 - \beta \; \forall n \geq n_0$.
Then we have $G_{N_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, \epsilon) > \alpha, G_{M_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, \epsilon) < 1 - \alpha \; \forall n \geq n_0$.
Since $\alpha \in (0, 1)$ is arbitrary, so $\lim_{n\to\infty} G_{N_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, \epsilon) = 1$, $G_{M_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, \epsilon) = 0$.
Again $\epsilon > 0$ is choossen arbitrarily, so $\lim_{n\to\infty} G_{N_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, t) = 1, G_{M_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, t) = 0 \; \forall t > 0$.
So T is sequentially intuitionistic G-fuzzy continuous at x_0.
Conversely suppose that T is sequentially intuitionistic G-fuzzy continuous at $x_0 \in X$.
If possible suppose that T is not intuitionistic G-fuzzy continuous at x_0. Thus $\exists \epsilon > 0$ and $\alpha \in (0, 1)$ such that for any $\delta > 0$ and $\beta \in (0, 1), \exists x', y', z' \in X$ such that $G_{N_1}(x' - x_0, y' - x_0, z' - x_0, \delta) > \beta$ and $G_{M_1}(x' - x_0, y' - x_0, z' - x_0, \delta) < 1 - \beta$ but $G_{N_2}(Tx' - Tx_0, Ty' - Tx_0, Tz' - Tx_0, \epsilon) \leq \alpha$ and $G_{M_2}(Tx' - Tx_0, Ty' - Tx_0, Tz' - Tx_0, \epsilon) \geq 1 - \alpha$.
Thus for $\beta^{(n)} = 1 - \frac{1}{n+1}, \delta^{(n)} = \frac{1}{n+1}$ where $n \in \mathbb{N}, \exists x_n, y_n, z_n \in X$ such that $G_{N_1}(x_n - x_0, y_n - x_0, z_n - x_0, \frac{1}{n+1}) > 1 - \frac{1}{n+1}, G_{M_1}(x_n - x_0, y_n - x_0, z_n - x_0, \frac{1}{n+1}) < \frac{1}{n+1}$

but $G_{N_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, \epsilon) \leq \alpha, G_{M_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, \epsilon) \geq 1 - \alpha$.
Now for $\delta > 0, \exists k_0 \in \mathbb{N}$ such that $\frac{1}{k_0+1} < \delta$. Again $\frac{1}{n+1} < \delta \; \forall n \geq k_0$.
Then $G_{N_1}(x_n - x_0, y_n - x_0, z_n - x_0, \delta) \geq G_{N_1}(x_n - x_0, y_n - x_0, z_n - x_0, \frac{1}{n+1}) > 1 - \frac{1}{n+1}$,
$G_{M_1}(x_n - x_0, y_n - x_0, z_n - x_0, \delta) \leq G_{M_1}(x_n - x_0, y_n - x_0, z_n - x_0, \frac{1}{n+1}) < \frac{1}{n+1} \forall n \geq k_0$.
This implies that

$$\begin{aligned}
&\lim_{n\to\infty} G_{N_1}(x_n - x_0, y_n - x_0, z_n - x_0, \delta) = 1,\\
&\lim_{n\to\infty} G_{M_1}(x_n - x_0, y_n - x_0, z_n - x_0, \delta) = 0;\\
\text{but } &G_{N_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, \epsilon) \leq \alpha,\\
&G_{M_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, \epsilon) \geq 1 - \alpha.
\end{aligned}$$

Thus $\{T(x_n)\}$ does not converge to $T(x_0)$ whereas $\{x_n\}$ converges to x_0. Hence we arrive at a contradiction. Therefore T is intuitionistic G-fuzzy continuous at x_0. □

Definition 3.6. Let $T : (X, G_{N_1}, G_{M_1}, *_1, \diamond_1) \to (Y, G_{N_2}, G_{M_2}, *_2, \diamond_2)$ be a linear operator where $(X, G_{N_1}, G_{M_1}, *_1, \diamond_1)$ and $(Y, G_{N_2}, G_{M_2}, *_2, \diamond_2)$ are two intuitionistic G-fuzzy normed linear spaces. T is said to be strongly intuitionistic G-fuzzy bounded if $\exists M > 0$ such that $\forall x, y, z \in X$ and $\forall s \in [0, \infty)$,

$$G_{N_2}(Tx, Ty, Tz, s) \geq G_{N_1}(x, y, z, \frac{s}{M}), G_{M_2}(Tx, Ty, Tz, s) \leq G_{M_1}(x, y, z, \frac{s}{M}).$$

Definition 3.7. Let $T : (X, G_{N_1}, G_{M_1}, *_1, \diamond_1) \to (Y, G_{N_2}, G_{M_2}, *_2, \diamond_2)$ be a linear operator where $(X, G_{N_1}, G_{M_1}, *_1, \diamond_1)$ and $(Y, G_{N_2}, G_{M_2}, *_2, \diamond_2)$ are two intuitionistic G-fuzzy normed linear spaces. T is said to be weakly intuitionistic G-fuzzy bounded if for any $\alpha \in (0, 1), \exists M_\alpha > 0$ such that $\forall x, y, z \in X$ and $\forall t \in (0, \infty)$,

$$\begin{aligned}
&G_{N_1}(x, y, z, \frac{t}{M_\alpha}) \geq \alpha, G_{M_1}(x, y, z, \frac{t}{M_\alpha}) \leq 1 - \alpha\\
\implies &G_{N_2}(Tx, Ty, Tz, t) \geq \alpha, G_{M_2}(Tx, Ty, Tz, t) \leq 1 - \alpha.
\end{aligned}$$

Theorem 3.8. *Let $T : (X, G_{N_1}, G_{M_1}, *_1, \diamond_1) \to (Y, G_{N_2}, G_{M_2}, *_2, \diamond_2)$ be a linear operator where $(X, G_{N_1}, G_{M_1}, *_1, \diamond_1)$ and $(Y, G_{N_2}, G_{M_2}, *_2, \diamond_2)$ are two intuitionistic G-fuzzy normed linear spaces. If T is strongly intuitionistic G-fuzzy bounded then T is weakly intuitionistic G-fuzzy bounded.*

Proof. Suppose that T is strongly intuitionistic G-fuzzy bounded. Then $\exists M > 0$ such that $\forall x, y, z \in X$ and $\forall t > 0$
we have
$G_{N_2}(Tx, Ty, Tz, t) \geq G_{N_1}(x, y, z, \frac{t}{M}), G_{M_2}(Tx, Ty, Tz, t) \leq G_{M_1}(x, y, z, \frac{t}{M})$.

Let $\alpha \in (0,1)$. Then $\exists M_\alpha (= M) > 0$ such that
$G_{N_1}(x,y,z,\frac{t}{M_\alpha}) \geq \alpha, G_{M_1}(x,y,z,\frac{t}{M_\alpha}) \leq 1-\alpha$ implies
$G_{N_2}(Tx,Ty,Tz,t) \geq \alpha, G_{M_2}(Tx,Ty,Tz,t) \leq 1-\alpha$.
Therefore T is weakly intuitionistic G-fuzzy bounded. □

***Remark* 3.9.** Converse need not be true. This can be verified by the following example.

Example 3.10. Let $(X, \|\cdot,\cdot,\cdot\|)$ be a G-fuzzy normed linear space. We define $G_{N_1}, G_{M_1}, G_{N_2}, G_{M_2} : X^3 \times [0,\infty) \to \mathbb{R}$ as follows:

$$G_{N_1}(x,y,z,t) = \begin{cases} \frac{t^2-\|x,y,z\|^2}{t^2+\|x,y,z\|^2}, & \text{if } t > \|x,y,z\| \\ 0, & \text{if } t \leq \|x,y,z\| \end{cases}$$

$$G_{M_1}(x,y,z,t) = \begin{cases} \frac{2\|x,y,z\|^2}{t^2+\|x,y,z\|^2}, & \text{if } t > \|x,y,z\| \\ 1, & \text{if } t \leq \|x,y,z\| \end{cases}$$

$$G_{N_2}(x,y,z,t) = \begin{cases} \frac{t}{t+\|x,y,z\|}, & \text{if } t > 0 \\ 0, & \text{if } t = 0, \end{cases}$$

$$G_{M_2}(x,y,z,t) = \begin{cases} \frac{\|x,y,z\|}{t+\|x,y,z\|}, & \text{if } t > 0 \\ 1, & \text{if } t = 0. \end{cases}$$

First we prove that $(X, G_{N_1}, G_{M_1}, \min, \max)$ is an intuitionistic G-fuzzy normed linear space.

1. It is obvious that $G_{N_1}(x,y,z,t) + G_{M_1}(x,y,z,t) = 1 \ \forall t \geq 0$. So $(\mathcal{I}GFN1)$ holds.

2. From definition $(\mathcal{I}GFN2)$ is obvious.

3. Suppose $x, y, z \in X$. Then

$$\begin{aligned} & G_{N_1}(x,y,z,t) = 1 \ \forall t > 0 \\ \iff & \frac{t^2 - \|x,y,z\|^2}{t^2 + \|x,y,z\|^2} = 1 \ \forall t > 0 \\ \iff & t^2 - \|x,y,z\|^2 = t^2 + \|x,y,z\|^2 \\ \iff & \|x,y,z\| = 0 \\ \iff & x = y = z = \theta. \end{aligned}$$

   Hence $(\mathcal{I}GFN3)$ holds.

4. Suppose $\alpha(\neq 0) \in \mathbb{K}$. Then

$$G_{N_1}(\alpha x, \alpha y, \alpha z, t) = \frac{t^2 - \|\alpha x, \alpha y, \alpha z\|^2}{t^2 + \|\alpha x, \alpha y, \alpha z\|^2} = \frac{t^2 - |\alpha|^2\|x, y, z\|^2}{t^2 + |\alpha|^2\|x, y, z\|^2}$$
$$= \frac{(\frac{t}{|\alpha|})^2 - \|x, y, z\|^2}{(\frac{t}{|\alpha|})^2 + \|x, y, z\|^2}$$
$$= G_{N_1}(x, y, z, \frac{t}{|\alpha|}).$$

Thus $(\mathcal{I}GFN4)$ is satisfied.

5. It is clear that $G_{N_1}(x, y, z, t) = G_{N_1}(p(x, y, z), t)$ for every permutation p of x, y, z. So $(\mathcal{I}GFN5)$ holds.

6. If either of t_1 or t_2 is 0, then
$G_{N_1}(x + x', y + y', z + z', t_1 + t_2) \geq \min\{G_{N_1}(x, y, z, t_1), G_{N_2}(x', y', z', t_2)\}$ is obvious.
Now suppose $t_1, t_2 > 0$. Further we suppose that $t_1 \leq \|x, y, z\|$.
Then by definition it follows that
$G_{N_1}(x + x', y + y', z + z', t_1 + t_2) \geq \min\{G_{N_1}(x, y, z, t_1), G_{N_2}(x', y', z', t_2)\}$.
Similarly if $t_2 \leq \|x', y', z'\|$ then
$G_{N_1}(x + x', y + y', z + z', t_1 + t_2) \geq \min\{G_{N_1}(x, y, z, t_1), G_{N_2}(x', y', z', t_2)\}$.
Next we suppose that $t_1 > \|x, y, z\|, t_2 > \|x', y', z'\|$. Then we have
$\|x + x', y + y', z + z'\| \leq \|x, y, z\| + \|x', y', z'\| < t_1 + t_2$. So

$$G_{N_1}(x, y, z, t_1) = \frac{t_1^2 - \|x, y, z\|^2}{t_1^2 + \|x, y, z\|^2},$$
$$G_{N_1}(x', y', z', t_2) = \frac{t_2^2 - \|x', y', z'\|^2}{t_2^2 + \|x', y', z'\|^2},$$
$$G_{N_1}(x + x', y + y', z + z', t_1 + t_2) = \frac{(t_1 + t_2)^2 - \|x + x'.y + y', z + z'\|^2}{(t_1 + t_2)^2 + \|x + x', y + y', z + z'\|^2}.$$

Further we assume that $\min\{G_{N_1}(x, y, z, t_1), G_{N_1}(x', y', z', t_2)\} = G_{N_1}(x, y, z, t_1)$. So

$$\frac{t_1^2 - \|x, y, z\|^2}{t_1^2 + \|x, y, z\|^2} \leq \frac{t_2^2 - \|x', y', z'\|^2}{t_2^2 + \|x', y', z'\|^2}$$
$$\implies t_2^2\|x, y, z\|^2 - t_1^2\|x', y', z'\|^2 \geq 0 \text{.......................................} (A)$$
$$\implies t_2\|x, y, z\| - t_1\|x', y', z'\| \geq 0 \text{..} (B)$$

Now we have

$$\begin{aligned}
&G_{N_1}(x+x', y+y', z+z', t_1+t_2) - G_{N_1}(x, y, z, t_1)\\
=&\frac{(t_1+t_2)^2 - \|x+x', y+y', z+z'\|^2}{(t_1+t_2)^2 + \|x+x', y+y', z+z'\|^2} - \frac{t_1^2 - \|x, y, z\|^2}{t_1^2 + \|x, y, z\|^2}\\
=&\frac{2(t_1+t_2)^2\|x, y, z\|^2 - t_1^2\|x+x', y+y', z+z'\|^2}{\{(t_1+t_2)^2 + \|x+x', y_y', z+z'\|^2\}(t_1^2 + \|x, y, z\|^2)}\\
\geq&\frac{2(t_1^2 + 2t_1t_2 + t_2^2)\|x, y, z\|^2 - 2t_1^2(\|x, y, z\| + \|x', y', z'\|)^2}{\{(t_1+t_2)^2 + \|x+x', y_y', z+z'\|^2\}(t_1^2 + \|x, y, z\|^2)}\\
=&\frac{2(t_2^2\|x, y, z\|^2 - t_1^2\|x', y', z'\|^2) + 4t_1\|x, y, z\|(t_2\|x, y, z\| - t_1\|x', y', z'\|)}{\{(t_1+t_2)^2 + \|x+x', y_y', z+z'\|^2\}(t_1^2 + \|x, y, z\|^2)}\\
\geq&0 \text{ (using } A, B).
\end{aligned}$$

Hence we conclude that $G_{N_1}(x+x', y+y', z+z', t_1+t_2) \geq \min\{G_{N_1}(x, y, z, t_1), G_{N_1}(x', y', z', t_2)\}$. So $(\mathcal{I}GFN6)$ is verified.

