International Journal of Pure and Applied Physics.
ISSN 0973-1776 Volume 13, Number 1 (2017), pp. 125-132

http://www.ripublication.com

Exact Solutions of Certain Nonlinear Diffusion-Reaction Equations with a Nonlinear Convective Term

Bhupendra Singh[1] and Ranjit Kumar[2 *]

[1]*Department of Physics, Atmaram Sanatan Dharma College,*
University of Delhi, Delhi 110021, India
[2]*Department of Physics, Dyal Singh College,*
University of Delhi, Delhi 110003, India

Abstract

Investigation of exact solutions of certain types of nonlinear diffusion-reaction (D-R) equations which involve not only the quadratic and quartic nonlinearities but also a nonlinear convective flux term has been carried out. These equations arise in a variety of contexts in physical problems, particularly in the density-dependent diffusion processes.

PACS numbers: 05.45. Yv; 02.30.Ik; 02.30.Jr

Keywords: Nonlinear diffusion equation, Nonlinear convective ux term, Modified extended tanh-function method.

I. INTRODUCTION

With a view to having a better accuracy in the theoretical studies of several phenomena in Nature an account of nonlinearity in corresponding models has become desirable. A large number of applications of nonlinear diffusion-reaction (D-R) equations have already been known in the literature [1-3]. In particular, nonlinear D-R equations with polynomial nonlinearity in their analogous forms have turned out to be

[*] *Corresponding Autor's*
E-mail : du.ranjit@gmail.com

of great interest in recent years [4-13]. Besides this polynomial type nonlinearity, the nonlinearities arising from the density-dependent diffusion are also studied in the literature [14-17]. In these studies, however, either the specific solutions of the nonlinear D-R equations are obtained or else numerical methods are used for this purpose. An exact solution of these equations, if become available, will further add to the scope of their applications in various studies.

The purpose of this work is to find the exact solutions of certain types of nonlinear D-R equations which involve quadratic and quartic nonlinearities with a nonlinear 'convective flux term' using modified extended tanh-function method which has been introduced in [18]. In particular, we investigate the exact solutions of the nonlinear D-R equations,

$$C_t + kCC_x = DC_{xx} + \alpha C - \beta C^2, \tag{1}$$

and

$$C_t + kC^2C_x = DC_{xx} + \alpha C - \beta C^4, \tag{2}$$

where $C = C(x;\ t)$ is the concentration or the density variable depending on the phenomenon under study; D is the diffusion coefficient, and k, α, β are real constants. With regard to the physical and mathematical contents of Eqs. (1) and (2) the following remarks are in the order:

(i) Equations (1) and (2) describe a transport phenomenon in which both diffusion and convection processes are of equal importance, i.e., the nonlinear diffusion could be thought of as equivalent to the nonlinear convection effects. In particular, the second term on the left hand side, $kCC_x\left(\text{or } kC^2Cx\right)$ is the replacement of the conventional $vCx-$term [4], where v can be a function of both x and t in general, but it is considered as a constant in most of the earlier works [4,11,12] except for some recent investigations [13].

Thus, by choosing this term as kCC_x (or kC^2C_x) in the present work, we are considering higher order generalization of the standard D-R equation in the spirit of point (ii) below.

(ii) It may be mentioned that the presence of vC_x-type convective flux term in the D-R equation [11] makes the system nonconservative, whereas a nonlinear convection term in (1) or in (2) arise as a natural extension of a conservation law [1]. This can be demonstrated by writing the D-R equation in the case of density-dependent diffusion models [1] as

$$C_t + h_x\left(C\right) = DC_{xx} + f\left(C\right), \tag{3}$$

where h(C) is some function of C. In Eq. (3), the choices of $f\left(C\right) = \alpha C - \beta C^2$ and of $hx\left(C\right) = \frac{\left(\partial h\left(C\right)\right)}{\partial C} C_x = kCC_x$ (or kC^2C_x) while lead to Eq. (1) (or Eq. (2)), the L.H.S. of (3) is expressible in the form of a divergence, namely $\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)\left(C, h\left(C\right)\right)$.

The 'convective velocity' now becomes $\frac{\partial h}{\partial C}$ and the same is chosen here as kC (or kC^2) with k a positive or negative constant.

(iii) No doubt, the form (3) of Eq. (1) has been studied [1] in a variety of situations like in the studies of ion-exchange columns, chromatography etc., but Eq. (2), which we are investigating here perhaps for the first time, could also be a potential candidate from this point of view for such studies. Further note that the presence of a nonlinear convection term in the D-R equations can lead to a dramatic effect on their solutions.

