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Study the Relations for Different Components of Isospin with Quark States

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Abstract

Starting with the quaternionic formulation of isospin SU (2) group, we have derived the relations for different components of isospin with quark states. Extending this formalism to the case of SU (3) group we have considered the theory of octonion variables. Accordingly, the octonion splitting of SU(3) have been reconsidered and various commutation relations for SU(3) group and its shift operators are also derived for different isospin multiplets i.e I, U, V spins, with quark states. The isospin symmetries are good approximation to simplify the interaction among hadrons. As such, normed algebras namely the algebra of complex numbers, quaternions and octanions play an important role for physical interpretation of standard model for electro-weak interactions and quantum chromodynamics (QCD).

Keywords: SU(2), SU(3), Quaternions, Octonions and Gell-Mann matrices.

INTRODUCTION

According to celebrated Hurwitz theorem [1] there exists four- division algebra consisting of $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions) and $\mathbb{O}$ (octonions). All four algebras are alternative with antisymmetric associators. Quaternions were very first example of hyper complex numbers introduced by Hamilton [2,3] and octonions by Caley [4] and Graves [5] having the significant impacts on mathematics and physics. Quaternions are also described in terms of Pauli spin- matrices for the non-Abelian gauge theory in order to unify [6]

electromagnetism and weak forces within the electroweak $SU(2) \times U(1)$ sector of standard model. Yet another complex system (i.e. Octonion) [7,8,9,10,11] also plays an important role in understanding the physics beyond the strong interaction between color degree of freedom of quarks and their interaction. A detailed introduction of the various aspects and applications of exceptional, Jordan, Division, Clifford and non commutative as well as non associative algebras related to octonions has recently been discussed by Castro[12] by extending the Octonionic geometry (gravity) developed earlier by Marques-Oliveira [13,14]. Recently we have also developed [15] the quaternionic formulat6ion of Yang-Mill's field equations and octonion reformulation of quantum chromo dynamics (*QCD*) where the resemblance between quaternions and *SU*(2) and that of octanions and *SU*(3) gauge symmetries has been discussed. The color group $SU(3)_C$ is embedded within the octonionic structure of the exceptional groups while the *SU*(3) flavor group has been discussed in terms of triality property of the octonion algebra. Keeping in view, the utility of quaternions and octonions, in this paper, we have made an attempt to discuss the quaternionic formulation of isospin *SU*(2) group and octonion reformulation of *SU*(3) flavor group (parallel to the *SU*(3) color group [15]. Accordingly, we have derived the relations for different components of isospin with quark states. Extending this formalism to the case of *SU*(3) flavor group, the octonion splitting of *SU*(3) group have been reconsidered and various commutation relations for *SU*(3) group and its shift operators are also derived and verified in terms of different iso-spin multiplets i.e. *I, U* and *V*- spins.

COMPONENT OF ISOSPIN WITH QUARK STATES

Let us consider the standard Lie algebra in terms of the quaternions by adopting the basic formula of the isotopic formalism for nucleons. Accordingly we may write a spinors ψ_a and ψ_b as

$$\psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}; \tag{1}$$

Where as the adjoint spinor is described as

$$\psi = \left(\overline{\psi_a}\ \overline{\psi_b}\right); \tag{2}$$

where

$$\psi = (\psi_0 + e_1\psi_1) + e_2(\psi_2 - e_1\psi_3) = \psi_a + e_2\psi_b. \tag{3}$$

In equations (1) and (2), $\psi_a = (\psi_0 + e_1\psi_1)$ and $\psi_b = (\psi_2 + e_1\psi_3)$ are considered over the field of a quaternion described in terms of the four dimensional representations of real numbers and two dimensional representations of complex numbers. Accordingly, under *SU*(2) gauge symmetry, the quaternion spinor transforms as [15, 16],