7. Let us take $x, y, z \in X$. Then $\|x, y, z\| \geq \|x+y, \theta, z\|$. This implies that $t^2 + \|x, y, z\|^2 \geq t^2 + \|x+y, \theta, z\|^2$. Again $t^2 - \|x, y, z\|^2 \leq t^2 - \|x+y, \theta, z\|^2$ i.e $t^2 - \|x+y, \theta, z\|^2 \geq t^2 - \|x, y, z\|^2$. Thus the following holds:

$$\frac{t^2 - \|x+y, \theta, z\|^2}{t^2 + \|x+y, \theta, z\|^2} \geq \frac{t^2 - \|x, y, z\|^2}{t^2 + \|x, y, z\|^2}.$$

Hence $(\mathcal{I}GFN7)$ is verified.

8. For $x, y, z \in X$,

$$\lim_{t\to\infty} G_{N_1}(x, y, z, t) = \lim_{t\to\infty} \frac{t^2 - \|x, y, z\|^2}{t^2 + \|x, y, z\|^2} = \lim_{t\to\infty} \frac{1 - \frac{\|x,y,z\|^2}{t^2}}{1 + \frac{\|x,y,z\|^2}{t^2}} = 1.$$

Thus $(\mathcal{I}GFN8)$ holds.

9. From definition it is quite clear that $G_{M_1}(x, y, z, t) = 1 - G_{N_1}(x, y, z, t)$ for every $x, y, z \in X$ and $t \geq 0$. Thus form above part of this proof we can conclude that $(\mathcal{I}GFN9), (\mathcal{I}GFN10), (\mathcal{I}GFN11), (\mathcal{I}GFN12), (\mathcal{I}GFN14), (\mathcal{I}GFN15)$ are obvious.
We only prove $(\mathcal{I}GFN13)$.

10. For $s = t = 0; s = 0, t > 0; s > 0, t = 0$ it is obvious that $G_{M_2}(x+x', y+y', z+z', s+t) \leq \max\{G_{M_2}(x, y, z, s), G_{M_2}(x', y', z', t)\}$.

Now suppose $s, t > 0$. So $s + t > 0$. Here

$$G_{M_2}(x, y, z, s) = 1 - G_{N_2}(x, y, z, s),$$
$$G_{M_2}(x', y', z', t) = 1 - G_{N_2}(x', y', z', t),$$
$$G_{M_2}(x'_x, y + y', z + z', s + t) = 1 - G_{N_2}(x + x', y + y', z + z', s + t).$$

Without loss of generality let us assume that
$\max\{G_{M_2}(x, y, z, s), G_{M_2}(x', y', z', t)\} = G_{M_2}(x, y, z, s)$.

$$\begin{aligned}
&\text{So } G_{M_2}(x, y, z, s) \geq G_{M_2}(x', y', z', t)\\
\implies &1 - G_{N_2}(x, y, z, s) \geq 1 - G_{N_2}(x', y', z', t)\\
\implies &G_{N_2}(x, y, z, s) \leq G_{N_2}(x', y', z', t)\\
\implies &\min\{G_{N_2}(x, y, z, s), G_{N_2}(x', y', z', t)\} = G_{N_2}(x, y, z, s).
\end{aligned}$$

Already we have
$G_{N_2}(x + x', y + y', z + z', s + t) \geq \min\{G_{N_2}(x, y, z, s), G_{N_2}(x', y', z', t)\} = G_{N_2}(x, y, z, s)$. This implies that

$$\begin{aligned}
&1 - G_{N_2}(x + x', y + y', z + z', s + t) \leq 1 - G_{N_2}(x, y, z, s)\\
\implies &G_{M_2}(x + x', y + y', z + z', s + t) \leq G_{M_2}(x, y, z, s)\\
\implies &G_{M_2}(x + x', y + y', z + z', s + t) \leq \max\{G_{M_2}(x, y, z, s), G_{M_2}(x', y', z', t)\}.
\end{aligned}$$

Similarly if $\max\{G_{M_2}(x, y, z, s), G_{M_2}(x', y', z', t)\} = G_{M_2}(x', y', z', t)$ then also we have
$G_{M_2}(x+x', y+y', z+z', s+t) \leq \max\{G_{M_2}(x, y, z, s), G_{M_2}(x', y', z', t)\}$. Thus $(\mathcal{I}GFN13)$ holds.

Hence $(X, G_{N_1}, G_{M_1}, \min, \max)$ is an intuitionistic G-fuzzy normed linear space.
Similarly it can be shown that $(X, G_{N_2}, G_{M_2}, \min, \max)$ is an intuitionistic G-fuzzy normed linear space.
Now we define $T : (X, G_{N_1}, G_{M_1}, \min, \max) \to (X, G_{N_2}, G_{M_2}, \min, \max)$ by $T(x) = x$.
Let us consider $M_\alpha = \frac{1}{1-\alpha}$ $\forall \alpha \in (0, 1)$.

Thus for $t > \|x, y, z\|$ we have

$$\begin{aligned}
& G_{M_1}(x, y, z, \frac{t}{M_\alpha}) \geq \alpha \\
\Longrightarrow & \frac{t^2(1-\alpha)^2 - \|x, y, z\|^2}{t^2(1-\alpha)^2 + \|x, y, z\|^2} \geq \alpha \\
\Longrightarrow & t^2(1-\alpha)^2 - \|x, y, z\|^2 \geq \alpha t^2(1-\alpha)^2 + \alpha\|x, y, z\|^2 \\
\Longrightarrow & t^2(1-\alpha)^2 - \alpha t^2(1-\alpha)^2 \geq (1+\alpha)\|x, y, z\|^2 \\
\Longrightarrow & t^2(1-\alpha)^3 \geq (1+\alpha)\|x, y, z\|^2 \\
\Longrightarrow & \|x, y, z\| \leq \frac{t(1-\alpha)\sqrt{1-\alpha}}{\sqrt{1+\alpha}} \\
\Longrightarrow & t + \|x, y, z\| \leq t + \frac{t(1-\alpha)\sqrt{1-\alpha}}{\sqrt{1+\alpha}} \\
\Longrightarrow & t + \|x, y, z\| \leq t\frac{(1-\alpha)\sqrt{1-\alpha} + \sqrt{1+\alpha}}{\sqrt{1+\alpha}} \\
\Longrightarrow & \frac{t}{t + \|x, y, z\|} \geq \frac{(1+\alpha)}{(1-\alpha)\sqrt{1-\alpha} + \sqrt{1+\alpha}}.
\end{aligned}$$

$$\begin{aligned}
\text{Now } & \frac{\sqrt{1+\alpha}}{(1-\alpha)\sqrt{1-\alpha} + \sqrt{1+\alpha}} \geq \alpha \\
\Longleftrightarrow & \sqrt{1+\alpha} \geq \alpha(1-\alpha)\sqrt{1-\alpha} + \alpha\sqrt{1+\alpha} \\
\Longleftrightarrow & (1-\alpha)\sqrt{1+\alpha} \geq \alpha(1-\alpha)\sqrt{1-\alpha} \\
\Longleftrightarrow & \sqrt{1+\alpha} \geq \alpha\sqrt{1-\alpha} \\
\Longleftrightarrow & 1+\alpha \geq \alpha^2(1-\alpha) \\
\Longleftrightarrow & 1+\alpha+\alpha^3 \geq \alpha^2.
\end{aligned}$$

This is true $\forall\alpha \in (0, 1)$. Thus we have $\frac{t}{t+\|x,y,z\|} \geq \alpha\ \forall\alpha \in (0, 1)$.
So $G_{M_2}(Tx, Ty, Tz, t) \geq \alpha$ for $t > \|x, y, z\|$.
For $t \leq \|x, y, z\|, G_{M_1}(x, y, z, t) = 0$.
So $G_{M_1}(x, y, z, \frac{t}{M_\alpha}) \geq \alpha$ implies $G_{M_2}(Tx, Ty, Tz, t) \geq \alpha$.
We have $G_{N_1}(x, y, z, \frac{t}{M_\alpha}) = 1 - G_{M_1}(x, y, z, \frac{t}{M_\alpha}), G_{N_2}(Tx, Ty, Tz, t) = 1 - G_{M_2}(Tx, Ty, Tz, t)$.
Suppose $G_{N_1}(x, y, z, \frac{t}{M_\alpha}) \leq 1 - \alpha$. So $G_{M_1}(x, y, z, \frac{t}{M_\alpha}) \geq \alpha$. Then $G_{M_2}(Tx, Ty, Tz, t) \geq \alpha$. Hence $G_{N_2}(Tx, Ty, Tz, t) \leq 1 - \alpha$.
Therefore $G_{N_1}(x, y, z, \frac{t}{M_\alpha}) \leq 1 - \alpha$ implies $G_{N_2}(Tx, Ty, Tz, t) \leq 1 - \alpha$.
Thus T is weakly intuitionistic G-fuzzy bounded linear operator.

Next for $t > \|x, y, z\|, (x, y, z) \neq (\theta, \theta, \theta)$,

$$
\begin{aligned}
& G_{M_2}(Tx, Ty, Tz, t) \geq G_{M_1}(x, y, z, \frac{t}{M}) \\
\Longrightarrow & \frac{t}{t + \|x, y, z\|} \geq \frac{t^2 - \|x, y, z\|^2}{t^2 + \|x, y, z\|^2} \\
\Longrightarrow & tM^2\|x, y, z\|^2 \geq -tM^2\|x, y, z\|^2 + t^2\|x, y, z\| - M^2\|x, y, z\|^3 \\
\Longrightarrow & 2tM^2\|x, y, z\|^2 + M^2\|x, y, z\|^3 \geq t^2\|x, y, z\| \\
\Longrightarrow & 2tM^2\|x, y, z\| + M^2\|x, y, z\|^2 \geq t^2 \\
\Longrightarrow & M^2 \geq \frac{t^2}{2t\|x, y, z\| + \|x, y, z\|^2} \\
\Longrightarrow & M \geq \frac{t}{\sqrt{2t\|x, y, z\| + \|x, y, z\|^2}} \\
\Longrightarrow & M = \infty \text{ as } t \to \infty.
\end{aligned}
$$

Hence T is not strongly intuitionistic G-fuzzy bounded.

Theorem 3.11.
Let $(X, G_{N_1}, G_{M_1}, *_1, \diamond_1)$ and $(Y, G_{N_2}, G_{M_2}, *_2, \diamond_2)$ be two intuitionistic G-fuzzy normed linear spaces and $T : X \to Y$ be a linear operator. Then

(i) T is strongly intuitionistic G-fuzzy continuous everywhere if T is strongly intuitionistic G-fuzzy continuous at a point $x_0 \in X$.

(ii) T is strongly intuitionistic G-fuzzy continuous iff T is strongly intuitionistic G-fuzzy bounded.