Now, by defining a variable $\xi = x - wt$, Eqs. (1) and (2) can be expressed respectively as

$$DC^n + wC' - kCC' + \alpha C - \beta C^2 = 0 \tag{4}$$

$$DC^n + wC' - kC^2C' + \alpha C - \beta C^4 = 0 \tag{5}$$

For the solutions of (4) and (5) we employ the modified extended tanh-function method of Ali [18] and thus make an ansatz for $C\left(\xi\right)$ as

$$C\left(\xi\right) = a_0 + \sum_{i=1}^{l}\left(a_i\phi^i + b_i\phi^{-i}\right), \tag{6}$$

where a_i and b_i are all real constants to be determined, l is a positive integer which can be determined by balancing the highest order derivative term with the highest order nonlinear term in these equations, and $\phi\left(\xi\right)$ satisfies the following auxiliary ordinary differential equation

$$\frac{d\phi}{d\xi} = b + \phi^2. \tag{7}$$

Here b is another real constant to be determined later. In fact Eq. (7) has the general solutions:

(I) If $b < 0$

$$\phi(\xi) = -\sqrt{-b}\tanh\left(\sqrt{-b}\xi\right), \text{ or } \phi(\xi) = -\sqrt{-b}\coth\left(\sqrt{-b}\xi\right)$$

(II) If b > 0

$$\phi(\xi) = \sqrt{b}\tan\left(\sqrt{-b}\xi\right), \text{ or } \phi(\xi) = -\sqrt{b}\cot\left(\sqrt{-b}\xi\right)$$

(III) If $b = 0$

$$\phi(\xi) = \frac{-1}{\xi}$$

II. EXACT SOLUTIONS OF EQ. (1)

As a result of using the balancing procedure in Eq. (4) one immediately obtains $l = 1$. This suggests the choice of $C(\xi)$ in Eq. (6) as

$$C(\xi) = a_0 + a_1\phi(\xi) + \frac{b_1}{\phi(\xi)} \tag{8}$$

Now, substituting (8) along with (7) in Eq. (4) and then setting the coefficients of $\phi^j(\xi)(j = 0,1,...,6)$ to zero in the resultant expression, one obtains a set of algebraic equations involving a_0, a_1, $b1$, b and w as

$$-kb_1^2 b - 2Db_1 b^2 = 0$$

$$wb_1 b - ka_0 b_1 b + \beta b_1^2 = 0,$$

$$-kb_1^2 - 2Db_1 b - \alpha b_1 + 2\beta a_0 b_1 = 0,$$

$$-wa_1 b + wb_1 + ka_0 b_1 - \alpha a_0 + \beta a_0^2 + 2\beta a_1 b_1 + ka_0 a_1 b = 0,$$

$$ka_1^2 b - 2bDa_1 - \alpha a_1 + 2\beta a_0 a_1 = 0,$$

$$-wa_1 + ka_0 a_1 + \beta a_1^2 = 0,$$

$$ka_1^2 - 2Da_1 = 0 \tag{9}$$

which can be solved for the four unknowns a_0, a_1, b_1, b and *w* to give

$$a_0 = \frac{\alpha}{2\beta}; a_1 = \frac{2D}{k}; b_1 = \frac{\alpha^2 k}{32\beta^2 D}; b = \frac{-\alpha^2 k^2}{64\beta^2 D^2}; w = \frac{k^2\alpha + 4D\beta^2}{2\beta k}, \tag{10}$$

and finally, the solution $C(\xi)$ of Eq. (4) turns out to be

$$C(\xi) = \frac{\alpha}{2\beta}\left[1 + \frac{1}{2}\tanh\left(\frac{\alpha k}{8\beta D}\xi\right) + \frac{1}{2}\coth\left(\frac{\alpha k}{8\beta D}\xi\right)\right] \tag{11}$$

Note that in obtaining the above solutions of (9) we have chosen the case when $a_1 \neq 0$ and $b_1 \neq 0$. Now if $a_1 = 0$ and $b_1 \neq 0$, then from (9) one obtains $b = \frac{-\alpha^2 k^2}{16\beta^2 D^2}$ and $b_1 = \frac{\alpha^2 k}{8\beta^2 D}$ and the solution of (4) takes the form

$$C(\xi) = \frac{\alpha}{2\beta}\left[1 + \frac{1}{2}\coth\left(\frac{\alpha k}{4\beta D}\xi\right)\right], \tag{12}$$

which is not a physically acceptable case. Similarly, if $a_1 \neq 0$ and $b_1 = 0$, the corresponding solution of (4) becomes-

$$C(\xi) = \frac{\alpha}{2\beta}\left[1 + \frac{1}{2}\tanh\left(\frac{\alpha k}{4\beta D}\xi\right)\right], \tag{13}$$

which is fact turns out to be a solitary wave solution of Eq. (4)

