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Re-arranging the Information Entropy

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Abstract

The position space information entropy of harmonic oscillator is evaluated using supersymmetric quantum mechanics techniques. Using isospectral Hamiltonian approach, a family of isospectral potentials is constructed having same energy eigenvalues as that of the original potential. Some interesting features of the entropy densities are graphically demonstrated and the information entropy content is obtained in position space for the potential. It is shown that the information entropy content in each level can be re-arranged as a function of deformation parameter.

1. INTRODUCTION

Information entropy is an important quantity employed to study the quantum mechanical systems. It forms a framework for the proper understanding of quantum communication and quantum computation. It provides a measure of information about the probability distribution in position and momentum space. The Shannon information entropy of the single particle distribution in position space is

$$S_{pos} = -\int \rho(r)\, ln\rho(r) dr \tag{1}$$

and in corresponding momentum space

$$S_{mom} = \int \rho(\,p) ln\rho(p) dp \tag{2}$$

where $\rho(\rho)$ denote the momentum space particle density. The position space and momentum space information entropies for various systems like the hyperbolic

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double-well potential [1], a squared tangent potential well [2], the hyperbolic potential function [3], an infinite circular potential well [4], the position dependent mass Schrodinger equation [5], the Poschl-Teller potential [6], isospectral potential [7], the Kratzer [8], the harmonic oscillator potential [9], Eckart potential [10] and ultracold trapped interacting bosons [11] have been recently studied. Information entropy plays a crucial role in a stronger formulation of the uncertainty relations. The information theoretic uncertainty relations were first conjectured by Hirschman [12] and Everett [13] in the context of many worlds interpretation and proved by Bialynicki-Birula and Mycielski (BBM) [14]. From the general properties of Fourier transform, it was proved in a d-dimensional system for wave functions normalized to unity,

$$S_{pos} + S_{mom} \geq d(1 + ln\,\pi) \qquad (3)$$

Though S_{pos} and S_{mom} are individual unbounded, their sum is bounded from below. The total sum of information entropy in position space and momentum space is minimum for the ground state of harmonic oscillator. The physical meaning of the inequality is that an increase of S_{mom} corresponds to a decrease of S_{pos} and vice-versa, which indicates that a diffused density distribution $\rho(\rho)$ in momentum space is associated with a localized density distribution $\rho(r)$ in configuration space. A framework for deriving uncertainty relations of the above type, between general dynamical variables, not necessarily canonically conjugate ones, have been given in reference [17]. A more general formulation of information theoretic uncertainty relations, which incorporates a pair of arbitrary quantum measurements have also been given [18]. It is interesting to know the value of information entropy which is a measure of the spatial spread of the wave function for the various states of different systems. For the simple harmonic oscillator, the information entropies were exactly calculated for the ground state in both coordinate and momentum state, for which the BBM inequality is saturated [19]. The information entropies in various contexts, e.g. mathematical physics, atomic and molecular physics, information theory, chemistry, statistics and statistical mechanics and other areas of physics have been extensively analyzed in recent times[20-25].

We use the isospectral Hamiltonian approach to study the isospectral wave functions and their entropies. Two Hamiltonians are said to be strictly isospectral, if they have exactly same energy eigen value spectrum and S-matrix [26-28], whereas the wave functions and their dependent quantities are different. Though the idea of generating isospectral Hamiltonians using the Gelfand-Levitan approach or the Darboux procedure were known for some time, the supersymmetric quantum mechanical techniques make the procedure look simpler [29-31]. When one deletes a bound state of a given potential V(x) and re-introduce the state, it involves solving a first order differential equation. Thus, a set of one-dimensional family of potentials $\hat{V}(x,c)$can be constructed which have the exactly same energy spectrum as that of V(x). In general, for any one dimensional potential with n bound states, one can construct an n-parameter family of strictly isospectral potentials, i.e. potentials with eigenvalues, reflection and transmission coefficients identical to those for original potential. This aspect has been utilized profitably in many physical situations, which are of interest to various fields [32-37]. In this paper, we consider the harmonic oscillator potential and calculate the position space information entropy exactly for ground state and excited states. Using isospectral Hamiltonian approach, the deformed potential and their wave functions are constructed and used to calculate the information entropy for the isospectral potential. The information density of the isospectral potential is graphically demonstrated in position space. In the last section, we conclude with brief discussion.