Proof. (i) Since T is strongly intuitionistic G-fuzzy continuous at $x_0 \in X$, thus for each $\epsilon > 0, \exists \delta > 0$ such that $\forall x \in X$ we have
$G_{N_2}(Tx - Tx_0, Ty - Tx_0, Tz - Tx_0, \epsilon) \geq G_{N_1}(x - x_0, y - x_0, z - x_0, \delta)$ and
$G_{M_2}(Tx - Tx_0, Ty - Tx_0, Tz - Tx_0, \epsilon) \leq G_{M_1}(x - x_0, y - x_0, z - x_0, \delta)$.
Now we take $x' \in X$ and replacing x, y, z by $x + x_0 - x', y + x_0 - x', z + x_0 - x'$ respectively we have

$$
\begin{aligned}
& G_{N_2}(T(x + x_0' - x') - T(x_0), T(y + x_0 - x') - T(x_0), T(z + x_0 - x') - T(x_0), \epsilon)) \\
\geq & G_{N_1}(x + x_0 - x' - x_0, y + x_0 - x' - x_0, z + x_0 - x' - x_0, \delta), \\
& G_{M_2}(T(x + x_0 - x') - T(x_0), T(y + x_0 - x') - T(x_0), T(z + x_0 - x') - T(x_0), \epsilon) \\
\leq & G_{M_1}(x + x_0 - x' - x_0, y + x_0 - x' - x_0, z + x_0 - x' - x_0, \delta).
\end{aligned}
$$

So it follows that

$$\begin{aligned}&G_{N_2}(Tx+Tx_0-Tx'-Tx_0, Ty+Tx_0-Tx'-Tx_0, Tz+Tx_0-Tx'-Tx_0, \epsilon)\\ \geq &G_{N_1}(x-x', y-x', z-x', \delta),\\ &G_{M_2}(Tx+Tx_0-Tx'-Tx_0, Ty+Tx_0-Tx'-Tx_0, Tz+Tx_0-Tx'-Tx_0, \epsilon)\\ \leq &G_{M_1}(x-x', y-x', z-x', \delta).\end{aligned}$$

Thus we have

$$\begin{aligned}G_{N_2}(Tx-Tx', Ty-Tx', Tz-Tx', \epsilon) &\geq G_{N_1}(x-x', y-x', z-x', \delta),\\ G_{M_2}(Tx-Tx', Ty-Tx', Tz-Tx', \epsilon) &\leq G_{M_1}(x-x', y-x', z-x', \delta).\end{aligned}$$

So T is strongly intuitionistic G-fuzzy continuous at x'. Since $x' \in X$ is arbitrary which implies that T is strongly intuitionistic G-fuzzy continuous on X.

(ii) First we suppose that T is strongly intuitionistic G-fuzzy bounded. Then $\exists M > 0$ such that

$$\begin{aligned}G_{N_2}(Tx, Ty, Tz, t) &\geq G_{N_1}(x, y, z, \frac{t}{M}),\\ G_{M_2}(Tx, Ty, Tz, t) &\leq G_{M_1}(x, y, z, \frac{t}{M}).\end{aligned}$$

So $G_{N_2}(Tx-\theta', Ty-\theta', Tz-\theta', \epsilon) \geq G_{N_1}(x-\theta, y-\theta, z-\theta, \delta)$
and $G_{M_2}(Tx-\theta', Ty-\theta', Tz-\theta', \epsilon) \leq G_{M_1}(x-\theta, y-\theta, z-\theta, \delta)$ where $\delta = \frac{\epsilon}{M} > 0$.
This implies that T is strongly intuitionistic G-fuzzy continuous at θ and hence by (i) of Theorem 3.13, T is strongly intuitionistic G-fuzzy continuous on X.
Conversely suppose that T is strongly intuitionistic G-fuzzy continuous on X. So T is strongly intuitionistic G-fuzzy continuous at θ.
Then for $\epsilon = 1, \exists \delta > 0$ such that

$$\begin{aligned}&G_{N_2}(Tx-T\theta, Ty-T\theta, Tz-T\theta, 1) \geq G_{N_1}(x-\theta, y-\theta, z-\theta, \delta),\\ &G_{M_2}(Tx-T\theta, Ty-T\theta, Tz-T\theta, 1) \leq G_{M_1}(x-\theta, y-\theta, z-\theta, \delta)\ \forall x, y, z \in X.\\ \text{So } &G_{N_2}(Tx, Ty, Tz, 1) \geq G_{N_1}(x, y, z, \delta),\\ &G_{M_2}(Tx, Ty, Tz, 1) \leq G_{M_1}(x, y, z, \delta)\ \forall x, y, z \in X.\end{aligned}$$

Suppose that $(x, y, z) \neq (\theta, \theta, \theta)$ and $t > 0$.

Let us put $u = \frac{x}{t}, v = \frac{y}{t}, w = \frac{z}{t}$.Then

$$\begin{aligned} G_{N_2}(Tx, Ty, Tz, t) =& G_{N_2}(T(tu), T(tv), T(tw), t) \\ =& G_{N_2}(tTu, tTv, tTw, t) \\ =& G_{N_2}(Tu, Tv.Tw, 1) \\ \geq& G_{N_1}(u, v, w, \delta) \\ =& G_{N_1}(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}, \delta) \\ =& G_{N_1}(x, y, z, t\delta) \\ =& G_{N_1}(x, y, z, \frac{t}{M}) \text{ where } M = \frac{1}{\delta} > 0. \end{aligned}$$

If $(x, y, z) \neq (\theta, \theta, \theta)$ and $t = 0$ then $G_{N_2}(Tx, Ty, Tz, t) = 0 = G_{N_1}(x, y, z, \frac{t}{M})$.
If $(x, y, z) = (\theta, \theta, \theta)$ and $t > 0$ then $G_{N_2}(Tx, Ty, Tz, t) = 1 = G_{N_1}(x, y, z, \frac{t}{M})$.
If $(x, y, z) = (\theta, \theta, \theta)$ and $t = 0$ then $G_{N_2}(Tx, Ty, Tz, t) = 0 = G_{N_1}(x, y, z, \frac{t}{M}$.
Thus we have $G_{N_2}(Tx, Ty, Tz, t) \geq G_{N_1}(x, y, z, \frac{t}{M})\ \forall x, y, z \in X$.
Similarly $G_{M_2}(Tx, Ty, Tz, t) \leq G_{M_1}(x, y, z, \frac{t}{M})\ \forall x, y, z \in X$.
Therefore T is strongly intuitionistic G-fuzzy bounded.

□

Theorem 3.12.
Let $(X, G_{N_1}, G_{M_1}, *_1, \diamond_1)$ and $(Y, G_{N_2}, G_{M_2}, *_2, \diamond_2)$ be two intuitionistic G-fuzzy normed linear spaces and $T : X \to Y$ be a linear operator. If T is sequentially intuitionistic G-fuzzy continuous at a point $x_0 \in X$ then T is sequentially intuitionistic G-fuzzy continuous on X.

Proof. Let x be an arbitrary point of X. Let $\{x_n\}, \{y_n\}, \{z_n\}$ be 3 sequences in X such that

$$\lim_{n\to\infty} G_{N_1}(x_n - x, y_n - x, z_n - x, t) = 1\ \forall t > 0,$$
$$\lim_{n\to\infty} G_{M_1}(x_n - x, y_n - x, z_n - x, t) = 0\ \forall t > 0.$$

So $\lim_{n\to\infty} G_{N_1}((x_n - x + x_0) - x_0, (y_n - x + x_0) - x_0, (z_n - x + x_0) - x_0, t) = 1\ \forall t > 0,$
$\lim_{n\to\infty} G_{M_1}((x_n - x + x_0) - x_0, (y_n - x + x_0) - x_0, (z_n - x + x_0) - x_0, t) = 0\ \forall t > 0.$

Since T is sequentially intuitionistic G-fuzzy continuous at x_0, so we have

$$\lim_{n\to\infty} G_{N_2}(T(x_n - x + x_0) - T(x_0), T(y_n - x + x_0) - T(x_0), T(z_n - x + x_0) - T(x_0), t) = 1,$$
$$\lim_{n\to\infty} G_{M_2}(T(x_n - x + x_0) - T(x_0), T(y_n - x + x_0) - T(x_0), T(z_n - x + x_0) - Tx_0), t) = 0$$

$\forall t > 0$. Thus it follows that

$$\lim_{n\to\infty} G_{N_2}(Tx_n - Tx, Ty_n - Tx, Tz_n - Tx, t) = 1 \; \forall t > 0,$$
$$\lim_{n\to\infty} G_{M_2}(Tx_n - Tx, Ty_n - Tx, Tz_n - Tx, t) = 0 \; \forall t > 0.$$

Therefore T is sequentially intuitionistic G-fuzzy continuous at x and hence T is sequentially intuitionistic G-fuzzy continuous on X. □

Theorem 3.13.
Let $(X, G_{N_1}, G_{M_1}, *_1, \diamond_1)$ and $(Y, G_{N_2}, G_{M_2}, *_2, \diamond_2)$ be two intuitionistic G-fuzzy normed linear spaces and $T : X \to Y$ be a linear operator.

(i) If T is weakly intuitionistic G-fuzzy continuous at a point $x_0 \in X$ then T is weakly intuitionistic G-fuzzy continuous on X.

(ii) T is weakly intuitionistic G-fuzzy continuous on X iff T is weakly intuitionistic G-fuzzy bounded.

Proof. 1. Since T is weakly intuitionistic G-fuzzy continuous at x_0 so for each $\epsilon > 0$ and $\alpha \in (0, 1), \exists \delta = \delta(\alpha, \epsilon) > 0$ such that

$$\begin{aligned} & G_{N_1}(x - x_0, y - x_0, z - x_0, \delta) \geq \alpha, G_{M_1}(x - x_0, y - x_0, z - x_0, \delta) \leq 1 - \alpha \\ \Longrightarrow\ & G_{N_2}(Tx - Tx_0, Ty - Tx_0, Tz - Tx_0, \epsilon) \geq \alpha, \\ & G_{M_2}(Tx - Tx_0, Ty - Tx_0, Tz - Tx_0, \epsilon) \leq 1 - \alpha \; \forall x, y, z \in X. \end{aligned}$$

Let us take $x' \in X$. Replacing x, y, z by $x + x_0 - x', y + x_0 - x', z + x_0 - x'$ respectively we have $G_{N_1}(x+x_0-x'-x_0, y+x_0-x'-x_0, z+x_0-x'-x_0, \delta) \geq \alpha$, $G_{M_1}(x + x_0 - x' - x_0, y + x_0 - x' - x_0, z + x_0 - x' - x_0, \delta) \leq 1 - \alpha$ implies $G_{N_2}(T(x + x_0 - x') - Tx_0, T(y + x_0 - x') - Tx_0, T(z + x_0 - x') - Tx_0, \epsilon) \geq \alpha$, $G_{M_2}(T(x+x_0-x')-Tx_0, T(y+x_0-x')-Tx_0, T(z+x_0-x')-Tx_0, \epsilon) \leq 1-\alpha$.

$$\begin{aligned} & \text{So } G_{N_1}(x - x', y - x', z - x', \delta) \geq \alpha, G_{M_1}(x - x', y - x', z - x', \delta) \leq 1 - \alpha \\ \Longrightarrow\ & G_{N_2}(Tx - Tx', Ty - Tx', Tz - Tx', \epsilon) \geq \alpha, \\ & G_{M_2}(Tx - Tx', Ty - Tx', Tz - Tx', \epsilon) \leq 1 - \alpha. \end{aligned}$$

Hence T is weakly intuitionistic G-fuzzy continuous at x'. Since $x' \in X$ is choosen arbitrarily so T is weakly intuitionistic G-fuzzy continuous on X.