III. EXACT SOLUTIONS OF EQ. (2)

As before, we use the balancing procedure for Eq. (5) and obtain $l = 1$. NOw using Eqs. (8) and (7) in Eq. (5) we get a set of algebraic equations for the unknowns a_0, a_1, b_1, b and *w* as

$$\beta b_1^4 - k b_1^3 b = 0,$$

$$4\beta a_0 b_1^3 - 2k a_0 b_1^2 b - 2D b_1 b^2 = 0,$$

$$w b_1 b - k a_0^2 b_1 b - k a_1 b_1^2 b + 6\beta a_0^2 b_1^2 + 4\beta a_1 b_1^3 - k b_1^3 = 0,$$

$$-2k a_0 b_1^2 - 2D b_1 b - \alpha b_1 + 4\beta a_0^3 b_1 + 12\beta a_0 a_1 b_1^2 = 0$$

$$-w a_1 b + w b_1 + k a_0^2 a_1 b - k a_0^2 b_1 + k a_1^2 b_1 b - k a_1 b_1^2 -$$

$$\alpha a_0+\beta a_0^4+12\beta a_0^2 a_1 b_1+6\beta a_1^2 b_1^2=0,$$

$$2ka_0a_1^2b-2Da_1b-\alpha a_1+4\beta a_0^3 a_1+12\beta a_0 a_1^2 b_1=0,$$

$$-wa_1+ka_0^2a_1+ka_1^3b+ka_1^2b_1+b\beta a_0^2a_1^2+4\beta a_1^3 b_1=0,$$

$$2ka_0a_1^2-2Da_1+4\beta a_0a_1^3=0,$$

$$ka_1^3+\beta a_1^4=0 \tag{14}$$

After solving these equations one obtains

$$a_0=\frac{\beta D}{k^2};a_1=\frac{-k}{\beta};w=-\frac{k^6\alpha+16\beta^4D^3}{4\beta^2Dk^3};b=\frac{\beta^4D^2}{4k^6}-\frac{\alpha}{16D};b_1=\frac{\beta^3D^2}{4k^5}-\frac{\alpha k}{16\beta D}, \tag{15}$$

along with a constraining relation, $\alpha=\pm\frac{8b^4D^3}{k^6}$, among the constant coefficients in (5). Using the above constraining relation we get two values, of b and b_1. It may be mentioned that while solving the set of Eqs. (9) and (14), some of the equations merely reduce to identities. Finally the solution $C(\xi)$ of Eq. (5) becomes

$$C(\xi)=\frac{\beta D}{k^2}\left[1+\frac{1}{2}\tanh\left(\frac{\beta^2D}{2k^3}\xi\right)+\frac{1}{2}\coth\left(\frac{\beta^2D}{2k^3}\xi\right)\right] \tag{16}$$

for $b=\frac{-\beta^4D^2}{4k^6}$ and $b_1=\frac{-\beta^3D^2}{4k^5}$, and

$$C(\xi)=\frac{\beta D}{k^2}\left[1-\frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}\beta^2D}{2k^3}\xi\right)-\frac{\sqrt{3}}{2}\cot\left(\frac{\sqrt{3}\beta^2D}{2k^3}\xi\right)\right] \tag{17}$$

for $b=\frac{3\beta^4D^2}{4k^6}$ and $b_1=\frac{3\beta^3D^2}{4k^5}$. For other pair of values of a_1,b_1 and w (not listed in Eq. (15)) but are permissible, the solutions of (5) become

$$C(\xi)=\frac{\beta D}{k^2}\left[1+\frac{1}{2}\tanh\left(\frac{\beta^2D}{2k^3}\xi\right)\right],\left(a_1=\frac{-k}{\beta},b_1=0,w=\frac{-6\beta^2D^2}{k^3}\right), \tag{18}$$

$$C(\xi)=\frac{\beta D}{k^2}\left[1-\frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}\beta^2D}{2k^3}\xi\right)\right],\left(a_1=\frac{-k}{\beta},b_1=0,w=\frac{-2\beta^2D^2}{k^3}\right), \tag{19}$$

$$C(\xi)=\frac{\beta D}{k^2}\left[1+\coth\left(\frac{\beta^2D}{k^3}\xi\right)\right],\left(a_1=0,b_1\frac{kb}{\beta},w=\frac{-6\beta^2D^2}{k^3}\right), \tag{20}$$

$$C(\xi)=\frac{\beta D}{k^2}\left[1+\sqrt{3}\cot\left(\frac{\sqrt{3}\beta^2 D}{k^3}\xi\right)\right],\left(a_1=0,b_1\frac{kb}{\beta},w=\frac{-3\beta^2 D^2}{k^3}\right) \tag{21}$$

Here, $C(\xi)$ of (18) represents a solitary wave solution.