2. ISOSPECTRAL HAMILTONIAN APPROACH

The isospectral Hamiltonian are constructed using the framework of supersymmetric quantum mechanics. If we choose the ground state wave function (ψ_0) to be zero, then the Hamiltonian can be factorized [31] as $H_1 = A^\dagger A$ where (in units $\hbar = 2m = 1$), $A = \frac{d}{dx} + W(x)$ and $A^\dagger = -\frac{d}{dx} + W(x)$ are the supersymmetric operators and $W(x) = -\frac{d}{dx}[\ln \psi_0(x)]$ is called the superpotential. We have

$$H_1\psi_n = A^\dagger A\psi_n = \varepsilon_n\psi_n, \tag{4}$$

$$AA^\dagger(A\psi_n) = \varepsilon_n(A\psi_n),$$

$$H_2(A\psi_n) = \varepsilon_n(A\psi_n). \tag{5}$$

Here H_2 is the supersymmetric partner Hamiltonian of H_1, with eigenfunction $\chi_n = A\psi_n$. It is obvious that H_2 has the same eigenvalue spectrum as that of H_1, but for the case $A\psi_0 = 0$, which is the case of supersymmetry broken. Explicitly, The relation between Hamiltonians reads,

$$E_n^{(2)} = E_{n+1}^{(1)}; \quad E_0^{(1)} = 0,$$

$$\psi_n^{(2)} = [E_{n+1}^{(1)}]^{-\frac{1}{2}} A\psi_{n+1}^{(1)},$$

$$\psi_{n+1}^{(1)} = [E_n^{(2)}]^{-\frac{1}{2}} A^\dagger\psi_n^{(2)},$$

The superpotential relates the supersymmetric partner potentials $V_1(x)$ and $V_2(x)$ as

$$V_{1,2}(x) = W^2(x) \mp \frac{dW}{dx}. \tag{6}$$

It is well known that for the potential $V_2(x)$, the original potential $V_1(x)$ is not unique . The argument is as follows. Suppose H_2 has another factorization $BB^\dagger$, where $B = \frac{d}{dx} + \hat{W}(x)$, then, $H_2 = AA^\dagger = BB^\dagger$ but $H_1 = B^\dagger B$ is not $A^\dagger A$ rather it defines a certain new Hamiltonian. For superpotential $\hat{W}(x)$, the partner potential $V_2(x)$ is

$$V_2(x) = \hat{W}^2(x) + \hat{W}'(x). \tag{7}$$

Consider the most general solution as $\hat{W}(x) = W(x) + \phi(x)$, which demands that,

$$\phi^2(x) + 2W(x)\phi(x) + \phi'(x) = 0. \tag{8}$$

The solution of the above equation is $\phi(x) = \frac{d}{dx}\ln[I(x)+c]$, where $I(x) = \int_{-\infty}^{x} \psi_0^2(x')dx'$ and c is a constant. Therefore, we obtain,

$$\hat{W}(x) = W(x) + \frac{d}{dx}\ln[I(x)+c]. \tag{9}$$

The corresponding one-parameter family of potentials $\hat{V}_1(x,c)$ is given as

$$\hat{V}_1(x,\lambda) = V_1(x) - 2\frac{d^2}{dx^2}(\ln(I(x)+c)). \tag{10}$$

When $c \to \pm\infty$, then $\hat{V}_1(x,c) \to V_1(x)$. The normalized ground state wave function corresponding to the potential $\hat{V}_1(x,c)$ reads,

$$\hat{\psi}_0(x,c) = \frac{\sqrt{c(1+c)}\,\psi_0(x)}{I(x)+c}, \tag{11}$$

where $c \notin (0,-1)$. The equations (10) and (11) represent the one-parameter family of isospectral potentials and wave functions, which shall be used to obtain the information entropy of harmonic oscillator as a function of deformation parameter.