$$\psi \longmapsto \psi^1 = U\psi \tag{4}$$

where *U* is 2×2 unitary matrix and satisfies

$$U^{+}U = UU^{\dagger} = UU^{-1} = U^{-1}U = 1. \quad (5)$$

On the other hand, the quaternion conjugate spinor transforms as

$$\bar{\psi} \longmapsto \bar{\psi}^{1} = \bar{\psi}\, U^{-1}. \quad (6)$$

Hence, the combination $\psi\,\bar{\psi} = \bar{\psi}\,\psi = \psi\,\bar{\psi}^{1} = \bar{\psi}^{1}\psi$ is an invariant quantity. So any unitary matrix may then be written as

$$U = \exp(\acute{\imath}\,\hat{H}); \quad (7)$$

Where $\hat{H}$ is Hermitian $\hat{H}^{+} = \hat{H}$. Thus, we may express the Hermitian 2 × 2 matrix in terms of four real numbers, a_1, a_2, a_3 and Θ as

$$\hat{H} = \Theta\hat{1} + \mathcal{T}_j\, a_j = \Theta 1 + \acute{\imath} e_j a_j; \quad (8)$$

Where $\hat{1}$ is the 2 X 2 unit matrix, $\mathcal{T}_j$ are well known 2 X 2 Pauli-spin matrices and e_1, e_2, e_3 are the quaternion units related to Pauli-spin matrices as

$$e_0 = 1;\ \ e_j = -\acute{\imath}\mathcal{T}_j. \quad (9)$$

Here e_j ($\forall_j$ = 1,2,3), the basis elements of quaternion algebra, satisfy the following multiplication rule

$$e_j e_k = -\delta_{jk} + \epsilon_{jkl}\, e_l\ (\forall_j, k, l = 1,2,3). \quad (10)$$

where δ_{jk} and ϵ_{jkl} are respectively known as Kronecker delta and three index Levi-Civita symbols respectively. We may now describe the *SU*(2) isospin in terms of quaternions as

$$I_a = \frac{ie_a}{2}\ (\forall_a = 1,2,3) \quad (11)$$

and

$$I_{\pm} = \frac{i}{2}(e_1 \pm ie_2) \quad (12)$$

So, we may write the quaternion basis elements in terms of *SU*(2) isospin as

$$e_1 = \frac{1}{i}(I_+ + I_-);\ e_2 = \frac{1}{i}(I_+ - I_-);\ e_3 = \frac{1}{i}(I_3); \quad (13)$$

which satisfy the following commutation relation

$$[I_+ + I_-] = ie_3;\ [I_3 + I_{\pm}] = \pm\frac{i}{2}(e_1 \pm ie_2). \quad (14)$$

Here *SU*(2) group acts upon the fundamental representation of *SU*(2) doublets of up (*u*) and down (*d*) quark spinors

$$|u\rangle = \begin{pmatrix}1\\0\end{pmatrix};\ |d\rangle = \begin{pmatrix}0\\1\end{pmatrix}. \quad (15)$$

So, we for up quarks

$$I_+ |u\rangle = \frac{i(e_1 + ie_2)}{2}|u\rangle = 0;\ I_- |u\rangle = \frac{i(e_1 - ie_2)}{2}|u\rangle$$
$$= \frac{1}{2}|d\rangle;\ I_3 |u\rangle = \frac{ie_3}{2}|u\rangle = \frac{1}{2}|u\rangle; \quad (16)$$

and for down quarks we have

$$I_+ |d\rangle = \frac{i(e_1 + ie_2)}{2}|d\rangle = \frac{1}{2}|u\rangle;\ I_- |d\rangle = \frac{i(e_1 - ie_2)}{2}|d\rangle = \frac{1}{2}|d\rangle = 0;\ I_3|d\rangle$$
$$= \frac{ie_3}{2}|d\rangle = -\frac{1}{2}|d\rangle; \quad (17)$$

Conjugates of equations (16) and (17) are now be described as

$$\langle d|I_+ = \langle d|\frac{i(e_1 + ie_2)}{2} = 0;\ \langle d|I_- = \langle d|\frac{i(e_1 - ie_2)}{2} = \frac{1}{2}\langle u|;\ \langle d|I$$
$$= \langle d|\frac{ie_3}{2} = \frac{1}{2}\langle d|; \quad (18)$$

and

$$\langle u|I_+ = \langle u|\frac{i(e_1 + ie_2)}{2} = \frac{1}{2}\langle d|;\ \langle u|I_- = \langle u|\frac{i(e_1 - ie_2)}{2} = 0;\ \langle u|I_3$$
$$= \langle u|\frac{ie_3}{2} = \frac{1}{2}\langle u|; \quad (19)$$

