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Sharp Ordering in Ternary Semiring

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Abstract

Aim of this article is to develop the concept of Sharp ordering as a generalization of minus ordering in Ternary Semiring. A relationship between the minus ordering and Sharp ordering is estabilished. We derive some equivalent conditions under sharp ordering.
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1. INTRODUCTION

In 1932, Lehmer [1] explored ternary algebraic system theory. He studied "triplexes", which are ternary algebraic systems that finally turn out to be ternary groups. A generalization of ternary rings are called ternary semirings, as presented by Dutta *et al* [2]. Ternary semiring is a superset of semiring. Negative integers forms a ternary semiring but it does not form a semiring. In 1920, E.H. Moore established the concept of the generalized inverse of a matrix, defining the unique generalized inverse through the use of matrices projectors."Penrose demonstrated in 1955 that Moore's inverse is the unique matrix that can satisfy four equations [3]. The notion of regular element was first introduced by Von Neumann in 1936 [4]. Meenakshi AR et al [6] explicate the minus ordering for fuzzy matrices and established that he minus ordering is a partial

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ordering in the set of all regular matrices. Meenakshi [7] has characterized the minus ordering on matrices in terms of their generalized inverses. Moreover, they established fuzzy matrices space ordering , which is a partial ordering on the set of all idempotent matrices in fuzzy matrices.

In this paper we studied the concept of sharp ordering for an elements in ternary semiring as a special case of minus ordering. We establish a set of necessary condition for elements with specified left and right cosets to be under sharp order.

2. PRELIMINARIES

In this work, $\mathcal{T}$ represents the Ternary semiring; it is abbreviated as T.S.R. throughout.

Definition 2.1. *The term "**ternary semiring**" refers to $\mathcal{T} \neq \emptyset$, a binary addition, and a ternary multiplication called juxtaposition if $(\mathcal{T}, +)$ is a semigroup that commutes that fulfilling the requirements listed below:*

(i) $(abc)de = a(bcd)e = ab(cde)$,

(ii) $(a + b)cd = acd + bcd$,

(iii) $a(b + c)d = abd + acd$,

(iv) $ab(c + d) = abc = abd$, *for all* $a, b, d, d, e \; in \; \mathcal{T}$.

Note:1 *An example of a ternary semiring in nature is z^-, which has ternary multiplication and regular addition.*

Definition 2.2. *If an element a exists in $\mathcal{T}$ such that $ababa = a$, then a is considered regular.*
The element b is represented as a^- and is also known as the generalizd or 1-inverse of the element $'a'$.

If every element in a T.S.R is regular, then we say T.S.R is regular.

Definition 2.3. *If there is an element a in $\mathcal{T} \quad \ni \quad babab = b$, then a is considered anti-regular.*
The name for element b is 2-inverse of a.

The both conditions in Definition (2.2) and Definition (2.3) satisfied by a single element b is called 1-2 inverse and is denoted by a^+.
In Definition (2.2), if the $'b'$ is unit then $'a'$ is called unit regular.

Definition 2.4. *For any $t_1, t_2 \in \mathcal{T}$ and $i \in I$, an additive subsemigroup I of $\mathcal{T}$ is considered a left (right, lateral) ideal of $\mathcal{T}$ if t_1t_2i (similarly it_1t_2, t_1it_2) $\in I$. I is said to as an ideal $\mathcal{T}$ if I is both a left, right, and lateral ideal of $\mathcal{T}$.*

Definition 2.5. *The group inverse $\mathfrak{a}^{\#}$ of a is defined by*

(i) $a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = a$

(ii) $\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#} = \mathfrak{a}^{\#}$

(iii) $a\mathfrak{a}^{\#}e = e\mathfrak{a}^{\#}a$

Definition 2.6. *An element $a \in \mathcal{T}$, is called Idempotent if $a^3 = a$.*

Definition 2.7. *A Ternary semiring $\mathcal{T}$ is said to be commutative Ternary semiring if $abc = bca = cab = bac = cba = acb \forall a, b, c \in \mathcal{T}$*

Example 2.1. *$(Z_0, +, .)$ is a ternary semiring of infinite order which is commutative.*

Example 2.2. *The set $2I$ of all evern integers is a commutative ternary semiring with respect to ordinary addition and ternary multiplication [ ] defined by $[abc] = abc$for all $a, b, c \in \mathcal{T}$.*

Example 2.3. *The set $\mathcal{T} = \{e, a, b\}$ is Ternary semiring with respect to binary addition and ternary multiplication but non-commutative Ternary semiring.*

$+$	e	a	b
e	e	a	b
a	a	a	a
b	b	a	b

$\cdot$	e	a	b
e	e	a	b
a	a	a	a
b	b	b	b

In this paper $\mathcal{T}$ denote the Ternary Semiring with multiplicative cancellation.
Here we denote $\mathcal{T}^{\#}$ is set if all group invertible elements of $\mathcal{T}$ for which the group inverse $\mathfrak{a}^{\#}$ exists.

3. SHARP ORDERING IN TERNARY SEMIRING

Definition 3.1. *For $a \in \mathcal{T}$ and $b \in \mathcal{T}$ the sharp ordering denoted as $\overset{\#}{<}$ is defined as $a \overset{\#}{<} b \Leftrightarrow \mathfrak{a}^{\#}aa = \mathfrak{a}^{\#}ab = \mathfrak{a}^{\#}ba$; $a\mathfrak{a}^{\#}a = a\mathfrak{a}^{\#}b = b\mathfrak{a}^{\#}a$; $aa\mathfrak{a}^{\#} = ab\mathfrak{a}^{\#} = ba\mathfrak{a}^{\#}$.*

Example 3.1. *Let $\mathcal{T} = 0, 1, 2, 3, 4$ be a ternary semiring with respect to modulo addition and modulo multiplication 5, the Sharp- ordering defined as follows:*

Let $a = 3; \mathfrak{a}^{\#} = 2; b = 3$ *satisfy* $a\mathfrak{a}^{\#}a = a\mathfrak{a}^{\#}b = b\mathfrak{a}^{\#}a; ; aa\mathfrak{a}^{\#} = ba\mathfrak{a}^{\#} = ab\mathfrak{a}^{\#}; \mathfrak{a}^{\#}aa = \mathfrak{a}^{\#}ba = \mathfrak{a}^{\#}ab$

$+_5$	*0*	*1*	*2*	*3*	*4*
0	*0*	*1*	*2*	*3*	*4*
1	*1*	*2*	*3*	*4*	*0*
2	*2*	*3*	*4*	*0*	*1*
3	*3*	*4*	*0*	*1*	*2*
4	*4*	*0*	*1*	*2*	*3*

$\cdot_5$	*0*	*1*	*2*	*3*	*4*
0	*0*	*0*	*0*	*0*	*0*
1	*0*	*1*	*2*	*3*	*4*
2	*0*	*2*	*4*	*1*	*3*
3	*0*	*3*	*1*	*4*	*2*
4	*0*	*4*	*3*	*2*	*1*

Sharp ordering is the special case of minus ordering.

Example 3.2.