3. INFORMATION ENTROPY OF HARMONIC OSCILLATOR

The one dimensional Harmonic oscillator is an interesting mathematical and physical problem in quantum mechanics. The potential is given as,

$$V(x) = \frac{1}{2}\lambda^2 x^2 \tag{12}$$

One dimensional Harmonic has only one quantum number and the energy of the system is $E_n = \lambda(n + \frac{1}{2})$. The position space eigenfunctions in ground and excited states are

$$\psi_0(x) = \left(\frac{\lambda}{\pi}\right)^{1/4} e^{-\frac{1}{2}\lambda x^2} \tag{13}$$

$$\psi_n(x) = \left(\frac{\lambda}{\pi}\right)^{1/4}\left(\frac{1}{2^n n!}\right)^{1/2} e^{-\frac{1}{2}\lambda x^2} H_n\left(\sqrt{\lambda}x\right) \tag{14}$$

Where $H_n(x)$ are the Hermite polynomials. The corresponding eigenfunctions in momentum space are

$$\psi_n(p) = \left(\frac{1}{\lambda\pi}\right)^{1/4}\left(\frac{1}{2^n n!}\right)^{1/2} e^{-\frac{p^2}{2\lambda}} H_n\left(p/\sqrt{\lambda}\right) \tag{15}$$

For the ground state, information entropies in the position and momentum spaces are [18]

$$S_{pos} = \frac{1}{2} - \frac{1}{2}\ln\left[\frac{\lambda}{\pi}\right] \tag{16}$$

$$S_{mom} = \frac{1}{2} + \frac{1}{2}\ln[\lambda\pi] \tag{17}$$

For excited states

$$S_{pos} = \ln\left[\sqrt{\frac{\pi}{\lambda}}\, 2^n n!\right] + n + \frac{1}{2} - \frac{1}{\sqrt{\pi}\, 2^n n!} J_1 \tag{18}$$

$$S_{mom} = \ln\left[\sqrt{\pi\lambda}\, 2^n n!\right] + n + \frac{1}{2} - \frac{1}{\sqrt{\pi}\, 2^n n!} J_1 \tag{19}$$

where

$$J_1 = \int_{-\infty}^{\infty} e^{-t^2} [H_n(t)]^2 \ln [H_n(t)]^2 \, dt$$

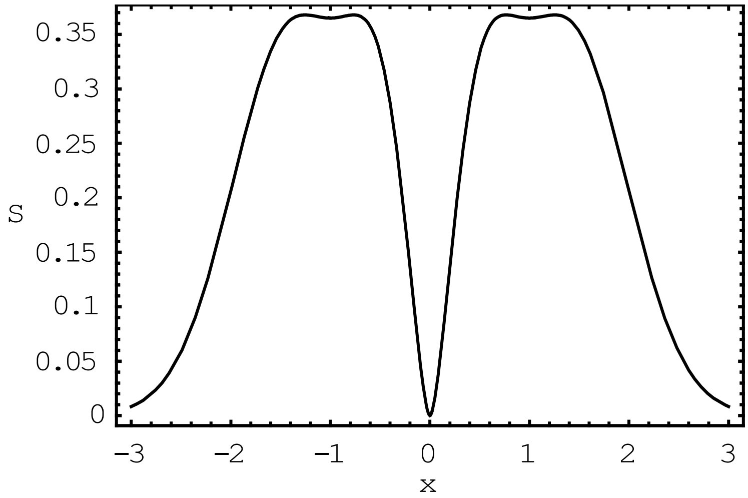

Figure 1: Position space information density of harmonic oscillator in the first excited state.

The entropy density for position space is plotted in the figure 1. The position space information density plot has a dip on both sides at its peak as shown in figure 2.