Where e_1, e_2 and e_3 are defined in (11) and (12). The effect of quaternion operator on up $|u\rangle$ and down $|d\rangle$ quarks states leads to

$$ie_1 |u\rangle = |d\rangle;\ ie_1 |d\rangle = |u\rangle;$$
$$e_2 |u\rangle = |d\rangle;\ e_2 |d\rangle = -|u\rangle$$
$$ie_3 |u\rangle = |u\rangle;\ ie_3 |d\rangle = -|d\rangle. \quad (20)$$

So, we may write

$$e_1 \begin{pmatrix} u \\ d \end{pmatrix} = i \begin{pmatrix} u \\ d \end{pmatrix}; \quad (21)$$

$$e_2 \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} d \\ -u \end{pmatrix}; \quad (22)$$

$$e_3 \begin{pmatrix} u \\ d \end{pmatrix} = i \begin{pmatrix} u \\ -d \end{pmatrix}; \quad (23)$$

transformation a neutron (down quark) state into a proton (up quark) state or vice versa. Only e_2 gives real doublets of up and down quarks.

OCTONIONS AND GELMANN Λ MATRICES

Octonions describe the widest normed algebra after the algebra of real numbers, complex numbers and quaternions. The octonions are non associative and non commutative normed division algebra over the algebra of real numbers. A set of

octets (e_1, e_2, e_3, e_4, e_5, e_6, e_7) are known as the octonion basis elements and satisfy the following multiplication rules

$$e_0 = 1;\ e_0 e_A = e_A e_0 = e_A;$$
$$e_A e_B = -\delta_{ABe0} + f_{ABCeC}.\ (\forall\ A, B, C = 1, 2, \ldots.., 7). \quad (24)$$

The structure constants f_{ABC} is completely anti-symmetric and takes the value 1 for the following combinations.

$f_{ABC} = +1;\ \forall(ABC) = (123), (471), (257), (165), (624), (543), (736).$ (25)

The relation between Gell-Mann λ matrices and octonion units are given as [15]

$$\frac{[e_{a+3}, e_7]}{[\lambda_{a+3}, \lambda_7]} = \frac{e_a}{2i\lambda_a} \quad (26)$$

$$\frac{[e_7, e_a]}{\lambda_7, \lambda_a]} = \frac{e_{a+3}}{2i\lambda_{a+3}}; \quad (27)$$

$$\frac{[e_a, e_{a+3}]}{\lambda_a, \lambda_{a+3}]} = \frac{e_7}{2i\lambda_7}; \quad (28)$$

Where a= 1,2,3. So, we have may describe [15] the following relationship between Gell-Mann λ matrices and octonion units as,

$\lambda_1 = -e_1\ k1;\ \lambda_2 = -ie_2\ k_2;\ \lambda_3 = -ie_3\ k_3;\ \lambda_4 = -ie_4\ k_4;\ \lambda_5 = -ie_5\ k_5;\ \lambda_6 = -e_6\ k_6;\ \lambda_7 = -ie_7\ k_7;$ (29)

and λ_8 are also related with e_3 as

$$\lambda_8 = -ik_8 e_3 \text{ where } k_8 = \frac{8}{\sqrt{3}}; \quad (30)$$

where $k_a = -1$ ($\forall a = 1,2,3,\ldots,7$) are proportionality constants.