Theorem 3.1. *For* $a, b \in \mathcal{T}, a \overset{\#}{<} b$ *iff* $a = a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = a\mathfrak{a}^{\#}b\mathfrak{a}^{\#}a = ba\mathfrak{a}^{\#}b\mathfrak{a}^{\#}ab = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b$

Proof. $a \overset{\#}{<} b \Leftrightarrow \mathfrak{a}^{\#}aa = \mathfrak{a}^{\#}ba = \mathfrak{a}^{\#}ab; aa\mathfrak{a}^{\#} = ba\mathfrak{a}^{\#} = ab\mathfrak{a}^{\#}; a\mathfrak{a}^{\#}a = a\mathfrak{a}^{\#}b = b\mathfrak{a}^{\#}a$

$$a = a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = (a\mathfrak{a}^{\#}a)\mathfrak{a}^{\#}a = a\mathfrak{a}^{\#}b\mathfrak{a}^{\#}a \tag{3.1}$$

$$a = a\mathfrak{a}^{\#}(a\mathfrak{a}^{\#}a) = a\mathfrak{a}^{\#}(a\mathfrak{a}^{\#}b) \tag{3.2}$$

$$a = a\mathfrak{a}^{\#}(a\mathfrak{a}^{\#}a) \Rightarrow a(a)a = a(a\mathfrak{a}^{\#}(a\mathfrak{a}^{\#}a)a$$
$$a^3 = (aa\mathfrak{a}^{\#})a(\mathfrak{a}^{\#}aa) = ba\mathfrak{a}^{\#}a\mathfrak{a}^{\#}ab$$

$$a = b(a\mathfrak{a}^{\#}a)\mathfrak{a}^{\#}ab = ba\mathfrak{a}^{\#}b\mathfrak{a}^{\#}ab \tag{3.3}$$

$$a = a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = (b\mathfrak{a}^{\#}a)\mathfrak{a}^{\#}a = b\mathfrak{a}^{\#}(a\mathfrak{a}^{\#}a)$$

$$a = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b \tag{3.4}$$

$$a = a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = (a\mathfrak{a}^{\#}a)\mathfrak{a}^{\#}a = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a \tag{3.5}$$

From Equation (3.1,3.2,3.3,3.4and 3.5), we get

$$a = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = ba\mathfrak{a}^{\#}b\mathfrak{a}^{\#}ab = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = a\mathfrak{a}^{\#}b\mathfrak{a}^{\#}a$$

Converse,

$$\mathfrak{a}^{\#}\text{exists} \Rightarrow a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = a;\ \mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#} = \mathfrak{a}^{\#};\ a\mathfrak{a}^{\#}e = e\mathfrak{a}^{\#}a$$

$$\mathfrak{a}^{\#}aa = \mathfrak{a}^{\#}(a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b)a = \mathfrak{a}^{\#}ba \tag{3.6}$$

$$\mathfrak{a}^{\#}aa = \mathfrak{a}^{\#}a(a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b) = \mathfrak{a}^{\#}aa\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = \mathfrak{a}^{\#}a(a\mathfrak{a}^{\#}e)(a\mathfrak{a}^{\#}e)b$$
$$= \mathfrak{a}^{\#}ae\mathfrak{a}^{\#}ae\mathfrak{a}^{\#}ab = \mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}ab = \mathfrak{a}^{\#}ab \tag{3.7}$$

From (3.6) and (3.7), we get

$$\mathfrak{a}^{\#}aa = \mathfrak{a}^{\#}ba = \mathfrak{a}^{\#}ab$$

$$aa\mathfrak{a}^{\#} = (b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a)a\mathfrak{a}^{\#} = ba\mathfrak{a}^{\#} \tag{3.8}$$

$$aa\mathfrak{a}^{\#} = a(b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a)\mathfrak{a}^{\#} = ab\mathfrak{a}^{\#} \tag{3.9}$$

From (3.8) and (3.9)

$$aa\mathfrak{a}^{\#} = ba\mathfrak{a}^{\#} = ab\mathfrak{a}^{\#}$$

$$a\mathfrak{a}^{\#}a = (b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a)\mathfrak{a}^{\#}a = b\mathfrak{a}^{\#}a \tag{3.10}$$

$$a\mathfrak{a}^{\#}a = a\mathfrak{a}^{\#}(a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b) = a\mathfrak{a}^{\#}b \tag{3.11}$$

From (3.10) and (3.1)

$$a\mathfrak{a}^{\#}a = b\mathfrak{a}^{\#}a = a\mathfrak{a}^{\#}b$$

Therefore the converse is true. □

Theorem 3.2. *For* $a, b \in \mathcal{T}, a \overset{\#}{<} b$ *iff* $a = a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}b = b\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a = a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a = ba\mathfrak{b}^{\#}b\mathfrak{b}^{\#}ab$

Proof. Since $a = a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = a\mathfrak{a}^{\#}b\mathfrak{a}^{\#}a = ba\mathfrak{a}^{\#}b\mathfrak{a}^{\#}ab = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b$

For $\mathfrak{b}^{\#} \in b\{1\}$, $a = a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = (a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b)\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b$ (By using Theorem 3.1)

$$\begin{aligned} a &= a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}(b\mathfrak{b}^{\#}b\mathfrak{b}^{\#}b)\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b \\ &= a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b \\ &= a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}b(\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#})a\mathfrak{a}^{\#}b \\ &= a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b \end{aligned}$$

$$= a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a \tag{3.12}$$

$$a = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = b\mathfrak{b}^{\#}b\mathfrak{b}^{\#}b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = b\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a \tag{3.13}$$

$$\begin{aligned} a = a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b &= a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}(b\mathfrak{b}^{\#}b\mathfrak{b}^{\#}b) \\ &= (a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b)\mathfrak{b}^{\#}b\mathfrak{b}^{\#}b \end{aligned}$$

$$a = a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}b \ for \ each \ \mathfrak{b}^{\#} \in b\{1\} \tag{3.14}$$

$$\begin{aligned} a &= ba\mathfrak{a}^{\#}b\mathfrak{a}^{\#}ab \\ &= ba(\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#})b\mathfrak{a}^{\#}ab \\ &= ba\mathfrak{a}^{\#}a\mathfrak{a}^{\#}ab \\ &= ba\mathfrak{a}^{\#}(a\mathfrak{a}^{\#}b\mathfrak{a}^{\#}a)\mathfrak{a}^{\#}ab \\ &= ba\mathfrak{a}^{\#}a\mathfrak{a}^{\#}(b\mathfrak{b}^{\#}b\mathfrak{b}^{\#}b)\mathfrak{a}^{\#}a\mathfrak{a}^{\#}ab \end{aligned}$$

$$a = ba\mathfrak{b}^{\#}b\mathfrak{b}^{\#}ab \tag{3.15}$$

$\therefore a = a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}b = b\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a = a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a = ba\mathfrak{b}^{\#}b\mathfrak{b}^{\#}ab = a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a$
Converse, $\mathfrak{a}^{\#}$ exists, Let $n = \mathfrak{b}^{\#}a\mathfrak{b}^{\#}a\mathfrak{b}^{\#}$
$a\mathfrak{a}^{\#}a = anana = a(\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a\mathfrak{b}^{\#})a(\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a\mathfrak{b}^{\#})a = a\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a = a \Rightarrow n \in a\{1\}$
$ana = a(\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a\mathfrak{b}^{\#})a = a\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a\mathfrak{b}^{\#}(a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}b)$
$= a\mathfrak{b}^{\#}a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}b = a\mathfrak{b}^{\#}a\mathfrak{b}^{\#}b(\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a\mathfrak{b}^{\#})b$
$= a\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a\mathfrak{b}^{\#}b = anb$
$\therefore ana = anb$
$ana = a(\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a\mathfrak{b}^{\#})a = b\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a(\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a\mathfrak{b}^{\#})a$
$= b\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a = b(\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a\mathfrak{b}^{\#})b\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a$
$= b(\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a\mathfrak{b}^{\#})a = bna$
$\therefore ana = bna$
Similarly we get $aan = abn = ban; naa = nab = nba$
Hence $a \overset{\#}{<} b$ with respect to $n \in a\{1\}$ □

Theorem 3.3. *For $a, b \in \mathcal{T}^{\#}$, The following conditions are equivalent:*

(i) $a \overset{\#}{<} b$

(ii) $a^3 = aab = baa = aba$

(iii) $aa^{\#}a^{\#} = b^{\#}a^{\#}a = a^{\#}b^{\#}a; a^{\#}aa^{\#} = a^{\#}ab^{\#} = b^{\#}aa^{\#}; a^{\#}a^{\#}a = b^{\#}a^{\#}a = a^{\#}b^{\#}a$