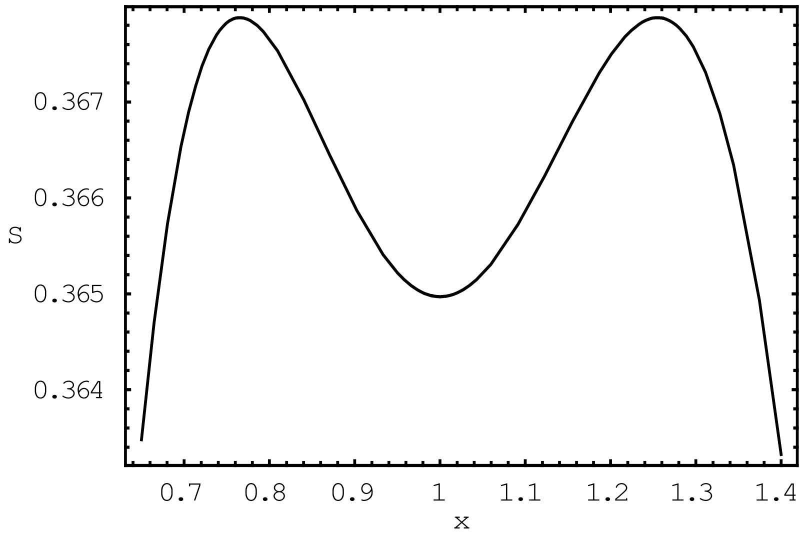

Figure 2: Local minima observed in position space information density of the harmonic oscillator in the first excited state.

4. INFORMATION ENTROPY OF ISOSPECTRAL HARMONIC OSCILLATOR

Using isospectral Hamiltonian approach, ground state wave function is obtained as

$$\hat{\psi}_0(x,c) = \frac{2\sqrt{c(c+1)}\left(\sqrt{\frac{\lambda}{\pi}}\right)^{1/2} e^{-\frac{1}{2}\lambda x^2}}{2c+1+erf(x\sqrt{\lambda})} \tag{20}$$

where erf(x) is the error function. The information entropy in position space for ground state of the isospectral potential is obtained after some calculation as

$$\hat{S}_0 = 4c(c+1)\left\{\frac{1}{c}\ln[2c] - \frac{1}{c+1}\ln[2c+2] - \frac{1}{4c(c+1)}\left(\ln\left[\frac{4c(c+1)}{\sqrt{\pi}}\right] - 2\right) + \int_{-\infty}^{\infty}\frac{x}{2c+1+erf(x)}dx\right\} \tag{21}$$

S_0 increases with the deformation parameter and saturates at undeformed value 1.70. The excited state isospectral wave function is calculated as

$$\hat{\psi}_n(x,c) = \left(\sqrt{\frac{\lambda}{\pi}}\right)^{1/2}\left(\frac{1}{2^n n!}\right)^{1/2} e^{\frac{-1\lambda x^2}{2}}\left[\frac{8n e^{-\lambda x^2} H_{n-1}(x\sqrt{\lambda})}{\sqrt{\pi}(2n+1)(2c+1+erf(x\sqrt{\pi})}] + H_n(x\sqrt{\lambda})\right] \tag{22}$$

The energy density in position space for excited states of the harmonic potential is

$$\hat{E} = \left(\sqrt{\frac{\lambda}{\pi}}\right)\left(\frac{1}{2^{n-1}n!}\right)e^{-\lambda x^2}\left[\frac{8n e^{-\lambda x^2} H_{n-1}(x\sqrt{\lambda})}{\sqrt{\pi}(2n+1)(2c+1+erf(x\sqrt{\lambda})} + H_n(x\sqrt{\pi})\right]^2$$
$$\ln\left[\sqrt{\frac{\lambda}{\pi}}\right]^{1/2}\left(\frac{1}{2^n n!}\right)^{1/2} e^{-\frac{1}{2}\lambda x^2}\left[\frac{8n e^{-\lambda x^2} H_{n-1}(x\sqrt{\lambda})}{\sqrt{\pi}(2n+1)(2c+1+erf(x\sqrt{\lambda})} + H_n(x\sqrt{\pi})\right] \tag{23}$$

The information density in position space for first excited state of the harmonic potential is plotted in figure 3 for different values of deformation parameter.

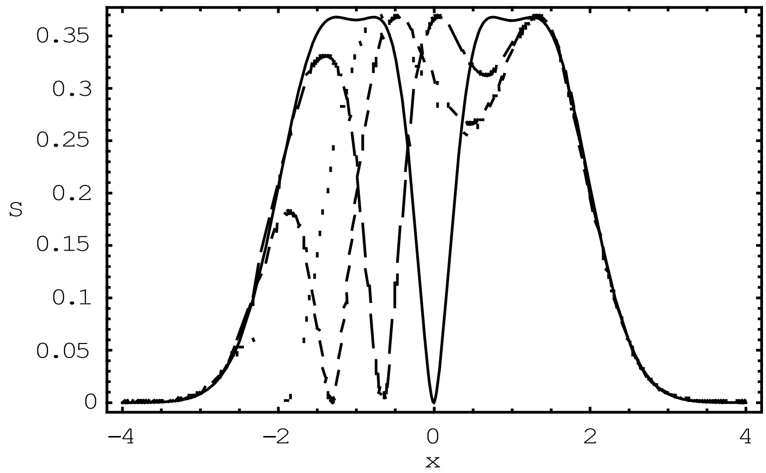

Figure 3: Position space information density of harmonic oscillator in the first excited state for deformation parameter c = 0.2 (dashed line), c = 0.02 (small dashed line), c = 0.002 (dotted line) and solid line for undeformed case.