2. First we suppose that T is weakly intuitionistic G-fuzzy bounded. Thus for any $\alpha \in (0, 1), \exists M_\alpha > 0$ such that

$$\begin{aligned} & G_{N_1}(x, y, z, \frac{t}{M_\alpha}) \geq \alpha, G_{M_1}(x, y, z, \frac{t}{M_\alpha}) \leq 1 - \alpha \\ \Longrightarrow\ & G_{N_2}(Tx, Ty, Tz, t) \geq \alpha, G_{M_2}(Tx, Ty, Tz, t) \leq 1 - \alpha \text{ for every } t > 0. \end{aligned}$$

This implies that

$$\begin{aligned} & G_{N_1}(x-\theta, y-\theta, z-\alpha, \delta) \geq \alpha, G_{M_1}(x-\theta, y-\theta, z-\theta, \delta) \leq 1-\alpha \\ \Longrightarrow\ & G_{N_2}(Tx-\theta', Ty-\theta, Tz-\theta', \epsilon) \geq \alpha, \\ & G_{M_2}(Tx-\theta', Ty-\theta', Tz-\theta', \epsilon) \leq 1-\alpha \end{aligned}$$

where $\delta = \frac{\epsilon}{M_\alpha} > 0$. Thus T is weakly intuitionistic G-fuzzy continuous at $x = \theta$. So T is weakly intuitionistic G-fuzzy continuous on X.
Conversely suppose that T is weakly intuitionistic G-fuzzy continuous on X. So T is weakly intuitionistic G-fuzzy continuous at $x = \theta$. Let us take $\epsilon = 1$. Then for a given $\alpha \in (0,1), \exists \delta = \delta(\alpha, 1) > 0$ such that

$$\begin{aligned} & G_{N_1}(x-\theta, y-\theta, z-\theta, \delta) \geq \alpha, G_{M_1}(x-\theta, y-\theta, z-\theta, \delta) \leq 1-\alpha \\ \Longrightarrow\ & G_{N_2}(Tx-T\theta, Ty-T\theta, Tz-T\theta, 1) \geq \alpha, \\ & G_{M_2}(Tx-T\theta, Ty-T\theta, Tz-T\theta, 1) \leq 1-\alpha. \end{aligned}$$

Suppose $(x, y, z) \neq (\theta, \theta, \theta)$ and $t > 0$. Putting $x = \frac{u}{t}, y = \frac{v}{t}, z = \frac{w}{t}$ we have

$$\begin{aligned} & G_{N_1}(\frac{u}{t}, \frac{v}{t}, \frac{w}{t}, \delta) \geq \alpha, G_{M_1}(\frac{u}{t}, \frac{v}{t}, \frac{w}{t}, \delta) \leq 1-\alpha \\ \Longrightarrow\ & G_{N_2}(\frac{Tu}{t}, \frac{Tv}{t}, \frac{Tw}{t}, 1) \geq \alpha, G_{M_2}(\frac{Tu}{t}, \frac{Tv}{t}, \frac{Tw}{t}, 1) \leq 1-\alpha. \\ \text{So } & G_{N_1}(u, v, w, t\delta) \geq \alpha, G_{M_2}(u, v, w, t\delta) \leq 1-\alpha \\ \Longrightarrow\ & G_{N_2}(Tu, Tv, Tw, t) \geq \alpha, G_{M_2}(Tu, Tv, Tw, t) \leq 1-\alpha. \end{aligned}$$

Hence it follows that

$$\begin{aligned} & G_{N_1}(u, v, w, \frac{t}{M_\alpha}) \geq \alpha, G_{M_1}(u, v, w, \frac{t}{M_\alpha}) \leq 1-\alpha \\ \Longrightarrow\ & G_{N_2}(Tu, Tv, Tw, t) \geq \alpha, G_{M_2}(Tu, Tv, Tw, t) \leq 1-\alpha \end{aligned}$$

where $M_\alpha = \frac{1}{\delta(\alpha,1)} > 0$.
If $(x, y, z) \neq (\theta, \theta, \theta)$ and $t = 0$ then $G_{N_1}(x, y, z, \frac{t}{M_\alpha}) = G_{N_2}(Tx, Ty, Tz, t) = 0$ and $G_{M_1}(x, y, z, \frac{t}{M_\alpha}) = G_{M_2}(Tx, Ty, Tz, t) = 1$.
If $(x, y, z) = (\theta, \theta, \theta)$ and $t > 0$ then $G_{N_1}(x, y, z, \frac{t}{M_\alpha}) = G_{M_2}(Tx, Ty, Tz, t) = 1$ and $G_{M_1}(x, y, z, \frac{t}{M_\alpha}) = G_{M_2}(Tx, Ty, Tz, t) = 0$.
If $(x, y, z) = (\theta, \theta, \theta)$ and $t = 0$ then
So $G_{N_1}(x, y, z, \frac{t}{M_\alpha}) = G_{N_2}(Tx, Ty, Tz, \theta) = 0$,
$G_{M_1}(x, y, z, \frac{t}{M_\alpha}) = G_{M_2}(Tx, Ty, Tz, t) = 1$ for $t = 0$.
Therefore it follows that for $\alpha \in (0,1), \exists M_\alpha > 0$ such that

$$\begin{aligned} & G_{N_1}(x, y, z, \frac{t}{M_\alpha}) \geq \alpha, G_{M_1}(x, y, z, \frac{t}{M_\alpha}) \leq 1-\alpha \\ \Longrightarrow\ & G_{N_2}(Tx, Ty, Tz, t) \geq \alpha, G_{M_2}(Tx, Ty, Tz, t) \leq 1-\alpha\ \forall x, y, z \in X, \forall t \in [0, \infty). \end{aligned}$$

Hence T is weakly intuitionistic G-fuzzy bounded.

□

- Let us denote the set of all strongly intuitionistic G-fuzzy bounded linear operators from one intuitionistic G-fuzzy normed linear space $(X, G_{N_1}, G_{M_1}, *_1, \diamond_1)$ to another intuitionistic G-fuzzy normed linear space $(Y, G_{N_2}, G_{M_2}, *_2, \diamond_2)$ by $\mathcal{I}GBF(X, Y)$.

Theorem 3.14. *Let $(X, G_{N_1}, G_{M_1}, \min, \max)$ and $(Y, G_{N_2}, G_{M_2}, \min, \max)$ be two intuitionistic G-fuzzy normed linear spaces. Then $\mathcal{I}GBF(X, Y)$ is a linear space.*

Proof. For $T_1, T_2 \in \mathcal{I}GBF(X, Y)$ and $\lambda \in \mathbb{K}$ we define
$(T_1 + T_2)(x) = T_1(x) + T_2(x), (\lambda T_1)(x) = \lambda T_1(x)$.
Since T_1, T_2 are strongly intuitionistic G-fuzzy bounded, then $\exists L_1, L_2 > 0$ such that

$$G_{N_2}(T_1x, T_1y, T_1z, t) \geq G_{N_1}(x, y, z, \frac{t}{L_1}), G_{M_2}(T_1x, T_1y, T_1z, t) \leq G_{M_1}(x, y, z, \frac{t}{L_1}); \quad (1)$$

$$G_{N_2}(T_2x, T_2y, T_2z, t) \geq G_{N_1}(x, y, z, \frac{t}{L_2}), G_{M_2}(T_2x, T_2y, T_2z, t) \leq G_{M_1}(x, y, z, \frac{t}{L_2}). \quad (2)$$

Now we consider two scalars α, β. Suppose $\alpha \neq 0$ and $\beta = 0$. Then $\forall x \in X$ we have,

$$\begin{aligned}
&G_{N_2}((\alpha T_1 + \beta T_2)x, (\alpha T_1 + \beta T_2)y, (\alpha T_1 + \beta T_2)z, t) \\
=&G_{N_2}(\alpha T_1 x, \alpha T_1 y, \alpha T_1 z, t) \\
=&G_{N_2}(T_1 x, T_1 y, T_1 z, \frac{t}{|\alpha|}) \\
\geq&G_{N_1}(x, y, z, \frac{t}{L_1|\alpha|})(\text{ from } (1)).
\end{aligned}$$

Similarly using (1) we have

$$G_{M_2}((\alpha T_1 + \beta T_2)x, (\alpha T_1 + \beta T_2)y, (\alpha T_1 + \beta T_2)z, t) \leq G_{M_1}(x, y, z, \frac{t}{L_1|\alpha|}).$$

Again if $\alpha = 0$ and $\beta \neq 0$ then by similar manner with the help of (2) we have

$$G_{N_2}((\alpha T_1 + \beta T_2)x, (\alpha T_1 + \beta T_2)y, (\alpha T_1 + \beta T_2)z, t) \geq G_{N_1}(x, y, z, \frac{t}{L_2|\beta|}),$$

$$G_{M_2}((\alpha T_1 + \beta T_2)x, (\alpha T_1 + \beta T_2)y, (\alpha T_1 + \beta T_2)z, t) \leq G_{M_1}(x, y, z, \frac{t}{L_2|\beta|}).$$

Now suppose that $\alpha, \beta \neq 0$. Then we have

$$\begin{aligned}
&G_{N_2}((\alpha T_1 + \beta T_2)x, (\alpha T_1 + \beta T_2)y, (\alpha T_1 + \beta T_2)z, t) \\
=&G_{N_2}(\alpha T_1 x + \beta T_2 x, \alpha T_1 y + \beta T_2 y, \alpha T_1 z + \beta T_2 z, t) \\
\geq& \min\{G_{N_2}(\alpha T_1 x, \alpha T_1 y, \alpha T_1 z, \frac{t}{2}), G_{N_2}(\beta T_2 x, \beta T_2 y, \beta T_2 z, \frac{t}{2})\} \\
=& \min\{G_{N_2}(T_1(\alpha x), T_1(\alpha y), T_1(\alpha z), \frac{t}{2}), G_{N_2}(T_2(\beta x), T_2(\beta y), T_2(\beta z), \frac{t}{2})\} \\
\geq& \min\{G_{N_1}(\alpha x, \alpha y, \alpha z, \frac{t}{2L_1}), G_{N_1}(\beta x, \beta y, \beta z, \frac{t}{2L_2})\} \\
=& \min\{G_{N_1}(x, y, z, \frac{t}{2L_1|\alpha|}), G_{M_1}(x, y, z, \frac{t}{2L_2|\beta|})\}.
\end{aligned}$$

Let us choose $L = \max\{2L_1|\alpha|, 2L_2|\beta|\}$. Thus $L > 0$ and $L \geq 2L_1|\alpha|, L \geq 2L_2|\beta|$. So $\frac{1}{L} \leq \frac{1}{2L_1|\alpha|}, \frac{1}{L} \leq \frac{1}{2L_2|\beta|}$. Then

$$G_{N_1}(x, y, z, \frac{t}{2L_1|\alpha|}) \geq G_{N_1}(x, y, z, \frac{t}{L}), G_{N_1}(x, y, z, \frac{t}{2L_2|\beta|}) \geq G_{N_1}(x, y, z, \frac{t}{L}).$$

Thus we have

$$\min\{G_{N_1}(x, y, z, \frac{t}{2L_1|\alpha|}), G_{N_1}(x, y, z, \frac{t}{2L_2|\beta|})\} \geq G_{N_1}(x, y, z, \frac{t}{L}).$$

Hence from above calculations it follows that

$$G_{N_2}((\alpha T_1 + \beta T_2)x, (\alpha T_1 + \beta T_2)y, (\alpha T_1 + \beta T_2)z, t) \geq G_{N_1}(x, y, z, \frac{t}{L}).$$

On the other hand

$$\begin{aligned}
&G_{M_2}((\alpha T_1 + \beta T_2)x, (\alpha T_1 + \beta T_2)y, (\alpha T_1 + \beta T_2)z, t) \\
=&G_{M_2}(T_1(\alpha x) + T_2(\beta x), T_1(\alpha y) + T_2(\beta y), T_1(\alpha z) + T_2(\beta z), t) \\
\leq& \max\{G_{M_2}(T_1(\alpha x), T_1(\alpha y), T_1(\alpha z), \frac{t}{2}), G_{M_2}(T_2(\beta x), T_2(\beta y), T_2(\beta z), \frac{t}{2})\} \\
\leq& \max\{G_{M_1}(\alpha x, \alpha y, \alpha z, \frac{t}{2L_1}), G_{M_1}(\beta x, \beta y, \beta z, \frac{t}{2L_2})\} \\
=& \max\{G_{M_1}(x, y, z, \frac{t}{2L_1|\alpha|}), G_{N_1}(x, y, z, \frac{t}{2L_2|\beta|})\}.
\end{aligned}$$

We have

$$G_{M_1}(x, y, z, \frac{t}{L}) \geq G_{M_1}(x, y, z, \frac{t}{2L_1|\alpha|}), G_{M_1}(x, y, z, \frac{t}{L}) \geq G_{M_1}(x, y, z, \frac{t}{2L_2|\beta|}).$$

Then

$$G_{M_1}(x,y,z,\frac{t}{L}) \geq \max\{G_{M_1}(x,y,z,\frac{t}{2L_1|\alpha|}), G_{M_1}(x,y,z,\frac{t}{2L_2|\beta|})\}$$
$$\implies \max\{G_{M_1}(x,y,z,\frac{t}{2L_1|\alpha|}), G_{M_1}(x,y,z,\frac{t}{2L_2|\beta|})\} \leq G_{M_1}(x,y,z,\frac{t}{L}).$$

Thus above relations implies that

$$G_{N_2}((\alpha T_1+\beta T_2)x,(\alpha T_1+\beta T_2)y,(\alpha T_1+\beta T_2)z,t) \leq G_{N_1}(x,y,z,\frac{t}{L}).$$

Therefore we have $\alpha T_1+\beta T_2 \in \mathcal{I}GBF(X,Y)$.
Hence $\mathcal{I}GBF(X,Y)$ is a linear space. □

4. INTUITIONISTIC G-FUZZY OPERATOR NORM

In this Section intuitionistic G-fuzzy operator norm is defined in the space of bounded linear operators from an intuitionistic G-fuzzy normed linnear space to another intuitionistic G-fuzzy normed linear space.