(iv) $a^{\#} \overset{\#}{<} a^{\#}$

Proof. $(i) \Leftrightarrow (ii)$ since $\mathfrak{a}^{\#}$ exists, $\mathfrak{a}^{\#^2}a^3 = \mathfrak{a}^{\#}\mathfrak{a}^{\#}aaa = \mathfrak{a}^{\#}(\mathfrak{a}^{\#}ae)aa = \mathfrak{a}^{\#}(ea\mathfrak{a}^{\#})aa$
$= (\mathfrak{a}^{\#}a)(\mathfrak{a}^{\#}a)a = a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = a(a\mathfrak{a}^{\#})(\mathfrak{a}^{\#}a) = aa\mathfrak{a}^{\#}(a\mathfrak{a}^{\#}) = aaa\mathfrak{a}^{\#}\mathfrak{a}^{\#} = a^3\mathfrak{a}^{\#^2} = a$
$aab = a(a^3\mathfrak{a}^{\#^2})b = a(aaa\mathfrak{a}^{\#}\mathfrak{a}^{\#})b = a(a(a\mathfrak{a}^{\#})a\mathfrak{a}^{\#}b) = aaa = a^3$
$baa = b(\mathfrak{a}^{\#^2}a^3)a = b(\mathfrak{a}^{\#}\mathfrak{a}^{\#}aaa)a = (b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a)aa = aaa = a^3$
$aba = ab(a^3\mathfrak{a}^{\#^2}) = ab(aaa\mathfrak{a}^{\#}\mathfrak{a}^{\#}) = abaa\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = aba(\mathfrak{a}^{\#}a)a\mathfrak{a}^{\#} = ab\mathfrak{a}^{\#}aa\mathfrak{a}^{\#}a =$

$a(b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a)a = aaa = a^3$
$\therefore\ aab = aba = baa = a^3$
Converse,

$$aa\mathfrak{a}^{\#} = (a^3\mathfrak{a}^{\#^2})a\mathfrak{a}^{\#} = (a^3)\mathfrak{a}^{\#^2}a\mathfrak{a}^{\#} = (baa)\mathfrak{a}^{\#^2}a\mathfrak{a}^{\#}$$
$$= baa\mathfrak{a}^{\#}\mathfrak{a}^{\#}a\mathfrak{a}^{\#} = ba(a\mathfrak{a}^{\#})\mathfrak{a}^{\#}a\mathfrak{a}^{\#} = ba\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#} = ba\mathfrak{a}^{\#}$$
$$aa\mathfrak{a}^{\#} = ba\mathfrak{a}^{\#} \tag{3.16}$$

$$aa\mathfrak{a}^{\#} = (a^3\mathfrak{a}^{\#^2})a\mathfrak{a}^{\#} = aba\mathfrak{a}^{\#^2}a\mathfrak{a}^{\#} = aba\mathfrak{a}^{\#}\mathfrak{a}^{\#}a\mathfrak{a}^{\#} = ab\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}$$
$$aa\mathfrak{a}^{\#} = ab\mathfrak{a}^{\#} \tag{3.17}$$

From Equation (3.16) and (3.17),

$$aa\mathfrak{a}^{\#} = ba\mathfrak{a}^{\#} = ab\mathfrak{a}^{\#} \tag{3.18}$$

$$a\mathfrak{a}^{\#}a = (a^3\mathfrak{a}^{\#^2})\mathfrak{a}^{\#}a = baa\mathfrak{a}^{\#^2}\mathfrak{a}^{\#}a = ba\mathfrak{a}^{\#}a\mathfrak{a}^{\#}\mathfrak{a}^{\#}a = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = b\mathfrak{a}^{\#}a \tag{3.19}$$

$$a\mathfrak{a}^{\#}a = a\mathfrak{a}^{\#}(\mathfrak{a}^{\#^2}a^3) = a\mathfrak{a}^{\#}\mathfrak{a}^{\#}\mathfrak{a}^{\#}aab = a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = a\mathfrak{a}^{\#}b \tag{3.20}$$

From Equations (3.19) and (3.20)

$$a\mathfrak{a}^{\#}a = a\mathfrak{a}^{\#}b = b\mathfrak{a}^{\#}a \tag{3.21}$$

$$\mathfrak{a}^{\#}aa = \mathfrak{a}^{\#}a(\mathfrak{a}^{\#^2}a^3) = \mathfrak{a}^{\#}a\mathfrak{a}^{\#}\mathfrak{a}^{\#}aba = \mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}ba = \mathfrak{a}^{\#}ba \tag{3.22}$$

$$Similarly,\ we\ get\ \mathfrak{a}^{\#}aa = \mathfrak{a}^{\#}ab \tag{3.23}$$

From Equations (3.22) and (3.23) we obtain $\mathfrak{a}^{\#}aa = \mathfrak{a}^{\#}ab = \mathfrak{a}^{\#}ba$
Hence we get $a \overset{\#}{<} b$
$(i) \Leftrightarrow (iii)$ Let $a \overset{\#}{<} b$, Now $(a^{\#})^{\#}a^{\#}a^{\#} = aa^{\#}a^{\#} = a^{\#}aa^{\#}$

$$(aaa^{\#})^{\#} = (a(ba^{\#}aa^{\#}a)a^{\#})^{\#} = aa^{\#}aa^{\#}ab^{\#}a^{\#} = ab^{\#}a^{\#}$$
$$\Rightarrow aa^{\#}a^{\#} = ab^{\#}a^{\#} \tag{3.24}$$

$$(aaa^{\#})^{\#} = (aba^{\#}aa^{\#}aa^{\#})^{\#} = (baa^{\#}aa^{\#}aa^{\#})^{\#} = aa^{\#}aa^{\#}aa^{\#}b^{\#} = aa^{\#}b^{\#} \tag{3.25}$$

From Equation (3.24) and (3.25)

$$aa^{\#}a^{\#} = ab^{\#}a^{\#} = aa^{\#}b^{\#} \tag{3.26}$$

$$\begin{gathered}(aa^{\#}a)^{\#} = ((aa^{\#}aa^{\#}a)a^{\#}a)^{\#} \\ \Rightarrow a^{\#}aa^{\#} = (ba^{\#}aa^{\#}aa^{\#}a)^{\#} = a^{\#}aa^{\#}aa^{\#}ab^{\#} \\ \Rightarrow a^{\#}aa^{\#} = a^{\#}ab^{\#}\end{gathered} \tag{3.27}$$

$$\begin{gathered}(aa^{\#}a)^{\#} = ((aa^{\#}aa^{\#}a)a^{\#}a)^{\#} \\ \Rightarrow a^{\#}aa^{\#} = (aa^{\#}aa^{\#}aa^{\#}b)^{\#} = b^{\#}aa^{\#}aa^{\#}aa^{\#} \\ \Rightarrow a^{\#}aa^{\#} = b^{\#}aa^{\#}\end{gathered} \tag{3.28}$$

From Equation (3.27) and (3.28)

$$a^{\#}aa^{\#} = b^{\#}aa^{\#} = a^{\#}ab^{\#} \tag{3.29}$$

$$\begin{gathered}(a^{\#}aa)^{\#} = (a^{\#}(aa^{\#}aa^{\#}a)a)^{\#} \\ \Rightarrow a^{\#}a^{\#}a = a^{\#}aa^{\#}(aa^{\#}b)a)^{\#} = a^{\#}b^{\#}aa^{\#}aa^{\#}a \\ \Rightarrow a^{\#}a^{\#}a = a^{\#}b^{\#}a\end{gathered} \tag{3.30}$$

$$\begin{gathered}(a^{\#}aa)^{\#} = (a^{\#}(aa^{\#}aa^{\#}a)a)^{\#} \\ \Rightarrow a^{\#}a^{\#}a = a^{\#}aa^{\#}a(a^{\#}ab)^{\#} = b^{\#}a^{\#}aa^{\#}aa^{\#}a \\ \Rightarrow a^{\#}a^{\#}a = b^{\#}a^{\#}a\end{gathered} \tag{3.31}$$