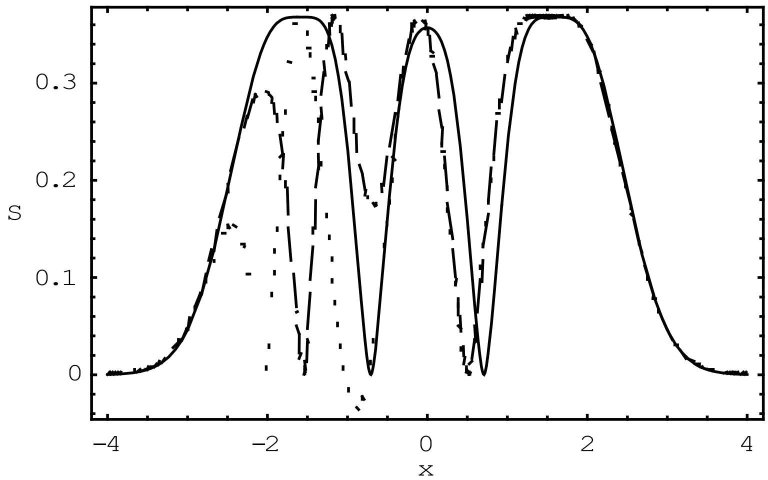

Figure 4: position space information density of harmonic oscillator in the second excited state for deformation parameter c = 0.02 (dashed line), c = 0.002 (dotted line) and solid line for undeformed case.

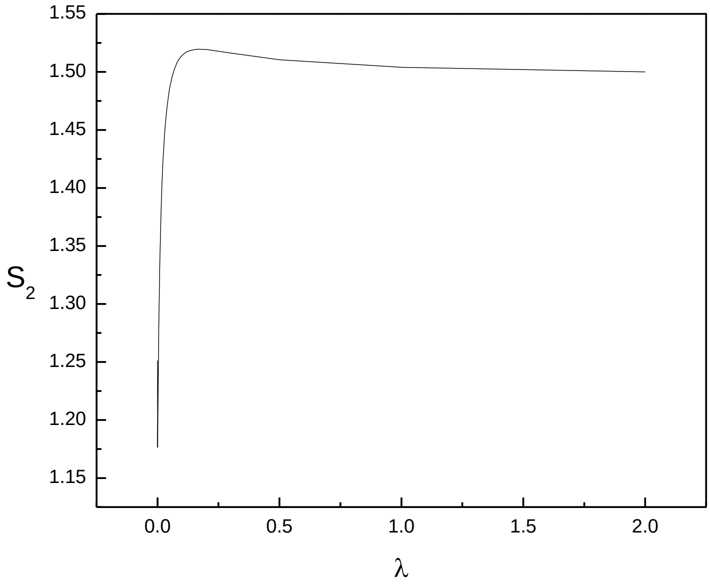

Figure 5: Position space information entropy of harmonic oscillator in the second excited state as a function of deformation parameter.

It is found that the energy dip in the left side of the plot decreases with the decrease in deformation parameter. The central minima also shifts to one side showing different behavior. The energy density in position state for second state of the harmonic potential is plotted in figure 4.

The analytical calculations for excited states of the potential are quite complex, so numerical results have been obtained. The excited state information entropy in position space for second excited state have been calculated numerically and plotted as a function of deformation parameter in figure 5. The Table 1 describes the information entropy in position space for different states of the potential corresponding to various values of the deformation parameter. It is observed that for higher excited states the information entropy first increases and then decreases to saturate at undeformed value for large values of deformation parameter. There is reduction in the values of position space information entropy for decreasing values of the deformation parameter.