Theorem 4.1. *Let $(X,G_N,G_M,*,\diamond)$ be an intuitionistic G-fuzzy normed linear space such that $*$ is lower-semi-continuous at $(1,1)$ and $\diamond$ is upper-semi-continuous at $(0,0)$. We define $\|\cdot,\cdot,\cdot\|^1_\alpha, \|\cdot,\cdot,\cdot\|^2_\alpha : X^3 \to \mathbb{R}_{\geq 0}$ as follows:*

$$\|x,y,z\|^1_\alpha = \wedge\{t>0 : G_N(x,y,z,t) \geq 1-\alpha\}, \|x,y,z\|^2_\alpha = \wedge\{t>0 : G_M(x,y,z,t) \leq \alpha\}$$

for every $\alpha \in (0,1)$ and for all $x,y,z \in X$. Let us denote $Q_1 = \{\|\cdot,\cdot,\cdot\|^1_\alpha : \alpha \in (0,1)\}$ and $Q_2 = \{\|\cdot,\cdot,\cdot\|^2_\alpha : \alpha \in (0,1)\}$. Then

(1) (X,Q_1) is a generating space of quasi-G-norm family.

(2) (X,Q_2) is a generating space of quasi-G-norm family.

Proof. (1) The proof of (1) is similar to that of Theorem 2.18.

(2) Now we prove (2).

(i) Let us take $x,y,z \in X$.

$$\|x,y,z\|^2_\alpha = 0 \ \forall \alpha \in (0,1)$$
$$\iff \wedge\{t>0 : G_M(x,y,z,t) \leq \alpha\} = 0 \ \forall \alpha \in (0,1)$$
$$\iff G_M(x,y,z,\epsilon) \leq \alpha \text{ for every } \epsilon > 0; \forall \alpha \in (0,1)$$
$$\iff G_M(x,y,z,t) = 0 \ \forall t > 0$$
$$\iff x = y = z = \theta.$$

(ii) Let $c \in \mathbb{K}$. For $c = 0$ we have $\|cx, cy, cz\|_\alpha^2 = 0 = |c|\|x, y, z\|_\alpha^2$.
Suppose $c \neq 0$. Then

$$\begin{aligned}\|cx, cy, cz\|_\alpha^2 &= \wedge \{t > 0 : G_M(cx, cy, cz, t) \leq \alpha\} \\ &= \wedge \{t > 0 : G_M(x, y, z, \frac{t}{|c|} \leq \alpha)\} \\ &= |c| \wedge \{t > 0 : G_M(x, y, z, t) \leq \alpha\} \\ &= |c|\|x, y, z\|_\alpha^2.\end{aligned}$$

(iii) It is clear that $\|x, y, z\|_\alpha^2 = \|p(x, y, z)\|_\alpha^2$ where p is a permutation of x, y, z.

(iv) Suppose $\alpha \in (0, 1)$. Then by the upper-semi-continuity of $\diamond$ at $(0, 0)$ there is $\beta \in (0, 1)$ such that $\beta \diamond \beta < \alpha$.
If $G_M(x, y, z, s) < \beta$ and $G_M(x', y', z', t) < \beta$ then we have,

$$\begin{aligned}G_M(x + x', y + y', z + z', s + t) &\leq G_M(x, y, z, s) \diamond G_M(x', y', z', t) \\ &\leq \beta \diamond \beta \\ &< \alpha.\end{aligned}$$

Thus $G_M(x + x', y + y', z + z', s + t) < \alpha$.
Now

$$\begin{aligned}&\wedge \{s > 0 : G_M(x, y, z, s) \leq \beta\} + \wedge\{t > 0 : G_M(x', y', z', t) \leq \beta\} \\ &= \wedge \{s + t > 0 : G_M(x + x', y + y', z + z', s + t) < \alpha\} \\ &\geq \wedge \{s + t > 0 : G_M(x + x', y + y', z + z', s + t) \leq \alpha\}.\end{aligned}$$

Then we have,

$$\|x + x', y + y', z + z'\|_\alpha^2 \leq \|x, y, z\|_\beta^2 + \|x', y', z'\|_\beta^2.$$

(v) Let us take $x, y, z \in X$. We know that $G_M(x + y, \theta, z, t) \leq G_M(x, y, z, t)$. So if $G_M(x, y, z, t) \leq \alpha$ then $G_M(x + y, \theta, z, t) \leq \alpha$. This implies that

$$\begin{aligned}&\{t > 0 : G_M(x, y, z, t) \leq \alpha\} \subset \{t > 0 : G_M(x + y, \theta, z, t) \leq \alpha\} \\ \implies &\wedge \{t > 0 : G_M(x, y, z, t) \leq \alpha\} \geq \wedge\{t > 0 : G_M(x_y, \theta, z, t) \leq \alpha\} \\ \implies &\|x, y, z\|_\alpha^2 \geq \|x + y, \theta, z\|_\alpha^2.\end{aligned}$$

(vi) Now suppose $\alpha_1, \alpha_2 \in (0, 1)$ and $\alpha_1 < \alpha_2$. If $G_M(x, y, z, t) \leq \alpha_1$ then $G_M(x, y, z, t) < \alpha_2$. Thus we have,

$$\begin{aligned}\wedge\{t > 0 : G_M(x, y, z, t) \leq \alpha_1\} &\geq \wedge \{t > 0 : G_M(x, y, z, t) < \alpha_2\} \\ &\geq \wedge \{t > 0 : G_M(x, y, z, t) \leq \alpha_2\}.\end{aligned}$$

So $\|x, y, z\|_{\alpha_1} \geq \|x, y, z\|_{\alpha_2}$.

Hence Q_2 is a quasi-G-norm family.
Consequently the theorem holds. □

Proposition 4.2. *Let $(X, G_N, G_M, *, \diamond)$ be an intuitionistic G-fuzzy normed linear space such that $G_N(x, y, z, \cdot), G_M(x, y, z, \cdot) : [0, \infty) \to [0, 1]$ are continuous at $t = 0$ for every $x, y, z \in X$ with $(x, y, z) \neq (\theta, \theta, \theta)$.*
Let us define $\|x, y, z\|^1_\alpha = \wedge\{t > 0 : G_N(x, y, z, t) \geq 1 - \alpha\}$ and
$\|x, y, z\|^2_\alpha = \wedge\{t > 0 : G_M(x, y, z, t) \leq \alpha\}$ for every $\alpha \in (0, 1)$.
Then $\|x, y, z\|^1_\alpha, \|x, y, z\|^2_\alpha > 0$ for every $\alpha \in (0, 1)$ and for every $x, y, z \in X$ with $(x, y, z) \neq (\theta, \theta, \theta)$.

Proof. If possible suppose $\exists \alpha_0, \beta_0 \in (0, 1)$ and $x_0, y_0, z_0, x'_0, y'_0, z'_0 \in X$ with $(x_0, y_0, z_0) \neq (\theta, \theta, \theta)$ and $(x'_0, y'_0, z'_0) \neq (\theta, \theta, \theta)$ such that $\|x_0, y_0, z_0\|_{\alpha_0} = 0$ and $\|x'_0, y'_0, z'_0\|_{\beta_0} = 0$.
Since $\|x_0, y_0, z_0\|_{\alpha_0} = 0$ so from definition it follows that $G_N(x_0, y_0, z_0, t) \geq 1 - \alpha_0$ for every $t > 0$. We have $G_N(x_0, y_0, z_0, 0) = 0$. Since $G_N(x_0, y_0, z_0, \cdot)$ is continuous at $t = 0$ so for any $\epsilon > 0$, $\exists \delta > 0$ such that $G_N(x_0, y_0, z_0, t) < \epsilon$ for every $0 < t < \delta$.
Let us take $\epsilon = \frac{1-\alpha_0}{2}$. Then $\exists \delta > 0$ such that $G_N(x_0, y_0, z_0, t) < \epsilon$ for every $0 < t < \delta$.
We choose $0 < t_0 < \delta$. Then $G_N(x_0, y_0, z_0, t_0) < \frac{1-\alpha_0}{2}$.
But already we have, $G_N(x_0, y_0, z_0, t_0) \geq 1 - \alpha_0$.
Hence we arrive at a contradiction.
Therefore $\|x, y, z\|^1_\alpha > 0 \; \forall \alpha \in (0, 1)$ for every $x, y, z \in X$ with $(x, y, z) \neq (\theta, \theta, \theta)$.
Again we have assumed that $\|x'_0, y'_0, z'_0\|_{\beta_0} = 0$. So $G_M(x'_0, y'_0, z'_0, t) \leq \beta_0 \; \forall t > 0$.
Now $G_M(x'_0, y'_0, z'_0, 0) = 1$. By the continuity of $G_M(x'_0, y'_0, z'_0, \cdot)$ at $t = 0$ it follows that for any $\epsilon > 0 \; \exists \delta > 0$ auch that $G_M(x'_0, y'_0, z'_0, t) > 1 - \epsilon$ for every $0 < t < \delta$.
Let us take $\epsilon' = 1 - \beta_0$. Then $\epsilon' > 0$. So $\exists \delta' > 0$ such that $G_M((x'_0, y'_0, z'_0, t) > 1 - \epsilon$ for any $0 < t < \delta'$. Let us take $0 < t'_0 < \delta'$. So $G_M(x'_0, y'_0, z'_0, t'_0) > \beta_0$. This leads to a contradiction.
Hence our assumption is wrong. As a consequence $\|x, y, z\|^2_\alpha > 0 \; \forall \alpha \in (0, 1)$ and for every $(x, y, z) \neq (\theta, \theta, \theta)$. □

Theorem 4.3. *Let $(X, G_N, G_M, \min, \max)$ be an intuitionistic G-fuzzy normed linear space. Let us define $\|x, y, z\|^1_\alpha = \wedge\{t > 0 : G_N(x, y, z, t) \geq 1 - \alpha\}$ for $\alpha \in (0, 1)$ and denote $Q_1 = \{\|\cdot, \cdot, \cdot\|^1_\alpha : \alpha \in (0, 1)\}$. Also we define $\|x, y, z\|^2_\alpha = \wedge\{t > 0 : G_M(x, y, z, t) \leq \alpha\}$ for $\alpha \in (0, 1)$ and $Q_2 = \{\|\cdot, \cdot, \cdot\|^2_\alpha : \alpha \in (0, 1)\}$. Then (X, Q_1) and (X, Q_2) are generating spaces of semi-G-norm family.*

Proof. The proof of (X, Q_1) follows from that of Theorem 2.19. Thus we may proceed for the proof of (X, Q_2).

1. Suppose $x, y, z \in X$ and $\alpha \in (0, 1)$.

$$\begin{aligned}
&\|x, y, z\|_\alpha^2 = 0 \ \forall \alpha \in (0, 1) \\
\iff &\wedge \{t > 0 : G_M(x, y, z, t) \leq \alpha\} = 0 \ \forall \alpha \in (0, 1) \\
\iff &G_M(x, y, z, \epsilon) \leq \alpha \text{ for every } \epsilon > 0; \forall \alpha \in (0, 1) \\
\iff &G_M(x, y, z, \epsilon) = 0 \text{ for every } \epsilon > 0 \\
\iff &G_M(x, y, z, t) = 0 \ \forall t > 0 \\
\iff &x = y = z = \theta.
\end{aligned}$$

2. Suppose $c \in \mathbb{K}$ and $x, y, z \in X$. For $c = 0$, $\|cx, cy, cz\|_\alpha^2 = 0 = |c|\|x, y, z\|_\alpha^2$ for every $\alpha \in (0, 1)$. Suppose $c \neq 0$. Then

$$\begin{aligned}
\|cx, cy, cz\|_\alpha^2 &= \wedge \{t > 0 : G_N(cx, cy, cz, t) \leq \alpha\} \\
&= \wedge \{t > 0 : G_M(x, y, z, \frac{t}{|c|}) \leq \alpha\} \\
&= |c| \wedge \{t > 0 : G_M(x, y, z, t) \leq \alpha\} \\
&= |c|\|x, y, z\|_\alpha^1.
\end{aligned}$$

3. $\|x, y, z\|_\alpha^2 = \|p(x, y, z)\|_\alpha^2$ for every permutation p of x, y, z.