From Equation (3.30) and (3.31)

$$a^{\#}a^{\#}a = b^{\#}a^{\#}a = a^{\#}b^{\#}a \tag{3.32}$$

Follows from Equations (3.26,3.29 and 3.32) we obtain
$aa^{\#}a^{\#} = b^{\#}a^{\#}a = a^{\#}b^{\#}a; a^{\#}aa^{\#} = a^{\#}ab^{\#} = b^{\#}aa^{\#}; a^{\#}a^{\#}a = b^{\#}a^{\#}a = a^{\#}b^{\#}a$
converse is true follows from $(a^{\#})^{\#} = a$
$(iii) \Leftrightarrow (iv)$ is obvious. □

Theorem 3.4. *Let $a, b \in \mathcal{T}^{\#}$ and $a \overset{-}{<} b$, The following conditions are equivalent:*

(i) $aab = baa = aba$

(ii) $a^2b^2a = ababa$ *and* $ab^2a^2 = ababa$

(iii) $\mathfrak{a}^{\#}(\mathfrak{b}^{\#})^2(\mathfrak{a}^{\#})^2 = (\mathfrak{a}^{\#})^2(\mathfrak{b}^{\#})^2\mathfrak{a}^{\#} = \mathfrak{a}^{\#}b\mathfrak{a}^{\#}b\mathfrak{a}^{\#}$

(iv) $(\mathfrak{b}^{\#})^2a^3 = a^3(\mathfrak{b}^{\#})^2 = a\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a$

(v) $(\mathfrak{b}^{\#})^2(\mathfrak{a}^{\#})^3 = (\mathfrak{a}^{\#})^3(\mathfrak{b}^{\#})^2$

Proof. $(i) \Leftrightarrow (ii)$
since $aab = aba$
$\Rightarrow (aab)ba = ababa$
$\Rightarrow a^2b^2a = ababa$
$baa = aba \Rightarrow ab(baa) = ababa$
$\Rightarrow ab^2aa^2 = ababa$
Converse, take $a^3b^2 = ababa$
$aab = a(a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a)b = aa\mathfrak{a}^{\#}a(\mathfrak{a}^{\#}aa)$

$$aab = a(a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a)a = a^3 \tag{3.33}$$

$$baa = b(a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a)a = (aa\mathfrak{a}^{\#})a\mathfrak{a}^{\#}a)a = a^3 \tag{3.34}$$

$aba = ab(a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a) = a(ba\mathfrak{a}^{\#})a\mathfrak{a}^{\#}a = aaa\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = a^3$ (By using Theorem 3.3) (3.35)

From Equations (3.33), (3.34)(3.35) we obtain $aab = baa = aba$.
$(i) \Leftrightarrow (iii)$
If $aab = baa = aba = a^3$
then $\mathfrak{a}^{\#}b^2(\mathfrak{a}^{\#})^2 = \mathfrak{a}^{\#}b^2\mathfrak{a}^{\#}a^2(\mathfrak{a}^{\#})^3 = \mathfrak{a}^{\#}b^2a^2(\mathfrak{a}^{\#})^4$
$= \mathfrak{a}^{\#}b(baa)(\mathfrak{a}^{\#})^4 = \mathfrak{a}^{\#}ba^3(\mathfrak{a}^{\#})^4$
$= \mathfrak{a}^{\#}baaa(\mathfrak{a}^{\#})^4 = \mathfrak{a}^{\#}a^4(\mathfrak{a}^{\#})^4$
$= \mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}$

$$\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#} = \mathfrak{a}^{\#} \tag{3.36}$$

$(\mathfrak{a}^{\#})^2b^2\mathfrak{a}^{\#} = (\mathfrak{a}^{\#})^4a^2b^2\mathfrak{a}^{\#}$
$(\mathfrak{a}^{\#})^4a^2bb\mathfrak{a}^{\#} = (\mathfrak{a}^{\#})^4a^3b\mathfrak{a}^{\#} = \mathfrak{a}^{\#}a^4(a^2b)\mathfrak{a}^{\#}$

$$(\mathfrak{a}^{\#})^2b^2\mathfrak{a}^{\#} = (\mathfrak{a}^{\#})^4aa^3\mathfrak{a}^{\#} = (\mathfrak{a}^{\#})^4a^4\mathfrak{a}^{\#} = \mathfrak{a}^{\#} \tag{3.37}$$

$\mathfrak{a}^{\#}b\mathfrak{a}^{\#}b\mathfrak{a}^{\#} = \mathfrak{a}^{\#}b\mathfrak{a}^{\#}ba^2(\mathfrak{a}^{\#})^3 = \mathfrak{a}^{\#}b\mathfrak{a}^{\#}a^3(\mathfrak{a}^{\#})^3$

$$\mathfrak{a}^{\#}b\mathfrak{a}^{\#}b\mathfrak{a}^{\#} = \mathfrak{a}^{\#}a^3(\mathfrak{a}^{\#})^3a^3(\mathfrak{a}^{\#})^3 = \mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#} \tag{3.38}$$

From Equations (3.46),(3.47) and (3.48)
we get $\mathfrak{a}^{\#}b^2(\mathfrak{a}^{\#})^2 = (\mathfrak{a}^{\#})^2b^2\mathfrak{a}^{\#} = \mathfrak{a}^{\#} = \mathfrak{a}^{\#}b\mathfrak{a}^{\#}b\mathfrak{a}^{\#} = \mathfrak{a}^{\#}$
$(i) \Leftrightarrow (iv)$ If $aab = baa = aba$ then $a\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a = a$
$(\mathfrak{b}^{\#})^2a^3 = (\mathfrak{b}^{\#})^2a^3a^2(\mathfrak{a}^{\#})^2$
$= (\mathfrak{b}^{\#})^2(baa)a^2(\mathfrak{a}^{\#})^2 = (\mathfrak{b}^{\#})^2ba^4(\mathfrak{a}^{\#})^2$
$= (\mathfrak{b}^{\#})^2b(baa)a(\mathfrak{a}^{\#})^2 = a(\mathfrak{b}^{\#})^2b^2a^3(\mathfrak{a}^{\#})^2$
$= (\mathfrak{b}^{\#})^2b^2a^3(\mathfrak{a}^{\#})^2$

$$= ((\mathfrak{b}^{\#})^2b^2) = a^3(\mathfrak{a}^{\#})^2 = a \tag{3.39}$$

$a^3(\mathfrak{b}^{\#})^2 = a^2((\mathfrak{a}^{\#})^2a^3)(\mathfrak{b}^{\#})^2$
$= a^2(\mathfrak{a}^{\#})^2(aab)(\mathfrak{b}^{\#})^2 = a^2(\mathfrak{a}^{\#})^2(a^2b)(\mathfrak{b}^{\#})^2$
$= a^2(\mathfrak{a}^{\#})^2a((\mathfrak{a}^{\#})^2a^3)b(\mathfrak{b}^{\#})^2$
$= a^2(\mathfrak{a}^{\#})^2a(\mathfrak{a}^{\#})^2(a^2b)b(\mathfrak{b}^{\#})^2$
$= a^2(\mathfrak{a}^{\#})^4a^3b^2(\mathfrak{b}^{\#})^2$
$= a^2(\mathfrak{a}^{\#})^4a^2((\mathfrak{a}^{\#})^2a^3)b^2(\mathfrak{b}^{\#})^2$
$= a^2(\mathfrak{a}^{\#})^4a^2(\mathfrak{a}^{\#})^2a^2b^3(\mathfrak{b}^{\#})^2$
$= a^2(\mathfrak{a}^{\#})^4a^2(\mathfrak{a}^{\#})^2a^2b$
$= a^2(\mathfrak{a}^{\#})^4a^2(\mathfrak{a}^{\#})^2a^3 = a^7(\mathfrak{a}^{\#})^6$

$$a^3(\mathfrak{b}^{\#})^2 = a^3a^3a(\mathfrak{a}^{\#})^3(\mathfrak{a}^{\#})^3 = a^3(\mathfrak{a}^{\#})^2 = a \tag{3.40}$$