Table 1: Position space information entropy of harmonic oscillator in the ground and excited states for some values of deformation parameter c.

c	S_0	S_1	S_2	S_3	S_4	S_5
0.00001	0.16	1.25	1.23	1.15	1.16	1.34
0.0001	0.29	1.23	1.19	1.15	1.34	1.69
0.001	0.46	1.21	1.18	1.34	1.68	1.90
0.01	0.68	1.18	1.34	1.63	1.80	1.85
0.1	0.93	1.25	1.51	1.65	1.73	1.80
1.0	1.05	1.33	1.50	1.61	1.70	1.77
5.0	1.07	1.34	1.49	1.61	1.70	1.77

5. CONCLUSION

The information entropy of quantum mechanical systems is a great scientific challenge as it provides a deeper insight into the internal structure of the systems. The information entropies of a class of systems is obtained, which belong to the harmonic oscillator potential. The properties of information density have been graphically analyzed for various values of the deformation parameter. In position space, the information entropy for isospectral potential has been calculated for all the energy levels. The analytical result is obtained for the ground state information entropy in position space whereas numerical calculations have been performed for the excited states. It is found that the information entropy is reduced for the smaller values of deformation parameter. For lower information entropy, the wave function will be more concentrated and the accuracy in predicting the localization of the particle will be higher. This approach can also be applied in the reduction of Heisenberg's uncertainty in position space.

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REFERENCE

[1] Sun, G. H., Dong, S. H., Launey, K. D., Dytrych, T., and Draayer, J.P., 2015, "Quantum information entropy for a hyperbolical potential function," Int. J. Quantum Chem., 115, pp. 891-899.

[2] Dong, S., Sun, G. H., Dong, S. H., and Draayer, J. P., 2014, "Quantum information entropies for a squared tangent potential well," Phys. Lett. A, 378, pp. 124-130.

[3] Valencia-Torres, R., Sun, G. H., and Dong, S. H., 2015, "Quantum information entropies for a hyperbolic potential function," Phys. Scr., 90, pp. 035205(1-9).

[4] Song, X. D., Sun, G. H., and Dong, S. H., 2015, "Shannon information entropy for an infinite circular well," Phys. Lett. A, 379, pp.1402-1408.

[5] Falaye, B. J., Serrano, F. A., and Dong, S. H., 2016, "Fisher information for the position-dependent mass Schrodinger system," Phys. Lett. A, 380, pp. 267-271.

[6] Sun, G. H., Aoki, M. A., and Dong, S. H., 2013, "Quantum information entropies of the eigenstates for the Poschl-Teller-like potential" Chin. Phys. B, 22(5), pp. 050302(1-5).

[7] Atre, R., Kumar, A., Kumar, C. N., and Panigrahi, P. K., 2004, "Quantum-information Entropies of the Eigenstates and the Coherent State of the Poschl-Teller Potential," Phys. Rev. A, 69, pp. 052107(1-6).

[8] Kumar, A., 2005, "Information entropy of isospectral Poschl-Teller potential," Ind. J. Pure & App. Phys., 43, pp. 958-963.

[9] Kumar, A., and Kumar, C. N., 2011, "Information Entropy for Isospectral Hydrogen atom," Int. Jour. of Eng. & Applied Sci., 7(1), pp. 57-61.

[10] Yahya, W. A., Oyewumi, K. J., and Sen, K. D., 2014, "Information and complexity measures for the ring shaped modified Kratzer potential," Ind. J. Chem. A, 53A(10), pp. 1307-1316.

[11] Haldar, S. K., Chakrabarti, B., Das, T. K., and Biswas, A., 2013, "Correlated many-body calculation to study characteristics of Shannon information entropy for ultracold trapped interacting bosons," Phys. Rev. A, 88, pp. 033602(1-11).

[12] Pooja, Kumar, R., Kumar, G., Kumar, R., Kumar, A., 2016, "Quantum Information Entropy of Eckart Potential," 2016, Int. J. Quantum Chem, 116, pp. 1413-1418.

[13] Panos, C. P., and Maassen, S. E., 1997, "Quantum entropy for nuclei," J.

Mod. Phys. E, 6, 497-505.

[14] Hirchman, I. I., 1957, "A note on entropy," Am. J. Math., 79, pp. 152-156.