4. Suppose $\alpha \in (0, 1)$ and $x, x', y, y', z, z' \in X$. If $G_M(x, y, z, t_1) \leq \alpha$ and $G_M(x', y', z', t_2) \leq \alpha$ then we have

$$\begin{aligned}
&G_M(x + x', y + y', z + z', t_1 + t_2) \leq \\
&\max\{G_M(x, y, z, t_1), G_M(x, y, z, t_2)\} \leq \alpha \\
\implies &G_M(x + x', y + y', z + z', t_1 + t_2) \leq \alpha.
\end{aligned}$$

   This implies that

$$\begin{aligned}
&\wedge \{t > 0 : G_M(x, y, z, t) \leq \alpha\} + \wedge \{s > 0 : G_M(x', y', z', s) \leq \alpha\} \\
= &\wedge \{t + s > 0 : G_M(x, y, z, t) \leq \alpha, G_M(x', y', z', s) \leq \alpha\} \\
\geq &\wedge \{t + s > 0 : G_M(x + x', y + y', z + z', t + s) \leq \alpha\}.
\end{aligned}$$

   So $\|x + x', y + y', z + z'\|_\alpha^2 \leq \|x, y, z\|_\alpha^2 + \|x', y', z'\|_\alpha^2$.

5. We have $G_M(x, y, z, t) \geq G_M(x + y, \theta, z, t) \ \forall x, y, z \in X; \ \forall t > 0$.
   If $G_M(x, y, z, t) \leq \alpha$ then $G_M(x + y, 0_X, z, t) \leq \alpha$ for $\alpha \in (0, 1)$.
   So $\{t > 0 : G_M(x, y, z, t) \geq 1 - \alpha\} \subseteq G_M(x + y, \theta, z, t) \geq 1 - \alpha\}$.
   This implies that

$$\wedge \{t > 0 : G_M(x, y, z, t) \leq \alpha\} \geq \wedge \{t > 0 : G_M(x + y, 0_X, z, t) \leq \alpha\}.$$

   Hence $\|x, y, z\|_\alpha \geq \|x + y, \theta, z\|_\alpha \ \forall \alpha \in (0, 1)$.

6. Let us take $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 < \alpha_2$. Thus we have, if $G_M(x,y,z,t) \leq \alpha_1$ then $G_M(x,y,z,t) \leq \alpha_2$.
   Hence $\wedge\{t > 0 : G_M(x,y,z,t) \leq \alpha_1\} \geq \wedge\{t > 0 : G_M(x,y,z,t) \leq \alpha_2\}$
   i.e. $\|x,y,z\|_{\alpha_1} \geq \|x,y,z\|_{\alpha_2}$.
   So $\|x,y,z\|_\alpha$ is non-increasing for $\alpha \in (0,1)$.

Hence (X, Q_2) is a generating space of semi-G-norm family. □

Theorem 4.4. *Let* $(X, G_{N_1}, G_{M_1}, \min, \max)$ *and* $(Y, G_{N_2}, G_{M_2}, \min, \max)$ *be two intuitionistic fuzzy normed linear spaces such that* $G_{N_1}(x,y,z,\cdot), G_{N_2}(x,y,z,\cdot) : [0,\infty) \to [0,1]$ *is continuous at* $t = 0$ *for* $(x,y,z) \neq (\theta,\theta,\theta)$. *For* $T_1, T_2, T_3 \in \mathcal{I}GBF(X,Y)$ *we define*

$$\|T_1,T_2,T_3\|^1_{1-\alpha} = \bigvee_{\substack{x,y,z\in X\\(x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_ix,T_iy,T_iz\|^{N_2}_{1-\alpha}}{\|x,y,z\|^{M_1}_{\alpha}}$$

and

$$\|T_1,T_2,T_3\|^2_{1-\alpha} = \bigwedge_{\substack{x,y,z\in X\\(x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_ix,T_iy,T_iz\|^{M_2}_{1-\alpha}}{\|x,y,z\|^{N_1}_{\alpha}}$$

where $\|x,y,z\|^{N_1}_\alpha = \wedge\{t > 0 : G_{N_1}(x,y,z) \geq 1-\alpha\}, \|x,y,z\|^{M_1}_\alpha = \wedge\{t > 0 : G_{M_1}(x,y,z,t \leq \alpha\}, \|Tx,Ty,Tz\|^{N_2}_\alpha = \wedge\{t > 0 : G_{N_2}(Tx,Ty,Tz) \geq 1-\alpha\}, \|Tx,Ty,Tz\|^{M_2}_\alpha = \wedge\{t > 0 : G_{M_2}(Tx,Ty,Tz,t) \leq \alpha\}$ *for every* $\alpha \in (0,1)$. *Let us denote* $\mathbb{Q}^1 = \{\|\cdot,\cdot,\cdot\|^1_\alpha : \alpha \in (0,1)\}, \mathbb{Q}^2 = \{\|\cdot,\cdot,\cdot\|^2_\alpha : \alpha \in (0,1)\}$. *Then* $\mathbb{Q}^1, \mathbb{Q}^2$ *are two semi-G-norm family on* $\mathcal{I}GBF(X,Y)$.

Proof. 1. Let us take $T_1, T_2, T_3 \in \mathcal{I}GBF(X,Y)$.

$$\begin{aligned}
&\text{Now } \|T_1,T_2,T_3\|^1_\alpha = 0\ \forall\alpha \in (0,1)\\
&\iff \bigvee_{\substack{x,y,z\in X\\(x,y,z)\neq(0_X,0_X,0_X)}} \sum_{i=1}^{3} \frac{\|T_ix,T_iy,T_iz\|^{N_2}_{\alpha}}{\|x,y,z\|^{M_1}_{1-\alpha}} = 0\ \forall\alpha \in (0,1)\\
&\iff \frac{\|T_ix,T_iy,T_iz\|^{N_2}_{\alpha}}{\|x,y,z\|^{M_1}_{1-\alpha}} = 0\ \forall\alpha \in (0,1)\ (i=1,2,3)\\
&\iff \|T_ix,T_iy,T_iz\|^{N_2}_{\alpha} = 0\ \forall x,y,z \in X (i=1,2,3)\\
&\iff T_ix = T_iy = T_iz = \theta'\ \forall x,y,z \in X (i=1,2,3)\\
&\iff T_i = \bigcirc (i=1,2,3).
\end{aligned}$$

2. It is obvious that $\|cT_1,cT_2,cT_3\|^1_\alpha = |c|\|T_1,T_2,T_3\|^1_\alpha\ \forall c \in \mathbb{K}$.

3. $\|T_1, T_2, T_3\|_\alpha^1$ is invariant under any permutation of T_1, T_2, T_3 for every $\alpha \in (0,1)$.

4. Suppose $T_1, T_1', T_2, T_2', T_3, T_3' \in \mathcal{I}GBF(X,Y)$. Then

$$\begin{aligned}
&\|T_1 + T_1', T_2 + T_2', T_3 + T_3'\|_\alpha^1 \\
&= \bigvee_{\substack{x,y,z \in X \\ (x,y,z) \neq (\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|(T_i + T_i')x, (T_i + T_i')y, (T_i + T_i')z\|_\alpha^{N_2}}{\|x,y,z\|_{1-\alpha}^{M_1}} \\
&= \bigvee_{\substack{x,y,z \in X \\ (x,y,z) \neq (\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_i x + T_i' x, T_i y + T_i' y, T_i z + T_i' z\|_\alpha^{N_2}}{\|x,y,z\|_{1-\alpha}^{M_1}}.
\end{aligned}$$

Since we have

$$\|T_i x + T_i' x, T_i y + T_i' y, T_i z + T_i' z\|_\alpha^{N_2} \leq \|T_i x, T_i y, T_i z\|_\alpha^{N_2} + \|T_i' x, T_i' y, T_i' z\|_\alpha^{N_2}$$

so it follows that

$$\begin{aligned}
&\|T_1 + T_1', T_2 + T_2', T_3 + T_3'\|_\alpha^1 \\
&\leq \bigvee_{\substack{x,y,z \in X \\ (x,y,z) \neq (\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_i x, T_i y, T_i z\|_\alpha^{N_2}}{\|x,y,z\|_{1-\alpha}^{M_1}} + \bigvee_{\substack{x,y,z \in X \\ (x,y,z) \neq (\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_i' x, T_i' y, T_i' z\|_\alpha^{N_2}}{\|x,y,z\|_{1-\alpha}^{M_1}} \\
&= \|T_1, T_2, T_3\|_\alpha^1 + \|T_1', T_2', T_3'\|_\alpha^1.
\end{aligned}$$

Hence we have,

$$\|T_1 + T_1', T_2 + T_2', T_3 + T_3'\|_\alpha^1 \leq \|T_1, T_2, T_3\|_\alpha^1 + \|T_1', T_2', T_3'\|_\alpha^1 \ \forall \alpha \in (0,1).$$

5. By definition,

$$\begin{aligned}
&\|T_1 + T_2, \bigcirc, T_3\|_\alpha^1 \\
&= \bigvee_{\substack{x,y,z \in X \\ (x,y,z) \neq (\theta,\theta,\theta)}} \frac{\|(T_1 + T_2)x, (T_1 + T_2)y, (T_1 + T_2)z\|_\alpha^{N_2}}{\|x,y,z\|_{1-\alpha}^{M_1}} + \bigvee_{\substack{x,y,z \in X \\ (x,y,z) \neq (\theta,\theta,\theta)}} \frac{\|T_3 x, T_3 y, T_3 z\|_\alpha^{N_2}}{\|x,y,z\|_{1-\alpha}^{M_1}} \\
&= \bigvee_{\substack{x,y,z \in X \\ (x,y,z) \neq (\theta,\theta,\theta)}} \frac{\|T_1 x + T_2 x, T_1 y + T_2 y, T_1 z + T_2 z\|_\alpha^{N_2}}{\|x,y,z\|_{1-\alpha}^{M_1}} + \bigvee_{\substack{x,y,z \in X \\ (x,y,z) \neq (\theta,\theta,\theta)}} \frac{\|T_3 x, T_3 y, T_3 z\|_\alpha^{N_2}}{\|x,y,z\|_{1-\alpha}^{M_1}} \\
&\leq \bigvee_{\substack{x,y,z \in X \\ (x,y,z) \neq (\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_i x, T_i y, T_i z\|_\alpha^{N_2}}{\|x,y,z\|_{1-\alpha}^{M_1}} \\
&= \|T_1, T_2, T_3\|_\alpha^1.
\end{aligned}$$

So $\|T_1 + T_2, \bigcirc, T_3\|_\alpha^1 \leq \|T_1, T_2, T_3\|_\alpha^1$ for every $\alpha \in (0,1)$.

6. Let us take $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 < \alpha_2$. Then we have,

$$\begin{aligned}
&\|x,y,z\|^{M_1}_{1-\alpha_1} \leq \|x,y,z\|^{M_1}_{1-\alpha_2} \\
\Longrightarrow\ &\frac{1}{\|x,y,z\|^{M_1}_{1-\alpha_1}} \geq \frac{1}{\|x,y,z\|^{M_1}_{1-\alpha_2}}. \\
&\text{Also } \|T_ix,T_iy,T_iz\|^{N_2}_{\alpha_1} \geq \|T_ix,T_iy,T_iz\|^{N_2}_{\alpha_2} (i=1,2,3).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{\|T_ix,T_iy,T_iz\|^{N_2}_{\alpha_1}}{\|x,y,z\|^{M_1}_{1-\alpha_1}} \geq \frac{\|T_ix,T_iy,T_iz\|^{N_2}_{\alpha_2}}{\|x,y,z\|^{M_1}_{1-\alpha_2}} \\
\Longrightarrow\ &\sum_{i=1}^{3} \frac{\|T_ix,T_iy,T_iz\|^{N_2}_{\alpha_1}}{\|x,y,z\|^{M_1}_{1-\alpha_1}} \geq \sum_{i=1}^{3} \frac{\|T_ix,T_iy,T_iz\|^{N_2}_{\alpha_2}}{\|x,y,z\|^{M_1}_{1-\alpha_2}} \\
\Longrightarrow\ &\bigvee_{\substack{x,y,z\in X \\ (x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_ix,T_iy,T_iz\|^{N_2}_{\alpha_1}}{\|x,y,z\|^{M_1}_{1-\alpha_1}} \geq \bigvee_{\substack{x,y,z\in X \\ (x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_ix,T_iy,T_iz\|^{N_2}_{\alpha_2}}{\|x,y,z\|^{M_1}_{1-\alpha_2}} \\
\Longrightarrow\ &\|T_1,T_2,T_3\|^1_{\alpha_1} \geq \|T_1,T_2,T_3\|^1_{\alpha_2}.
\end{aligned}$$

Hence $\|T_1,T_2,T_3\|^1_\alpha$ is non-increasing for $\alpha \in (0,1)$.