So, from Equations(3.49) and (3.50)$(\mathfrak{b}^{\#})^2a^3 = a^3(\mathfrak{b}^{\#})^2$
converse, Suppose that $(\mathfrak{b}^{\#})^2a^3 = a^3(\mathfrak{b}^{\#})^2 = a$ we duduce that
$baa = b((\mathfrak{b}^{\#})^2a^3)a = b(\mathfrak{b}^{\#})^2a(a\mathfrak{a}^{-}a\mathfrak{a}^{-}a)aa$
$= b(\mathfrak{b}^{\#})^2(ba\mathfrak{a}^{-})a\mathfrak{a}^{-}abaa = b(\mathfrak{b}^{\#})^2b(b\mathfrak{a}^{-}a)\mathfrak{a}^{-}baa$
$= b(\mathfrak{b}^{\#})^2b^2\mathfrak{a}^{-}a\mathfrak{a}^{-}baa$
$= b(\mathfrak{b}^{\#})^2b^2\mathfrak{a}^{-}a\mathfrak{a}^{-}b(a\mathfrak{a}^{-}a\mathfrak{a}^{-}a)a$
$= b(\mathfrak{b}^{\#})^2b^2\mathfrak{a}^{-}a\mathfrak{a}^{-}(ba\mathfrak{a}^{-})a\mathfrak{a}^{-}aa$
$= b\mathfrak{a}^{-}a\mathfrak{a}^{-}(aa\mathfrak{a}^{-})a\mathfrak{a}^{-}aa$
$= b\mathfrak{a}^{-}a\mathfrak{a}^{-}a(a\mathfrak{a}^{-}a\mathfrak{a}^{-}a)a$
$= aaa = a^3$
$aab = a(a^3(\mathfrak{b}^{\#})^2)b = aa(b\mathfrak{a}^{-}a\mathfrak{a}^{-}a)a(\mathfrak{b}^{\#})^2b$
$= aab\mathfrak{a}^{-}a(\mathfrak{a}^{-}ab)(\mathfrak{b}^{\#})^2b = aab\mathfrak{a}^{-}(a\mathfrak{a}^{-}a)b(\mathfrak{b}^{\#})^2b$
$= aab\mathfrak{a}^{-}a\mathfrak{a}^{-}bb(\mathfrak{b}^{\#})^2b = aab\mathfrak{a}^{-}a\mathfrak{a}^{-}b$
$= aab\mathfrak{a}^{-}a\mathfrak{a}^{-}b = aaa\mathfrak{a}^{-}a\mathfrak{a}^{-}b = aaa = a^3$
$a^3 = aab$
$aba = (a^3(\mathfrak{b}^{\#})^2)ba = a(b\mathfrak{a}^{-}a\mathfrak{a}^{-}a)a(\mathfrak{b}^{\#})^2ba$

$= ab\mathfrak{a}^{-}a(\mathfrak{a}^{-}aa)(\mathfrak{b}^{\#})^2ba$
$= ab\mathfrak{a}^{-}a(\mathfrak{a}^{-}ab)(\mathfrak{b}^{\#})^2ba$
$= ab\mathfrak{a}^{-}(a\mathfrak{a}^{-}a)b(\mathfrak{b}^{\#})^2ba$
$= ab\mathfrak{a}^{-}(a\mathfrak{a}^{-}b)b(\mathfrak{b}^{\#})^2ba$
$= ab\mathfrak{a}^{-}a\mathfrak{a}^{-}b^2(\mathfrak{b}^{\#})^2ba$
$= ab\mathfrak{a}^{-}a\mathfrak{a}^{-}ba = ab\mathfrak{a}^{-}a\mathfrak{a}^{-}aa$
$= aaa = a^3$
$\therefore aab = baa = aba$
$(iv) \Leftrightarrow (v)$ Since $a \overset{-}{<} b$ we have
$(\mathfrak{b}^{\#})^2(\mathfrak{a}^{\#})^3 = (\mathfrak{b}^{\#})^2(a^2((\mathfrak{a}^{\#})^3)(\mathfrak{a}^{\#})^2 = (\mathfrak{b}^{\#})^2a^2(a^2(\mathfrak{a}^{\#})^3)(\mathfrak{a}^{\#})^4$
$= (\mathfrak{b}^{\#})^2a^3(a(\mathfrak{a}^{\#})^3(\mathfrak{a}^{\#})^4)$
$= aa(\mathfrak{a}^{\#})^3(\mathfrak{a}^{\#})^4 = a^2(\mathfrak{a}^{\#})^3(\mathfrak{a}^{\#})^4$
$= \mathfrak{a}^{\#}(\mathfrak{a}^{\#})^4 = (\mathfrak{a}^{\#})^5$

$$\therefore (\mathfrak{b}^{\#})^2(\mathfrak{a}^{\#})^3 = (\mathfrak{a}^{\#})^5 \tag{3.41}$$

$(\mathfrak{a}^{\#})^3(\mathfrak{b}^{\#})^2 = (\mathfrak{a}^{\#})^2((\mathfrak{a}^{\#})^3a^2)(\mathfrak{b}^{\#})^2$
$= (\mathfrak{a}^{\#})^4((\mathfrak{a}^{\#})^3a^2)a^2(\mathfrak{b}^{\#})^2$
$= \mathfrak{a}^{\#})^4(\mathfrak{a}^{\#})^3a(a^3(\mathfrak{b}^{\#})^2)$
$= \mathfrak{a}^{\#})^4(\mathfrak{a}^{\#})^3aa = \mathfrak{a}^{\#})^4((\mathfrak{a}^{\#})^3a^2)$
$= (\mathfrak{a}^{\#})^4\mathfrak{a}^{\#} = (\mathfrak{a}^{\#})^5$

$$\therefore (\mathfrak{a}^{\#})^3(\mathfrak{b}^{\#})^2 = (\mathfrak{a}^{\#})^5 \tag{3.42}$$

so $(\mathfrak{a}^{\#})^3(\mathfrak{b}^{\#})^2 = (\mathfrak{b}^{\#})^2(\mathfrak{a}^{\#})^3$
Converse, suppose that $(\mathfrak{a}^{\#})^3(\mathfrak{b}^{\#})^2 = (\mathfrak{b}^{\#})^2(\mathfrak{a}^{\#})^3$
From Equation (3.51) $(\mathfrak{b}^{\#})^2(\mathfrak{a}^{\#})^3 = (\mathfrak{a}^{\#})^5$
Multiply $\mathfrak{a}^{\#}a^7$ on both sides
$(\mathfrak{b}^{\#})^2(\mathfrak{a}^{\#})^3(\mathfrak{a}^{\#}a^7) = (\mathfrak{a}^{\#})^5(\mathfrak{a}^{\#}a^7)$
$(\mathfrak{b}^{\#})^2(\mathfrak{a}^{\#})^4a^7 = (\mathfrak{a}^{\#})^5a^7$
$(\mathfrak{b}^{\#})^2aaa = a^3$
$(\mathfrak{b}^{\#})^2a^3 = a^3$=a
$\therefore (\mathfrak{b}^{\#})^2a^3 = a$
similarly we get $a^3(\mathfrak{b}^{\#})^2 = a$. □