[15] Everett, H., 1973, The Many Worlds Interpretation of Quantum Mechanics, Princeton University Press, Princeton, NJ.

[16] Bialynicki-Birula, I., and Mycielski, J., 1975, "Uncertainty relations for information entropy in wave mechanics," Commun. Math. Phys., 44, pp. 129-132.

[17] Sanchez-Ruiz, J., 1993, "Maassen-Uffink entropic uncertainty relation for angular momentum observables," Phys. Lett. A, 181, pp. 193-198.

[18] Krishna, M., and Parthasarthy, K.R., 2002 "An entropic uncertainty principle for quantum measurement," Sankhya: Ind. J. Stat., 64, pp. 842-851.

[19] Majernik, V., and Opatrny, T., 1996, "Entropic uncertainty relations for a quantum oscillator," J. Phys. A: Math. Gen., 29, pp. 2187-2197.

[20] Jizba, P., Dunningham, J. A., and Joo, J., 2015, "Role of information theoretic uncertainty relations in quantum theory," Annal. Phys., 355, pp. 87-114.

[21] March, N. H., Angilella, G. G. N., and Pucci, R., 2013, "Natural orbitals in relation to quantum information theory: from model light atoms through to emergent metallic properties," Int. J. Mod. Phys. B, 27, pp. 133021(1-26).

[22] Coles, P. J., Kaniewski, J., and Wehner, S., 2014, "Equivalence of wave–particle duality to entropic uncertainty, "Nat. Commun., 5, pp. 5814(1-8).

[23] Narayanan, K. R., and Srinivasa, A. R., 2012, "Shannon-entropy-based nonequilibrium entropic temperature of a general distribution," Phys. Rev. E, 85, pp. 031151(1-11).

[24] Rhee, A., Cheong, R., and Levchenko, A., 2012, "The application of information theory to biochemical signaling systems," Phys. Biol., 9, pp. 045011(1-11).

[25] Molina-Espiritu, M., Esquivel, R. O., Angulo, J. C., Antolin, J., Iuga, C., and Dehesa, J. S., 2013, "Information-theoretical analysis for the SN2 exchange reaction," Int. J. Quan. Chem., 113, pp. 2589-2599.

[26] Pursey, D. L., 1986, "New families of isospectral Hamiltonians" Phys. Rev. D, 33, pp. 1048-1055.

[27] Abraham, P. B., and Moses, H. E., 1980, "Changes in potentials due to change in the point spectram: Anharmonic oscillators with exact solutionss," Phys. Rev. A, 22, pp. 1333-1340.

[28] Kumar, A., 2014, "Isospectral Hulthen Potential," Int. J. Math., Comp., Phys. & Quant. Eng., 8(2), pp. 419-422.

[29] Mielnik, B., 1984, "Factorization method and new potentials with the oscillator spectrum," J. Math. Phys. 25, pp. 3387-3389.

[30] Neito, M. M., 1984, "Relationship between supersymmetry and the inverse methods in quantum mechanics," Phys. Lett. B, 145, pp. 208-210.

[31] Cooper, F., Khare, A., and Sukhatme, U., 1995, "Supersymmetry and quantum mechanics," Phys. Rep., 251, pp. 267-385.

[32] Chakrabarti, B., 2009, "Use of supersymmetric isospectral formalism to realistic quantum many body problems," Pramana: J. Phys., 73, pp. 405-416.

[33] Dey, B., and Kumar, C. N., 1994, "New set of kink bearing Hamiltonians," Int. J. Mod. Phys. A, 9, pp. 2699-2705.

[34] Kumar, A., 2014, "Calculation of Moments and uncertainty in position space," American Jour. Sci. & Tech., 1(5), 283-287.

[35] Kumar, A., 2013, "Generalization of Soliton Solutions," Int. Jour. Nonlinear Sc., 13(2), 170-176.

[36] Kumar, A., and Kumar, C. N., 2005, "Calculation of Franck-Condon Factors and r- centroids Using Isospectral Hamiltonian Approach," Ind. J. Pure & App. Phys. 43, 738- 742.

[37] Khare, A., and Kumar, C. N., 1993, "Landau level spectrum for charged particle in a class of non-uniform magnetic fields," Mod. Phys. Lett. A, 8, pp. 523-530.