IV. CONCLUDING REMARKS

With a view to extending the scope of applications of nonlinear D-R equations with quadratic and quartic nonlinearities, the role of certain types of nonlinear convective terms in these equations is investigated. In certain cases the existence of shock wave solutions is demonstrated for both the Eqs. (1) and (2). Certain observations from the solution (11) of Eq. (4) and solutions (16) and (17) of Eq. (5) are in order: (i) It can be seen that $C(\xi)$ in (16) or (17) decreases with *k*-a measure of contribution of nonlinear convective flux term in (2). On the other hand, $C(\xi)$ remains unaffected with respect to k in case of solution (11) of Eq. (1). Similarly, the nonlinear parameter β in Eqs. (1) and (2) plays just opposite role in these solutions. Interestingly, some of these features agree with the ones discussed in Ref. [1]. From the point of view of further applications it may be of interest to study the time dependence of the parameters appearing in these equations. Such study are in progress.

ACKNOWLEDGMENTS

R. Kumar would like to thank Dyal singh college, University of Delhi, for providing the computational facility during the course of this work. We would also like to thank Awadhesh Prasad and R. S. Kaushal for helpful discussions.

REFERENCES

[1] J. D. Murray, Mathematical Biology, Springer-Verlag, New York 1993.

[2] R. S. Kaushal and D. Parashar, Advanced Methods of Mathematical Physics (Narosa Pub. House, New Delhi, CRC Press, USA, Alpha Science Int. London, 2000).

[3] M. Lakshmanan and S. Rajasekhar, Nonlinear Dynamics: Integrability, Chaos, and Patterns, Springer-Verlag, Indian reprint, 2003.

[4] N. Moiseyeb and M. Gluck, 'Non-Hermitian Delocalization from Hermitian Hamiltonians', Phys. Rev. E 2001; 63:041103-041105.

[5] N. Hatano and D. R. Nelson, 'Localization Transitions in Non-Hermitian Quantum Mechanics', Phys. Rev. Lett. 1996; 77:570-573.

[6] D. R. Nelson and N. M. Shnerb, 'Non-Hermitian Localization and Population Biology', Phys. Rev. E 1998;58:1383-1403, and the references therein.

[7] N. M. Snerb and D. R. Nelson, `Winding Numbers, Complex Currents, and Non-Hermitian Localization', Phys. Rev. Lett. 1998;80:5172-5175.

[8] J. T. Chalker and Z. Jane Wang, 'Diffusion in a Random Velocity Field: Spectral Properties of a Non-Hermitian Fokker-Planck Operator', Phys. Rev. Lett. 1997;79:1797-1800.

[9] D. R. King, 'Exact Solutions to a Nonlinear Diffusion Equation', J. Phys. A: Math. & Gen. 1991;24:3213-3216.

[10] R.S. Banerjee, 'Exact Solutions of Some Nonlinear Equations', Int. J. Theo. Phys. 1993;32:879-884.

[11] R. S. Kaushal, 'The Diffusion-Reaction (D-R) Hamiltonian and the Solutions of Certain Types of Linear and Nonlinear D-R Equations in One Dimension', J. Phys. A: Math. & Gen. 2005;38:3897-3907.

[12] R. S. Kaushal, R. Kumar and A. Prasad, `On the Exact Solutions of Nonlinear Diffusion-Reaction Equations with Quadratic and Cubic Nonlinearities', Pramana-J. Phys. 2006;67:249-256.

[13] R. Kumar, R.S. Kaushal and A. Prasad, 'Soliton-like Solutions of Certain Types of Nonlinear Diffusion-Reaction Equations with Variable Coe_cient', Phys. Lett. A 2008;372:1862-1866.

[14] B. H. Gilding and R. Kersner, 'A Fisher/KPP-Type Equation with Density-Dependent Diffusion and Convection: Travelling-Wave Solutions', J. Phys. A: Math. & Gen. 2005;38:3367-3379.

[15] R. A. Van Gorder and K. Vajravelu, 'Analytical and Numerical Solutions of the Density Dependent Diffusion Nagumo Equation', Phys. Lett. A 2008; 372:5152-5158.

[16] J. Satsuma, `Explicit Solutions of Nonlinear Equations with Density Dependent Diffusion', J. Phys. Soc. of Japan 1987;56:1947-1950.

[17] S. Harris, 'Fisher Equation with Density-Dependent Diffusion: Special Solutions', J. Phys. A: Math. & Gen. 2004;37:6267-6268.

[18] A. H. A. Ali, 'The Modified Extended tanh-Function Method for Solving Coupled MKdV and Coupled Hirota-Satsuma Coupled KdV Equations', Phys. Lett. A 2007;363:420-425.