Theorem 3.5. *let* $a, b \in \mathcal{T}^{\#}$ *and* $a \overset{-}{<} b$ *then*

$a \overset{\#}{<} b \Leftrightarrow b\mathfrak{a}^{\#}b\mathfrak{a}^{\#}b = a$

Proof. $b\mathfrak{a}^{\#}b\mathfrak{a}^{\#}b = b(a^2(\mathfrak{a}^{\#})^3)b(a^2(\mathfrak{a}^{\#})^3)b$
$= (ba^2)(\mathfrak{a}^{\#})^3b(\mathfrak{a}^{\#})^3a^2b = a^3(\mathfrak{a}^{\#})^3a^3(\mathfrak{a}^{\#})^3b$
$= a^3(\mathfrak{a}^{\#})^3(\mathfrak{a}^{\#})^3a^3b = a^3(\mathfrak{a}^{\#})^3(\mathfrak{a}^{\#})^3a(a^2b)$
$= a^3(\mathfrak{a}^{\#})^3(\mathfrak{a}^{\#})^3aa^3 = a^3(\mathfrak{a}^{\#})^3a(\mathfrak{a}^{\#})^3)a^3$
$= a^3(\mathfrak{a}^{\#})^2\mathfrak{a}^{\#}a\mathfrak{a}^{\#}(\mathfrak{a}^{\#})^2a^3$
$a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = a$
$b\mathfrak{a}^{\#}b\mathfrak{a}^{\#}b = a$
converse, $b\mathfrak{a}^{\#}b\mathfrak{a}^{\#}b = a$. since $a \overset{-}{<} b \; \exists\, \mathfrak{a}^{-} \in \{1\}$ such that
$aa\mathfrak{a}^{-} = ab\mathfrak{a}^{-} = ba\mathfrak{a}^{-}; a\mathfrak{a}^{-}a = b\mathfrak{a}^{-}a = a\mathfrak{a}^{-}b; = \mathfrak{a}^{-}aa = \mathfrak{a}^{-}ab = \mathfrak{a}^{-}ba;$
we have $aa\mathfrak{a}^{\#} = (a\mathfrak{a}^{-}a\mathfrak{a}^{-}a)a\mathfrak{a}^{\#} = (b\mathfrak{a}^{-}a)\mathfrak{a}^{-}aa\mathfrak{a}^{\#}$
$= (b\mathfrak{a}^{\#}b\mathfrak{a}^{\#}b)\mathfrak{a}^{-}a\mathfrak{a}^{-}aa\mathfrak{a}^{\#}$
$= (b\mathfrak{a}^{\#}b\mathfrak{a}^{\#}b)\mathfrak{a}^{-}a\mathfrak{a}^{-}ab\mathfrak{a}^{\#}$
$= a\mathfrak{a}^{-}a\mathfrak{a}^{-}ab\mathfrak{a}^{\#} = ab\mathfrak{a}^{\#}$
$aa\mathfrak{a}^{\#} = (a\mathfrak{a}^{-}a\mathfrak{a}^{-}(b\mathfrak{a}^{\#}b\mathfrak{a}^{\#}b)\mathfrak{a}^{\#}$
$= a\mathfrak{a}^{-}a\mathfrak{a}^{-}aa\mathfrak{a}^{\#} = aa\mathfrak{a}^{-}a\mathfrak{a}^{-}a\mathfrak{a}^{\#}$
$= ba\mathfrak{a}^{-}a\mathfrak{a}^{-}a\mathfrak{a}^{\#} = ba\mathfrak{a}^{\#}$
hence $aa\mathfrak{a}^{\#} = ab\mathfrak{a}^{\#} = ba\mathfrak{a}^{\#}$
$a\mathfrak{a}^{\#}a = a\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a(\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#})a$
$= a\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a^3(\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#})a$
$= (a\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a)aa\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a$
$= aaa\mathfrak{a}^{\#}a = a(a\mathfrak{a}^{-}a\mathfrak{a}^{-}a)a\mathfrak{a}^{\#}a = ba\mathfrak{a}^{-}a\mathfrak{a}^{-}aa\mathfrak{a}^{\#}a$
$= baa\mathfrak{a}^{\#}a = ba^3\mathfrak{a}^{\#} = ba^3(\mathfrak{a}^{\#})^3a^2$
$= ba^5(\mathfrak{a}^{\#})^3 = b\mathfrak{a}^{\#}a$
hence $a\mathfrak{a}^{\#}a = b\mathfrak{a}^{\#}a$
similarly we get $a\mathfrak{a}^{\#}a = a\mathfrak{a}^{\#}b$ and
$\mathfrak{a}^{\#}aa = \mathfrak{a}^{\#}ba = \mathfrak{a}^{\#}ab$.
Hence $a \overset{\#}{<} b$. □

Lemma 3.1. $Teb \subseteq T\mathfrak{a}^{-}a \Leftrightarrow b^TeT \subseteq a^T(\mathfrak{a}^T)^{-}T$

Proof. $Teb \subseteq T\mathfrak{a}^{-}a \Leftrightarrow b = b\mathfrak{a}^{-}a\mathfrak{a}^{-}a$
$\Leftrightarrow b^T = (b\mathfrak{a}^{-}a\mathfrak{a}^{-}a)^T = a^T(\mathfrak{a}^{-})^Ta^T(\mathfrak{a}^{-})^Tb^T$
$\Leftrightarrow = a^T(\mathfrak{a}^T)^{-}a^T(\mathfrak{a}^T)^{-}b^T$
$\Leftrightarrow b^T = a^T(\mathfrak{a}^T)^{-}T$ □

Theorem 3.6. *For* $a, b \in \mathcal{T}$, $a \overset{\#}{<} b$ *iff* $aab = aba = baa$ *and* $a \overset{-}{<} b$

Proof. $a \overset{\#}{<} b \Rightarrow a \overset{-}{<} b, aab = aba = baa = a^3$ (By using Theorem 3.3)

Conversely, since $a \overset{-}{<} b \Rightarrow a = ab^-bb^-b = bb^-bb^-a = ab^-bb^-a = bab^-bb^-ab$ (by using[[14], Theorem 3.2)
$aab = (ab^-bb^-a)ab$
$= ab^-bb^-(aab)$
$= (ab^-bb^-b)aa$
$\therefore aab = aaa = a^3$
$baa = ba(ab^-bb^-a)$
$= baa(b^-bb^-a)$
$= aab(b^-bb^-a)$
$= aa(bb^-bb^-a)$
$= aaa = a^3$
$\therefore baa = a^3$
$aba = ab(ab^-bb^-a)$
$= (aab)b^-bb^-a$
$= aa(bb^-bb^-a) = aaa = a^3$
$\therefore aba = a^3$
Hence $aab = aba = baa = a^3 \Rightarrow a \overset{\#}{<} b$. □

Lemma 3.2. *For $a, b \in \mathcal{T}$, if a is regular then the following hold:*

(i) $Tea \subseteq Teb$ *iff* $a = ab^-bb^-b \; \forall \, b^- \in b\{1\}$

(ii) $aeT \subseteq beT$ *iff* $a = bb^-bb^-a \; \forall \, b^- \in b\{1\}$

Proof.

$$(i) Tea \subseteq Teb \Leftrightarrow a = ab^-b$$
$$\Leftrightarrow a = ab^-(bb^-bb^-b)$$
$$\Leftrightarrow a = ab^-bb^-b$$

hence (i) holds,

$$(ii) aeT \subseteq beT \Leftrightarrow a = bb^-a$$
$$\Leftrightarrow a = (bb^-bb^-b)b^-a$$
$$\Leftrightarrow a = bb^-bb^-a$$

hence (ii) holds.

□

Theorem 3.7. *For $a, b \in \mathcal{T}$ then the given are equivalent*

(i) $a \bar{<} b$

(ii) $a = ab^-bb^-b = bb^-bb^-a = ab^-ab^-a \ \forall\, b \in b\{1\}$

(iii) $Tea \subseteq Teb, aeT \subseteq beT$ *and* $ab^-ab^-a = a$

Proof. $(i) \Rightarrow (ii)$
Since $a \bar{<} b \Rightarrow a = aa^-aa^-b = ba^-aa^-a = aa^-ba^-a = baa^-ba^-ab$
$a = aa^-aa^-b = aa^-aa^-(bb^-bb^-b) = ab^-bb^-b$
$a = ba^-aa^-a = (bb^-bb^-b)a^-aa^-a = bb^-bb^-a$
Since $a \bar{<} b \Rightarrow b\{1\} \subseteq a\{1\}$ so $a = ab^-ab^-a$
$(ii) \Rightarrow (iii)$ consequence of Lemma 3.2
$(iii) \Rightarrow (i)$ let $n = b^-ab^-ab^-$
$anana = a(b^-ab^-ab^-)a(b^-ab^-ab^-)a = ab^-ab^-a = a$
$\Rightarrow n \in a\{1\}$

$$aan = (b^-ab^-ab^-)a(b^-ab^-ab^-)$$
$$= ba(b^-ab^-ab^-) = ban \ (\because aeT \subseteq beT \Rightarrow a = bab^-ab^-)$$
$$\therefore aan = ban \tag{3.43}$$

$$aan = a(b^-ab^-ab^-)(b^-ab^-ab^-)$$
$$= ab(b^-ab^-ab^-) = abn(\because Tea \subseteq Teb \Rightarrow a = b^-ab^-ab^-)$$
$$\therefore aan = abn \tag{3.44}$$