SU (3) FLAVOR GROUP AND I, U, V –SPIN FORMULATION

The Lie algebra of *SU* (3) exhibits most of the features of the larger Lie algebras. The elements of *SU*(3) group may be obtained in terms of 3 ✕3 Hermitian Gell Mann λ Hermitian matrices related to octonions where first three matrices describe the familiar isotopic spin generators from the *SU*(2) subgroup of *SU*(3). These generators connect up (*u*) and down quarks (*d*). The fourth and fifth generators and the sixth and seventh generators are denoted as the *V* – spin and the *U* – spin. *V* – spin connects the up (*u*) and strange quarks (*s*) while *U* – spin connects the down (*d*) quarks (*s*). the eighth generator is diagonal in nature responsible for hypercharge. The *I*, *U* and *V* – spin algebra fulfills the angular momentum algebra and turn out to the sub algebras of *SU* (3). Hence, the *SU*(3) multiplets are constructed in form of a *I* – multiplets, *V* multiplets and an *U* – multiplets. The *I*- spin, *U* – spin and *V* – spin algebra are closely related and are the elements of sub algebra of *SU*(3). As we may define *SU*(3) multiplets as

$I_1 = \frac{1}{2}\lambda_1 = \frac{ie_1}{2}$; $I_2 = \frac{1}{2}\lambda_2 = \frac{ie_2}{2}$; $I_3 = \frac{1}{2}\lambda_3 = \frac{ie_3}{2}$ (I – *Spin*)

$V_1 = \frac{1}{2}\lambda_4 = \frac{ie_4}{2}$; $V_2 = \frac{1}{2}\lambda_5 = \frac{ie_5}{2}$; $V_3 = \frac{ie_3}{4}$ ($8\sqrt{3}$ +1) (U– *Spin*)

$U_1 = \frac{1}{2}\lambda_6 = \frac{ie_6}{2}$; $U_2 = \frac{1}{2}\lambda_7 = \frac{ie_7}{2}$; $U_3 = \frac{ie_3}{4}$ ($8\sqrt{3}$ +1) (V – *Spin*) (31)

along with the hyper charge is written in terms of octonions as follows

$$Y = \frac{1}{\sqrt{3}}\lambda_8 = -\frac{8ie_3}{\sqrt{3}}. \quad (32)$$

Here I_1, I_2 and I_3 contain the 2 ✕2 isospin operators (i.e. quaternion units). U_3, V_3, I_3 and Y are linearly independent generators and are simultaneously diagonalized. It will to be noted that λ_1, λ_2, λ_3 agree with σ_1, σ_2, σ_3. The matrices V_1, V_2, U_3 and U_2 connect the nucleon and Λ and changes strangeness by one unit and isotopic spin by a half unit. The operators $I\pm$, $U\pm$, $V\pm$ are also the shift operators known as raising (I_+, U_+, V_+) and lowering (I_-, U_-, V_-) operators. $I_\pm$, $U_\pm$, $V_\pm$ are called as ladder operators. The complexified variants contain the third operators $T\pm$, $U\pm$, $V\pm$ which characterizes the states of $SU(3)$ multiplets. The operators $I_\pm$, $U_\pm$, $V_\pm$ are defined as

$$I_\pm = I_x \pm iI_y = \frac{1}{2}(\lambda_1 \pm i\lambda_2) = \frac{i}{2}(e_1 \pm ie_2). \quad (33)$$

$$V_\pm = V_x \pm iV_y = \frac{1}{2}(\lambda_4 \pm i\lambda_5) = \frac{i}{2}(e_4 \pm ie_5). \quad (34)$$

$$U_\pm = U_x \pm iU_y = \frac{1}{2}(\lambda_6 \pm i\lambda_7) = \frac{i}{2}(e_6 \pm ie_7). \quad (35)$$

tates of these multiplets as $|q_1\rangle\,|q_2\rangle\,|q_3\rangle$. So the quark states of I, U and V spin are described as

$$I_-|q_1\rangle = |q_2\rangle;\ I_+|q_2\rangle = |q_1\rangle. \quad (36)$$

$$U_-|q_2\rangle = |q_3\rangle;\ U_+|q_3\rangle = |q_2\rangle. \quad (37)$$

$$V_-|q_1\rangle = |q_3\rangle;\ V_+|q_3\rangle = |q_1\rangle. \quad (38)$$

Thus, the operators $I_\pm$, $U_\pm$, $V_\pm$ are viewed as operators which transforms one flavor into another flavor of quarks

$$\begin{aligned} I_\pm\,(I_3) &\longmapsto I_3 \pm 1; \\ V_\pm\,(V_3) &\longmapsto V_3 \pm 1; \\ U_\pm\,(U_3) &\longmapsto U_3 \pm 1. \end{aligned} \quad (39)$$

It means the action of $I_\pm$, $U_\pm$, and $V_\pm$ shifts the values of I_3, V_3 and U_3 by ± 1.