1. Let us take $T_1,T_2,T_3 \in \mathcal{I}GBF(X,Y)$.

$$\begin{aligned}
&\text{Now } \|T_1,T_2,T_3\|^2_\alpha = 0\ \forall \alpha \in (0,1) \\
\Longleftrightarrow\ &\bigvee_{\substack{x,y,z\in X \\ (x,y,z)\neq(0_X,0_X,0_X)}} \sum_{i=1}^{3} \frac{\|T_ix,T_iy,T_iz\|^{M_2}_\alpha}{\|x,y,z\|^{N_1}_{1-\alpha}} = 0\ \forall \alpha \in (0,1) \\
\Longleftrightarrow\ &\frac{\|T_ix,T_iy,T_iz\|^2_\alpha}{\|x,y,z\|^{M_2}_{1-\alpha}} = 0\ \forall \alpha \in (0,1)\ (i=1,2,3) \\
\Longleftrightarrow\ &\|T_ix,T_iy,T_iz\|^{M_2}_\alpha = 0\ \forall x,y,z \in X (i=1,2,3) \\
\Longleftrightarrow\ &T_ix = T_iy = T_iz = \theta'\ \forall x,y,z \in X (i=1,2,3) \\
\Longleftrightarrow\ &T_i = \bigcirc (i=1,2,3).
\end{aligned}$$

2. It is obvious that $\|cT_1,cT_2,cT_3\|^2_\alpha = |c|\|T_1,T_2,T_3\|^2_\alpha\ \forall c \in \mathbb{K}$.

3. $\|T_1,T_2,T_3\|^2_\alpha$ is invariant under any permutation of T_1,T_2,T_3 for every $\alpha \in (0,1)$.

4. Suppose $T_1, T_1', T_2, T_2', T_3, T_3' \in \mathcal{I}GBF(X,Y)$. Then

$$\begin{aligned}
&\|T_1+T_1', T_2+T_2', T_3+T_3'\|_\alpha^2 \\
&= \bigvee_{\substack{x,y,z\in X\\(x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|(T_i+T_i')x,(T_i+T_i')y,(T_i+T_i')z\|_\alpha^{M_2}}{\|x,y,z\|_{1-\alpha}^{N_1}} \\
&= \bigvee_{\substack{x,y,z\in X\\(x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_ix+T_i'x,T_iy+T_i'y,T_iz+T_i'z\|_\alpha^{M_2}}{\|x,y,z\|_{1-\alpha}^{N_1}}.
\end{aligned}$$

Since we have

$$\|T_ix+T_i'x,T_iy+T_i'y,T_iz+T_i'z\|_\alpha^{M_2} \leq \|T_ix,T_iy,T_iz\|_\alpha^{M_2} + \|T_i'x,T_i'y,T_i'z\|_\alpha^{M_2}$$

so it follows that

$$\begin{aligned}
&\|T_1+T_1', T_2+T_2', T_3+T_3'\|_\alpha^2 \\
&\leq \bigvee_{\substack{x,y,z\in X\\(x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_ix,T_iy,T_iz\|_\alpha^{M_2}}{\|x,y,z\|_{1-\alpha}^{N_1}} + \bigvee_{\substack{x,y,z\in X\\(x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_i'x,T_i'y,T_i'z\|_\alpha^{M_2}}{\|x,y,z\|_{1-\alpha}^{N_1}} \\
&= \|T_1,T_2,T_3\|_\alpha^2 + \|T_1',T_2',T_3'\|_\alpha^2.
\end{aligned}$$

Hence we have,

$$\|T_1+T_1', T_2+T_2', T_3+T_3'\|_\alpha^2 \leq \|T_1,T_2,T_3\|_\alpha^2 + \|T_1',T_2',T_3'\|_\alpha^2 \ \forall \alpha \in (0,1).$$

5. By definition,

$$\begin{aligned}
&\|T_1+T_2, \bigcirc, T_3\|_\alpha^2 \\
&= \bigvee_{\substack{x,y,z\in X\\(x,y,z)\neq(\theta,\theta,\theta)}} \frac{\|(T_1+T_2)x,(T_1+T_2)y,(T_1+T_2)z\|_\alpha^{M_2}}{\|x,y,z\|_{1-\alpha}^{N_1}} + \bigvee_{\substack{x,y,z\in X\\(x,y,z)\neq(\theta,\theta,\theta)}} \frac{\|T_3x,T_3y,T_3z\|_\alpha^{M_2}}{\|x,y,z\|_{1-\alpha}^{N_1}} \\
&= \bigvee_{\substack{x,y,z\in X\\(x,y,z)\neq(\theta,\theta,\theta)}} \frac{\|T_1x+T_2x,T_1y+T_2y,T_1z+T_2z\|_\alpha^{M_2}}{\|x,y,z\|_{1-\alpha}^{N_1}} + \bigvee_{\substack{x,y,z\in X\\(x,y,z)\neq(\theta,\theta,\theta)}} \frac{\|T_3x,T_3y,T_3z\|_\alpha^{M_2}}{\|x,y,z\|_{1-\alpha}^{N_1}} \\
&\leq \bigvee_{\substack{x,y,z\in X\\(x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_ix,T_iy,T_iz\|_\alpha^{M_2}}{\|x,y,z\|_{1-\alpha}^{N_1}} \\
&= \|T_1,T_2,T_3\|_\alpha^2.
\end{aligned}$$

So $\|T_1+T_2, \bigcirc, T_3\|_\alpha^2 \leq \|T_1,T_2,T_3\|_\alpha^2$ for every $\alpha \in (0,1)$.

6. Let us take $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 < \alpha_2$. Then we have,

$$\|x,y,z\|_{1-\alpha_1}^{N_1} \leq \|x,y,z\|_{1-\alpha_2}^{N_1}$$
$$\Longrightarrow \frac{1}{\|x,y,z\|_{1-\alpha_1}^{N_1}} \geq \frac{1}{\|x,y,z\|_{1-\alpha_2}^{N_1}}.$$
Also $\|T_ix,T_iy,T_iz\|_{\alpha_1}^{M_2} \geq \|T_ix,T_iy,T_iz\|_{\alpha_2}^{M_2} (i=1,2,3)$.

This implies that

$$\frac{\|T_ix,T_iy,T_iz\|_{\alpha_1}^{M_2}}{\|x,y,z\|_{1-\alpha_1}^{N_1}} \geq \frac{\|T_ix,T_iy,T_iz\|_{\alpha_2}^{M_2}}{\|x,y,z\|_{1-\alpha_2}^{N_1}}$$
$$\Longrightarrow \sum_{i=1}^{3} \frac{\|T_ix,T_iy,T_iz\|_{\alpha_1}^{M_2}}{\|x,y,z\|_{1-\alpha_1}^{N_1}} \geq \sum_{i=1}^{3} \frac{\|T_ix,T_iy,T_iz\|_{\alpha_2}^{M_2}}{\|x,y,z\|_{1-\alpha_2}^{N_1}}$$
$$\Longrightarrow \bigvee_{\substack{x,y,z\in X \\ (x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_ix,T_iy,T_iz\|_{\alpha_1}^{M_2}}{\|x,y,z\|_{1-\alpha_1}^{N_1}} \geq \bigvee_{\substack{x,y,z\in X \\ (x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_ix,T_iy,T_iz\|_{\alpha_2}^{M_2}}{\|x,y,z\|_{1-\alpha_2}^{N_1}}$$
$$\Longrightarrow \|T_1,T_2,T_3\|_{\alpha_1}^{2} \geq \|T_1,T_2,T_3\|_{\alpha_2}^{2}.$$

Hence $\|T_1,T_2,T_3\|_{\alpha}^{1}$ is non-increasing for $\alpha \in (0,1)$.

Hence $(\mathcal{I}GBF(X,Y),\mathbb{Q}^1)$ and $(\mathcal{I}GBF(X,Y),\mathbb{Q}^2)$ are two generating spaces of semi-G-norm family on $\mathcal{I}GBF(X,Y)$. □

Proposition 4.5. *Let $(X,G_{N_1},G_{M_1},\min,\max)$ and $(Y,G_{N_2},G_{M_2},\min,\max)$ be two intuitionistic G-fuzzy normed linear spaces. Suppose $T_1,T_2,T_3 \in \mathcal{I}GBF(X,Y)$. Let us define $\|T_1,T_2,T_3\|_{\alpha}^{1}, \|T_1,T_2,T_3\|_{\alpha}^{2} : \{\mathcal{I}GBF(X,Y)\}^3 \to \mathbb{R}$ as follows:*

$$\|T_1,T_2,T_3\|_{\alpha}^{1} = \bigvee_{\substack{x,y,z\in X \\ (x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_ix,T_iy,T_iz\|_{\alpha}^{N_2}}{\|x,y,z\|_{1-\alpha}^{M_1}},$$
$$\|T_1,T_2,T_3\|_{\alpha}^{2} = \bigvee_{\substack{x,y,z\in X \\ (x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_ix,T_iy,T_iz\|_{\alpha}^{M_2}}{\|x,y,z\|_{1-\alpha}^{N_1}}$$

where $\|x,y,z\|_{\alpha}^{N_1} = \wedge\{t>0 : G_{N_1}(x,y,z,t) \geq 1-\alpha\}, \|x,y,z\|_{\alpha}^{M_1} = \wedge\{t>0 : G_{M_1}(x,y,z,t) \leq \alpha\}, \|Tx,Ty,Tz\|_{\alpha}^{N_2} = \wedge\{t>0 : G_{N_2}(Tx,Ty,Tz,t) \geq 1-\alpha\}, \|Tx,Ty,Tz\|_{\alpha}^{M_2} = \wedge\{t>0 : G_{M_2}(Tx,Ty,Tz,t) \leq \alpha\}$ for every $\alpha \in (0,1)$. Then $\|T_1,T_2,T_3\|_{1-\alpha}^{1} \geq \|T_1,T_2,T_3\|_{\alpha}^{2}$ for every $\alpha \in (0,1)$.

Proof. Suppose $\alpha \in (0,1)$. If $G_{N_1}(x,y,z,t_0) \geq 1-\alpha$ for some $t_0 > 0$ then $G_{M_1}(x,y,z,t_0) \leq \alpha$. This implies that

$$\begin{aligned}
&\wedge\{t>0 : G_{N_1}(x,y,z,t) \geq 1-\alpha\} \geq \wedge\{t>0 : G_{M_1}(x,y,z,t) \leq \alpha\} \\
\implies &\|x,y,z\|_\alpha^{N_1} \geq \|x,y,z\|_\alpha^{M_1} \\
\implies &\|x,y,z\|_{1-\alpha}^{N_1} \geq \|x,y,z\|_{1-\alpha}^{M_1} \\
\implies &\frac{1}{\|x,y,z\|_{1-\alpha}^{M_1}} \geq \frac{1}{\|x,y,z\|_{1-\alpha}^{N_1}}.
\end{aligned}$$

By similar argument for each $i = 1,2,3$ we have

$$\|T_ix, T_iy, T_iz\|_\alpha^{N_2} \geq \|T_ix, T_iy, T_iz\|_\alpha^{M_2}.$$

Therefore from above it follows that for $(x,y,z) \neq (\theta,\theta,\theta)$,

$$\begin{aligned}
&\frac{\|T_ix, T_iy, T_iz\|_\alpha^{N_2}}{\|x,y,z\|_{1-\alpha}^{M_1}} \geq \frac{\|T_ix, T_iy, T_iz\|_\alpha^{M_2}}{\|x,y,z\|_{1-\alpha}^{N_1}} \\
\implies &\sum_{i=1}^{3} \frac{\|T_ix, T_iy, T_iz\|_\alpha^{N_2}}{\|x,y,z\|_{1-\alpha}^{M_1}} \geq \sum_{i=1}^{3} \frac{\|T_ix, T_iy, T_iz\|_\alpha^{M_2}}{\|x,y,z\|_{1-\alpha}^{N_1}} \\
\implies &\bigvee_{\substack{x,y,z\in X \\ (x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_ix, T_iy, T_iz\|_\alpha^{N_2}}{\|x,y,z\|_{1-\alpha}^{M_1}} \geq \sum_{i=1}^{3} \frac{\|T_ix, T_iy, T_iz\|_\alpha^{M_2}}{\|x,y,z\|_{1-\alpha}^{N_1}} \\
\implies &\bigvee_{\substack{x,y,z\in X \\ (x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_ix, T_iy, T_iz\|_\alpha^{N_2}}{\|x,y,z\|_{1-\alpha}^{M_1}} \geq \bigvee_{\substack{x,y,z\in X \\ (x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_ix, T_iy, T_iz\|_\alpha^{M_2}}{\|x,y,z\|_{1-\alpha}^{N_1}} \\
\implies &\|T_1,T_2,T_3\|_\alpha^1 \geq \|T_1,T_2,T_3\|_\alpha^2.
\end{aligned}$$