From Equations (3.43) and (3.44) we get

$$aan = abn = ban \tag{3.45}$$

$$naa = (b^-ab^-ab^-)(bab^-ab^-)a = nba \tag{3.46}$$

$$naa = (b^-ab^-ab^-)a(bab^-ab^-) = nab \tag{3.47}$$

$$\therefore naa = nab = nba \tag{3.48}$$

$$ana = a(b^-ab^-ab^-)(ab^-ab^-b) = ab^-ab^-ab^-b = anb \tag{3.49}$$

$$ana = (bb^-ab^-a)(b^-ab^-ab^-)a = bb^-ab^-ab^-a = bna \tag{3.50}$$

$$\therefore ana = bna = anb \tag{3.51}$$

Hence $a \bar{<} b$ with respect to $n \in a\{1\}$. □

Theorem 3.8. *For* $a, b \in \mathcal{T}$, $a \overset{\#}{<} b$ *then* $a \overset{-}{<} b$ *and* $b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = a$. *Conversely* $a \overset{-}{<} b$, $b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}beT = aeT$ *and* $Teb\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = Tea \Rightarrow a \overset{\#}{<} b$

Proof. Clearly $a \overset{\#}{<} b \Rightarrow a \overset{-}{<} b$ with respect to $\mathfrak{a}^{\#}$ and $b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = a$.
Assume that $a \overset{-}{<} b$ and $b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}beT = aeT$ hold.
By Theorem [3] since $a \overset{-}{<} b$ and $\mathfrak{b}^{\#} \in b\{1\}$
we have $a\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a = a, a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}b = b\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a = a$
$b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}beT = aeT \Rightarrow (a\mathfrak{a}^{\#}a\mathfrak{a}^{\#})b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b$
$\Rightarrow (a\mathfrak{b}^{\#}b\mathfrak{b}^{\#})a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = (a\mathfrak{b}^{\#}b\mathfrak{b}^{\#})b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b$
$\Rightarrow a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b$ (By Theorem 3.2)
$\Rightarrow (a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b)\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = (a\mathfrak{a}^{\#}a)\mathfrak{a}^{\#}b$

$$\Rightarrow a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b \tag{3.52}$$

Post multiply "$\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a$" on both sides
$\Rightarrow a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b(\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a) = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b(\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a)$
$\Rightarrow a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a$
Post multiply "$\mathfrak{a}^{\#}a$" on both sides
$\Rightarrow a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a(\mathfrak{a}^{\#}a) = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a(\mathfrak{a}^{\#}a)$

$$\Rightarrow a\mathfrak{a}^{\#}a = b\mathfrak{a}^{\#}a \tag{3.53}$$

Pre multiply "$a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}$" in Equation (3.32)
$(a\mathfrak{b}^{\#}b\mathfrak{b}^{\#})a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = (a\mathfrak{b}^{\#}b\mathfrak{b}^{\#})b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b$
$\Rightarrow a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = a$
Pre multiply "$a\mathfrak{a}^{\#}$" on both sides
$aa^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = a\mathfrak{a}^{\#}a$

$$\Rightarrow a\mathfrak{a}^{\#}b = a\mathfrak{a}^{\#}a \tag{3.54}$$

From Equations (3.53) and (3.54) we get
$\therefore a\mathfrak{a}^{\#}a = b\mathfrak{a}^{\#}a = a\mathfrak{a}^{\#}b$
Similarly, $Teb\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = Tea \Rightarrow b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b(\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a)$
Post multiply "$\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a$" on both sides
$\Rightarrow b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b(\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a) = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a(\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a)$
$\Rightarrow b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b$

$$\Rightarrow b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = a \tag{3.55}$$

Pre multiplying "$a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}$" in (3.55)on both sides
$(a\mathfrak{b}^{\#}b\mathfrak{b}^{\#})b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a = (a\mathfrak{b}^{\#}b\mathfrak{b}^{\#})b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b$

$$a = a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b \tag{3.56}$$

Pre multiplying "$\mathfrak{a}^{\#}$" and post multiplying "a" in (3.56)on both sides, $\mathfrak{a}^{\#}aa = \mathfrak{a}^{\#}(a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b)a$

$$\Rightarrow \mathfrak{a}^{\#}aa = \mathfrak{a}^{\#}ba \tag{3.57}$$

Pre multiplying "a" and post multiplying "$\mathfrak{a}^{\#}$" in (3.55)on both sides,
$\Rightarrow aa\mathfrak{a}^{\#} = a(b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a)\mathfrak{a}^{\#}$

$$\Rightarrow aa\mathfrak{a}^{\#} = ab\mathfrak{a}^{\#} \tag{3.58}$$

Multiply "$\mathfrak{a}^{\#}a$" in Equation(3.55)on both sides
$\mathfrak{a}^{\#}aa = \mathfrak{a}^{\#}a(b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b) = \mathfrak{a}^{\#}a(a\mathfrak{a}^{\#}a)\mathfrak{a}^{\#}b$
$= \mathfrak{a}^{\#}a(a\mathfrak{a}^{\#}b)\mathfrak{a}^{\#}b = (\mathfrak{a}^{\#}aa)\mathfrak{a}^{\#}b\mathfrak{a}^{\#}b = (\mathfrak{a}^{\#}ab)\mathfrak{a}^{\#}b\mathfrak{a}^{\#}b = \mathfrak{a}^{\#}ab$

$$\mathfrak{a}^{\#}aa = \mathfrak{a}^{\#}ab \tag{3.59}$$

Similarly post multiply "$a\mathfrak{a}^{\#}$" in Equation(3.55) on both sides
$aa\mathfrak{a}^{\#} = b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}ba\mathfrak{a}^{\#} = b\mathfrak{a}^{\#}a(\mathfrak{a}^{\#}ba)\mathfrak{a}^{\#} = b\mathfrak{a}^{\#}(a\mathfrak{a}^{\#}a)a\mathfrak{a}^{\#}$
$= b\mathfrak{a}^{\#}(b\mathfrak{a}^{\#}a)a\mathfrak{a}^{\#} = b\mathfrak{a}^{\#}b(\mathfrak{a}^{\#}aa) = b\mathfrak{a}^{\#}b\mathfrak{a}^{\#}ba\mathfrak{a}^{\#} = ba\mathfrak{a}^{\#}$

$$aa\mathfrak{a}^{\#} = ba\mathfrak{a}^{\#} \tag{3.60}$$

From Equations (3.58) and (3.59) we get $aa\mathfrak{a}^{\#} = ba\mathfrak{a}^{\#} = ab\mathfrak{a}^{\#}$
Similarly Equations (3.57) and (3.59) we get $\mathfrak{a}^{\#}aa = \mathfrak{a}^{\#}ab = \mathfrak{a}^{\#}ba$
Hence $a \overset{\#}{<} b$. □

Theorem 3.9. *For* $a, b \in \mathcal{T}$, $a \overset{\#}{<} b$ *then* $a \overset{-}{<} b$ *and* $\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} = \mathfrak{a}^{\#}$. *Conversely* $a \overset{-}{<} b$, $\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#}eT = \mathfrak{a}^{\#}eT$ *and* $Te\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} = Te\mathfrak{a}^{\#} \Rightarrow a \overset{\#}{<} b$