COMMUTATION RELATIONS FOR SHIFT OPERATORS

The I, U and U- spin algebras are closed. Let us obtain the commutation relations of shift operators $I_\pm$, $U_\pm$, $V_\pm$ [16] for $SU(3)$ group.

The commutation relation between the third components of I, *U*, and $I_{\pm}$ are given as

$$[I_3, U_{\pm}] = \pm \left[\frac{i}{2} e_1 \pm i e_2\right] = \pm I_{\pm};$$

$$[U_3, I_{\pm}] = \mp \left[\frac{i}{2} e_1 \pm i e_2\right] = \mp \frac{1}{2} I_{\pm};$$

$$[V_3, I_{\pm}] = \pm \left[\frac{1}{2} e_1 \pm i e_2\right] = \pm \frac{1}{2} I_{\pm}; \quad (40)$$

The octonions are related to raising I_+, U_+, V_+ and lowering I_-, U_-, V_- operators as

$$e_1 = I\,(I_+ + I_-);\ e_2 = I\,(I_+ - I_-);\ e_3 = 2I_3;$$

$$e_4 = i\,(V_+ + V_-);\ e_5 = (V_+ - V_-);$$

$$e_6 = i\,(U_+ + U_-);\ e_7 = (I_+ + V_-). \quad (41)$$

the commutation relations between I_+ and I_-,U_+ and U_- and V_+ and V_- are described as

$$[I_+, I_-] = ie_3 = 2I_3;$$

$$[U_+, U_-] = \frac{ie_3}{2}(8\sqrt{3}-1) = 2U_3;$$

$$[V_+, V_-] = \frac{ie_3}{4}(8\sqrt{3}+1) = 2V_3. \quad (42)$$

CONCLUSION

In the foregoing analysis, we have described the resemblance between real Lie algebra *SU*(2) and the quaternion formalism of isospin *SU*(2) group by using the isomorphism between quaternion basis elements and Pauli matrices. On the same footing, we have established the connection between octonion basis elements and Gell Mann λ matrices of *SU*(3) flavor group. Accordingly, we have discussed the various sub algebras of *SU*(3) flavor group constructed in terms three *SU*(2) algebras of *I*, *U* and *V* – spin multiplets. It therefore appears that the *SU*(3) flavor group may represent a new fundamental symmetry using octonions for classifying hadrons within *SU*(3) multiplets. The isospin symmetries are good approximation to simplify the interaction among hadrons. The motivation behind the present theory was to develop a simple compact and consistent algebraic formulation of *SU*(2) and *SU*(3) symmetries in terms of normed algebras namely quaternions and octonions. So, we have described the compact simplified notations instead of using the Pauli and Gellmann matrices respectively got *SU*(2) and *SU*(3) groups. Octonion representation of *SU*(3) flavor group directly establishes the one to one mapping between the non-associativity and

the theory of strong interactions. It also shows that the theory of hadrons (or quark-colour etc.) has the direct link with non-associativity (octonions) while the isotopic spin leads to non commutativity (quaternions). So, we have derived and verified the commutation relations of shift operators and generators of *SU*(3) flavor group using octonions.

As such normed algebras namely the algebra of complex numbers, quaternions and octonions play an important role for the physical interpretation of quantum electrodynamics (QED), standard model of EW interactions and quantum chromo dynamics (QCD). The coupling constant of the monopole-monopole interaction is as large as $1/4\alpha_E \simeq 34$. It has to be examined how the arising difficulties that concern unitarity and non-locality can be solved. A similar problem arose in meson theory. The coupling constant of the pion is as large as 14.4 . The arising problems were solved by the quark substructure of hadrons . So $U(1)\times U'(1)$ model has to be embedded in a Grand Unified theory and one may imagine the underlying group to be $SU(5) \times SU'(5)$, where the primed group describes the hypothetical magnetic photon, (chromo-)magnetic gluons and (iso-)magnetic W, Z, X, and Y bosons.

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