□

Theorem 4.6. *Let* $(X, G_{N_1}, G_{M_1}, \min, \max)$ *and* $(Y, G_{N_2}, G_{M_2}, \min, \max)$ *be two intuitionistic G-fuzzy normed linear spaces with* $G_{N_1}(x,y,z,\cdot)$ *is continuous at* $t = 0$. *For* $T_1, T_2, T_3 \in \mathcal{I}GBF(X,Y)$ *we have*

$$\|T_1,T_2,T_3\|_\alpha^1 = \bigvee_{\substack{x,y,z\in X \\ (x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_ix, T_iy, T_iz\|_\alpha^{N_2}}{\|x,y,z\|_{1-\alpha}^{M_1}},$$

$$\|T_1,T_2,T_3\|_\alpha^2 = \bigvee_{\substack{x,y,z\in X \\ (x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \frac{\|T_ix, T_iy, T_iz\|_\alpha^{M_2}}{\|x,y,z\|_{1-\alpha}^{N_1}}$$

for each $\alpha \in (0,1)$.
Now we define $G_{\mathcal{I}_N}, G_{\mathcal{I}_M} : \{\mathcal{I}GBF(X,Y)\}^3 \times [0,\infty) \to [0,1]$ *as follows:*

$$G_{\mathcal{I}_N}(T_1,T_2,T_3,t) = \begin{cases} \vee\{\alpha \in (0,1) : \|T_1,T_2,T_3\|^1_\alpha \leq t\}, & \text{if } t > 0, \\ 0, & \text{if } t = 0; \end{cases}$$

$$G_{\mathcal{I}_M}(T_1,T_2,T_3,t) = \begin{cases} \wedge\{\alpha \in (0,1) : \|T_1,T_2,T_3\|^2_{1-\alpha} \leq t\}, & \text{if } t > 0, \\ 1, & \text{if } t = 0. \end{cases}$$

Then the pair $(G_{\mathcal{I}_N}, G_{\mathcal{I}_M})$ *is said to be intuitionistic G-fuzzy operator norm on* $\mathcal{I}GBF(X,Y)$.

Proof. 1. First we have to show that $G_{\mathcal{I}_N}(T_1,T_2,T_3,t) + G_{\mathcal{I}_M}(T_1,T_2,T_3,t) \leq 1\ \forall t \geq 0$.
For $t = 0$ it is obvious.
Suppose $t_0 > 0$. Let us assume that $G_{\mathcal{I}_N}(T_1,T_2,T_3,t) = \alpha_0$.
Since $G_{\mathcal{I}_N}(T_1,T_2,T_3,t) = \alpha_0$ so there is a sequence $\{\alpha_n\}$ in $(0,1)$ such that $\alpha_n \to \alpha_0$ and $\|T_1,T_2,T_3\|_{\alpha_n} \leq t_0$. By Proposition 4.5 we have, $\|T_1,T_2,T_3\|^1_\alpha \geq \|T_1,T_2,T_3\|^2_\alpha$ for each $\alpha \in (0,1)$. Then $\|T_1,T_2,T_3\|^2_{\alpha_n} \leq \|T_1,T_2,T_3\|^1_{\alpha_n} \leq t_0$ i.e. $\|T_1,T_2,T_3\|_{\alpha_n} \leq t_0\ \forall n \in \mathbb{N}$. Thus by definition, $G_{\mathcal{I}_M}(T_1,T_2,T_3,t_0) \leq 1-\alpha_n\ \forall n \in \mathbb{N}$. So it follows that $G_{\mathcal{I}_M}(T_1,T_2,T_3,t) \leq 1-\alpha_0$.
Hence we have

$$\begin{aligned} & G_{\mathcal{I}_M}(T_1,T_2,T_3,t_0) \leq 1 - G_{\mathcal{I}_N}(T_1,T_2,T_3,t_0) \\ \Longrightarrow\ & G_{\mathcal{I}_M}(T_1,T_2,T_3,t_0) + G_{\mathcal{I}_N}(T_1,T_2,T_3,t_0) \leq 1. \end{aligned}$$

Since $t_0 > 0$ is arbitrary, so $G_{\mathcal{I}_N}(T_1,T_2,T_3,t) + G_{\mathcal{I}_M}(T_1,T_2,T_3,t) = 1\ \forall t > 0$.
From the proof of Theorem 2.20 it follows that $(\mathcal{I}GFN2 - \mathcal{I}GFN8)$ holds.
So we proceed for $(\mathcal{I}GFN9 - \mathcal{I}GFN15)$.

2. From definition we have, $G_{\mathcal{I}_M}(T_1,T_2,T_3,0) = 1$.

3. Suppose $T_1,T_2,T_3 \in \mathcal{I}GBF(X,Y)$ and $t > 0$.

$$\begin{aligned} & G_{\mathcal{I}_M}(T_1,T_2,T_3,t) = 0\ \forall t > 0 \\ \Longleftrightarrow\ & \wedge\{\alpha \in (0,1) : \|T_1,T_2,T_3\|^2_{1-\alpha} \leq t\} = 0\ \forall t > 0 \\ \Longleftrightarrow\ & \|T_1,T_2,T_3\|^2_{1-\alpha} \leq t\ \forall \alpha \in (0,1); \forall t > 0 \\ \Longleftrightarrow\ & \|T_1,T_2,T_3\|^2_{1-\alpha} = 0\ \forall \alpha \in (0,1) \\ \Longleftrightarrow\ & T_1 = T_2 = T_3 = \bigcirc. \end{aligned}$$

4. It is evident that $G_{\mathcal{I}_M}(\alpha T_1, \alpha T_2, \alpha T_3, t) = G_{\mathcal{I}_M}(T_1, T_2, T_3, \frac{t}{|\alpha|})$ for $\alpha(\neq 0) \in \mathbb{K}$.

5. $G_{\mathcal{I}_M}(T_1, T_2, T_3, t) = G_{\mathcal{I}_M}(p(T_1, T_2, T_3), t)$ for every permutation p of T_1, T_2, T_3.

6. Let us take $T_1, T_2, T_3, T_1', T_2', T_3' \in \mathcal{I}GBF(X, Y)$. It is to prove that

$$G_{\mathcal{I}_M}(T_1 + T_1', T_2 + T_2', T_3 + T_3', s + t) \leq$$

$$\max\{G_{\mathcal{I}_M}(T_1, T_2, T_3, s), G_{\mathcal{I}_M}(T_1', T_2', T_3', t)\ \forall s, t \geq 0.$$

For $s = t = 0; s = 0, t > 0; s > 0, t = 0$ this is obvious.
So let us assume that $s, t > 0$. Then $s + t > 0$.

$$\begin{aligned}
&G_{\mathcal{I}_M}(T_1, T_2, T_3, s) = \wedge\{\alpha \in (0,1) : \|T_1, T_2, T_3\|^2_{1-\alpha} \leq s\},\\
&G_{\mathcal{I}_M}(T_1', T_2', T_3', t) = \wedge\{\alpha \in (0,1) : \|T_1', T_2', T_3'\|^2_{1-\alpha} \leq t\},\\
&G_{\mathcal{I}_M}(T_1 + T_1', T_2 + T_2', T_3 + T_3', s + t)\\
&= \wedge\{\alpha \in (0,1) : \|T_1 + T_1', T_2 + T_2', T_3 + T_3'\|^2_{1-\alpha} \leq s + t\}.
\end{aligned}$$

It is to prove that

$$\begin{aligned}
&G_{\mathcal{I}_M}(T_1 + T_1', T_2 + T_2', T_3 + T_3', s + t) \leq\\
&\quad \max\{G_{\mathcal{I}_M}(T_1, T_2, T_3, s), G_{\mathcal{I}_M}(T_1', T_2', T_3', t)\}.....(C).
\end{aligned}$$

Let us assume that $G_{\mathcal{I}_M}(T_1, T_2, T_3, s) = \alpha_0, G_{\mathcal{I}_M}(T_1', T_2', T_3', t) = \beta_0$.
If $\alpha_0 = \beta_0 = 1$ then there is nothing to proof.
Suppose that $\max\{\alpha_0, \beta_0\} = \alpha_0$. Then $\beta_0 \leq \alpha_0$. Let $\beta_0 \leq \alpha_0 < \gamma < 1$. Then $\exists \alpha, \beta < \gamma$ such that $\|T_1, T_2, T_3\|^2_{1-\alpha} \leq s$ and $\|T_1', T_2', T_3'\|^2_{1-\beta} \leq t$. Let us take $\delta = \alpha \wedge \beta$. Then $\delta < \gamma$.
So $\|T_1, T_2, T_3\|^2_{1-\delta} \leq s, \|T_1', T_2', T_3'\|^2_{1-\delta} \leq t$.

$$\begin{aligned}
&\text{Now } \|T_1 + T_1', T_2 + T_2', T_3 + T_3'\|^2_{1-\delta} \leq \|T_1, T_2, T_3\|^2_{1-\delta} + \|T_1', T_2', T_3'\|^2_{1-\delta}\\
&\implies \|T_1 + T_1', T_2 + T_2', T_3 + T_3'\|^2_{1-\delta} \leq s + t\\
&\implies G_{\mathcal{I}_M}(T_1 + T_1', T_2 + T_2', T_3 + T_3', s + t) \leq \delta\\
&\implies G_{\mathcal{I}_M}(T_1 + T_1', T_2 + T_2', T_3 + T_3', s + t) < \gamma.
\end{aligned}$$

Here γ is arbitrary, so (C) is satisfied.

7. Suppose $\alpha \in (0,1)$ and $t > 0$. We know that $\|T_1, T_2, T_3\|^2_{1-\alpha} \leq t \implies \|T_1 + T_2, \bigcirc, T_3\|^2_{1-\alpha} \leq t$. Thus

$$\begin{aligned}
&\{\alpha \in (0,1) : \|T_1, T_2, T_3\|^2_{1-\alpha} \leq t\} \subset \{\alpha \in (0,1) : \|T_1 + T_2, \bigcirc, T_3\|^2_{1-\alpha} \leq t\}\\
&\implies \wedge\{\alpha \in (0,1) : \|T_1, T_2, T_3\|^2_{1-\alpha} \leq t\} \geq \wedge\{\alpha \in (0,1) : \|T_1 + T_2, \bigcirc, T_3\|^2_{1-\alpha} \leq t\}\\
&\implies G_{\mathcal{I}_M}(T_1, T_2, T_3, t) \geq G_{\mathcal{I}_M}(T_1 + T_2, \bigcirc, T_3, t).
\end{aligned}$$

8. Suppose $t \to \infty$. Then it is evident that for each $\alpha \in (0,1)$, $\|T_1, T_2, T_3\|^2_{1-\alpha} \leq t$. Thus we have,

$$\lim_{t\to\infty} \wedge \{\alpha \in (0,1) : \|T_1, T_2, T_3\|^2_{1-\alpha} \leq t\} = 0$$
$$\implies \lim_{t\to\infty} G_{\mathcal{I}_M}(T_1, T_2, T_3, t) = 0.$$

Therefore the pair $(G_{\mathcal{I}_N}, G_{\mathcal{I}_M})$ is an intuitionistic G-fuzzy norm with respect to t-norm $\min$ and t-co-norm $\max$ on $\mathcal{I}GBF(X,Y)$ which is said to be the intuitionistic G-fuzzy operator norm on $\mathcal{I}GBF(X,Y)$.

□

CONCLUSION

In the present paper our aim are to study different types of fuzzy bounded linear operators and fuzzy continuous operators in intuitioistic G-fuzzy normed linear spaces and set up some relations among them. We have proved that the set of all strongly intuitionistic G-fuzzy bounded linear operators from $(X, G_{N_1}, G_{M_1}, \min, \max)$ to $(Y, G_{N_2}, G_{M_2}, \min, \max)$ forms a linear space. Moreover, we have defined the intuitionistic G-fuzzy operator norm for intuitionistic G-fuzzy bounded linear operators.

Since bounded linear operators play a pivotal role in the development of functional analysis so our study ensure that there are lot of scopes for future works (e.g. Hahn-Banach theorem, Open Mapping Theorem, Closed Graph Theorem, Uniform Boundedness Principle and so on) will be carried out on intuitionistic G-fuzzy normed linear spaces.

CONFLICT OF INTEREST

There is no conflict of interest.

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