Proof. $a \overset{\#}{<} b \Rightarrow a \overset{-}{<} b$ with respect to '$\mathfrak{a}^{\#\prime}$ and $a^{\#} \overset{\#}{<} b^{\#}$ follows by Definition (3.1) and Theorem (3.3)
$a^{\#} \overset{\#}{<} b^{\#} \Rightarrow \mathfrak{a}^{\#} = \mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} = \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#} = \mathfrak{a}^{\#}a\mathfrak{b}^{\#}a\mathfrak{a}^{\#} = \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} = \mathfrak{b}^{\#}a\mathfrak{a}^{\#}b\mathfrak{a}^{\#}a\mathfrak{b}^{\#}$
Conversely, $a \overset{\#}{<} b \Rightarrow a\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a = a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}b = b\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a = a\mathfrak{b}^{\#}b\mathfrak{b}^{\#}a = ba\mathfrak{b}^{\#}b\mathfrak{b}^{\#}ab$ (by Theorem(3.2))

$\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#}eT = \mathfrak{a}^{\#}eT \Rightarrow \mathfrak{a}^{\#}(a\mathfrak{a}^{\#}a(\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#})) = \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#}$
$\Rightarrow \mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#}a\mathfrak{a}^{\#}\mathfrak{b}^{\#} = \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#}$

$$\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} = \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} \tag{3.61}$$

post multiplying "$a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}$" $\Rightarrow \mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#}(a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}) = \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#}(a\mathfrak{a}^{\#}a\mathfrak{a}^{\#})$
$\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#} = \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}$
$\mathfrak{a}^{\#} = \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}$
post multiplying "$a\mathfrak{a}^{\#}$" $\Rightarrow \mathfrak{a}^{\#}(a\mathfrak{a}^{\#}) = \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}(a\mathfrak{a}^{\#})$

$$\Rightarrow \mathfrak{a}^{\#}a\mathfrak{a}^{\#} = \mathfrak{b}^{\#}a\mathfrak{a}^{\#} \tag{3.62}$$

Pre multiply "$\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a$" $\Rightarrow (\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a)\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} = \mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#}$
$\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#} = \mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#}$
$\mathfrak{a}^{\#} = \mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#}$

$$(\mathfrak{a}^{\#}a)\mathfrak{a}^{\#} = (\mathfrak{a}^{\#}a)\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} \Rightarrow \mathfrak{a}^{\#}a\mathfrak{a}^{\#} = \mathfrak{a}^{\#}a\mathfrak{b}^{\#} \tag{3.63}$$

From Equations (3.62) and (3.63) we get
$\mathfrak{a}^{\#}a\mathfrak{a}^{\#} = \mathfrak{a}^{\#}a\mathfrak{b}^{\#} = \mathfrak{b}^{\#}a\mathfrak{a}^{\#}$

$$Te\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} = Te\mathfrak{a}^{\#} \Rightarrow \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} = \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#} \tag{3.64}$$

$\Rightarrow (\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a)\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} = (\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a)\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}$
$\Rightarrow \mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} = \mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}$
$\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} = \mathfrak{a}^{\#}$
$a(\mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#}) = a(\mathfrak{a}^{\#})\mathfrak{a}^{\#}$

$$\Rightarrow a\mathfrak{b}^{\#}\mathfrak{a}^{\#} = a\mathfrak{a}^{\#}\mathfrak{a}^{\#} \tag{3.65}$$

From Equation (3.64) $\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#}(a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}) = \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}(a\mathfrak{a}^{\#}a\mathfrak{a}^{\#})$
$\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#} = \mathfrak{a}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}$
$\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#} = \mathfrak{a}^{\#}$
$\mathfrak{a}^{\#}(\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#})a = \mathfrak{a}^{\#}(\mathfrak{a}^{\#})a$

$$\mathfrak{a}^{\#}\mathfrak{b}^{\#}a = \mathfrak{a}^{\#}\mathfrak{a}^{\#}a \tag{3.66}$$

From Equation (3.64) $\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} = \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#} \Rightarrow \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} = \mathfrak{a}^{\#}$
$(a\mathfrak{a}^{\#})\mathfrak{a}^{\#} = (a\mathfrak{a}^{\#})\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} = a(\mathfrak{a}^{\#}\mathfrak{a}^{\#}a)\mathfrak{a}^{\#}a\mathfrak{b}^{\#}$
$= a\mathfrak{a}^{\#}\mathfrak{a}^{\#}(a\mathfrak{a}^{\#}a)\mathfrak{b}^{\#} = a\mathfrak{a}^{\#}\mathfrak{a}^{\#}a\mathfrak{b}^{\#}a\mathfrak{b}^{\#} = a\mathfrak{a}^{\#}\mathfrak{b}^{\#}a\mathfrak{b}^{\#}a\mathfrak{b}^{\#} = a\mathfrak{a}^{\#}\mathfrak{b}^{\#}$

$$a\mathfrak{a}^{\#}\mathfrak{a}^{\#} = a\mathfrak{a}^{\#}\mathfrak{b}^{\#} \tag{3.67}$$

Similarly $\mathfrak{a}^{\#} = \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} \Rightarrow \mathfrak{a}^{\#}(\mathfrak{a}^{\#}a) = \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#}\mathfrak{a}^{\#}a$
$= \mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}\mathfrak{a}^{\#}a = \mathfrak{b}^{\#}a\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{a}^{\#}$
$= \mathfrak{b}^{\#}a\mathfrak{b}^{\#}a\mathfrak{b}^{\#}\mathfrak{a}^{\#}a = \mathfrak{b}^{\#}\mathfrak{a}^{\#}a$

$$\mathfrak{a}^{\#}\mathfrak{a}^{\#}a = \mathfrak{b}^{\#}\mathfrak{a}^{\#}a \tag{3.68}$$

From Equations (3.65) and (3.67) we get $a\mathfrak{a}^{\#}\mathfrak{a}^{\#} = a\mathfrak{a}^{\#}\mathfrak{b}^{\#} = a\mathfrak{b}^{\#}\mathfrak{a}^{\#}$
Similarly, from Equations (3.66) and (3.68) we get $\mathfrak{a}^{\#}\mathfrak{a}^{\#}a = \mathfrak{b}^{\#}\mathfrak{a}^{\#}a = \mathfrak{a}^{\#}\mathfrak{b}^{\#}a$
$\therefore \mathfrak{a}^{\#} \overset{\#}{<} \mathfrak{b}^{\#} \Rightarrow a \overset{\#}{<} b$ (by Theorem (3.3)
Hence $a \overset{\#}{<} b$. □

Theorem 3.10. *For $a, b \in \mathcal{T}$,the following are equivalent*

(i) $a \overset{\#}{<} b$

(ii) $a \overset{-}{<} b$ *and* $b\mathfrak{a}^{\#}a\mathfrak{a}^{\#}b = a$

(iii) $a \overset{-}{<} b$ *and* $\mathfrak{b}^{\#}a\mathfrak{a}^{\#}a\mathfrak{b}^{\#} = \mathfrak{a}^{\#}$

Theorem 3.11. *For $a, b \in \mathcal{T}$ if both a and b are symmetric then $a \overset{-}{<} b = b^3 \Rightarrow a^3 = a \overset{\#}{<} b$*

Proof. By [Theorem 3.3]$a \overset{-}{<} b \Rightarrow a = ab^{-}bb^{-}b = bb^{-}bb^{-}a = ab^{-}bb^{-}a = bab^{-}bb^{-}ab$
By [Theorem 3.4] $a \overset{-}{<} b$ with b idempotent $\Rightarrow a$ idempotent
$\because a = bb^{-}bb^{-}a \Rightarrow (bb)a = (bb)bb^{-}bb^{-}a = b^3b^{-}bb^{-}a = bb^{-}bb^{-}a = a = a^T$
$a = bba \Rightarrow a^T = (bba)^T = a^Tb^Tb^T = abb$
Hence $abb = bba = a = a^3$
$\Rightarrow a \overset{\#}{<} b$ (By Theorem (3.6)) □

4. CONCLUSION

The study of group inverses in a ternary semiring is continued in this publication. We have examined a number of fascinating characteristics of ideals in ternary semiring.

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