International Journal of Pure and Applied Physics.
ISSN 0973-1776 Volume 13, Number 4 (2017), pp. 271-280

http://www.ripublication.com

Cosmological Models with quadratic equation of state and dissipative effects

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Abstract

In this paper cosmological solutions in Einstein's theories of gravity in the presence of non linear equation of state (EoS), specially quadratic equation of state with dissipative effects are presented here. The evolution of the universe is explored considering the imperfect fluid described by Eckart and truncated causal theories (TIS) proposed by Israel and Stewart. The corresponding cosmological dynamics with the bulk viscosity and quadratic equation of state (EoS) are obtained for homogeneous isotropic Friedman-Robertson walker space-time. In this paper exact analytical solutions are studied in some special cases. We also discuss numerical solutions as the field equations to obtained evolution of the universe are highly nonlinear. To find numerical solution we plot deceleration parameter (q) verses Hubble parameter (H) for a given set of other parameters. The interesting feature obtained from numerical solutions, it has three phase of evolution of the universe. The models admit present accelerating phase as well as early inflationary phase with intermediate deceleration phase of the evolution of the universe in the presence of quadratic EoS with dissipative effect. The models may also help to predict future evolution of the universe. A study of Stability analysis of the models is discussed in this paper both in Eckart and Truncated Israel Stewart theory.

Keywords: dark energy, cosmology, viscosity, quadratic EoS

1. INTRODUCTION

The observational data [1, 2] strongly suggest that the dominated part of the present universe are the dark sector (composed by dark matter (~26.8 %) and dark energy (~ 68.3 %) and the rest (~4.9 %) are usual baryonic matter). The recent predictions from the observational Astronomy [1-2] indicate that the present universe is accelerating which might have emerged to the present state from an inflationary phase in the past followed by a deceleration phase.

Loop quantum gravity corrections result in a modified Friedman equation [3], with the modification appearing as a negative term which is quadratic in the energy density. Further motivation for considering a quadratic equation of state comes from recent studies of k-essence fluids as unified dark matter (UDM) models [4] has a nonlinear EoS [5]. Ananda and Bruni [6] discussed the cosmological models by considering different form of non-linear quadratic EoS. Dark energy universe with different EoS has been discussed by Nojiri [7], Capozziello [8] and proved that the quadratic EoS may describe dark energy or unified dark matter. Quadratic EoS is needed to explore in cosmological models to study of dark energy and general relativistic dynamics of the universe.

Some processes in cosmology and astrophysics lead to dissipative effects process i.e., viscosity. Viscosity may be arises due to the decoupling of neutrinos from the radiation era, the decoupling of matter from radiation during the recombination era, particle collisions involving gravitons, cosmological quantum particle creation processes and formation of galaxies [9].

A non negligible dissipative bulk stress on cosmological scales at the late universe phase might be important [10, 11]. To describe a relativistic theory of viscosity, Eckart [12] made the first attempt. Israel and Stewart [13] developed a fully relativistic formulation of the theory taking into account second order deviation terms in the theory, which is termed as "transient" or "extended" irreversible thermodynamics (EIT). Using the transport equations obtained from EIT, several works are done in literature [14, 15]. In this paper we consider the existence of the bulk viscosity in the quadratic EoS to study the cosmological dynamics of the universe. The layout of the paper as: in sec. 2, we give the relevant field equations. In sec. 3 cosmological solutions are presented. In sec. 4, 5 stability analysis and conclusion are given respectively.

2. RELEVANT FIELD EQUATION:

We consider the homogeneous and isotropic space-time metric given Friedmann Robertson-Walker (FRW)

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \qquad (1)$$

where $a(t)$ is the scale factor of the universe. The constant k defined curvature of the space time, k= 0, 1,-1 represents flat, closed and open spaces respectively. The field equations and conservation equation yield

$$H^2 = \frac{\rho}{3} - \frac{k}{a^2}, \qquad 2\dot{H} + 3H^2 = -p - \Pi - \frac{k}{a^2}, \qquad (2)$$

$$\dot{\rho} + 3(\rho + p)H = -3\Pi H. \qquad (3)$$

Where $H = \frac{\dot{a}}{a}$ is the Hubble parameter and an over dot represents derivative with respect to cosmic time (t). Where we consider the standard unit $8\pi G = c = 1$. Where p is the isotropic pressure [16] of the universe, ρ is the energy density of the universe and $\Pi\,(\leq 0)$ is the bulk viscous pressure. In EIT for TIS theory the bulk viscous stress Π satisfies the transport equation given by

$$\Pi + \tau\,\dot{\Pi} = -3\,\zeta\,H\,. \qquad (4)$$

Where $\zeta\ (\geq 0\,)$ is the coefficient of bulk viscosity and $\tau\ (\geq\ 0\,)$ is the relaxation time. In Eckart theory $\tau = 0$. We consider the cosmological fluid obeys quadratic EoS [6]

$$p\ = A\,\rho\ +\ B\,\rho^2. \qquad (5)$$

Where $A\ (\ \geq 0)$ and B are dimensional constant. The deceleration parameter(q) is related to H as

$$q\ = \frac{d}{dt}\left(\frac{1}{H}\right) - 1\,. \qquad (6)$$

The value of deceleration parameter is negative for accelerating and positive for decelerating phase of evolution of the universe. Using equations (2), (3) and (5) we obtain

$$\Pi\ =\ -2\dot{H} - 3\left[1 + \frac{k}{3\,a^2H^2} + A\left(1 + \frac{k}{a^2H^2}\right) + \ 3\,B\,H^2\left(1 + \frac{k}{a^2H^2}\right)^2\right]H^2. \qquad (7)$$

3. COSMOLOGICAL SOLUTIONS

The system of equations (2)-(7) are employed to obtain cosmological solutions. The coefficients ζ and τ follow the relations [17]

$$\zeta\ =\ \beta\,\rho^{\delta},\quad \tau =\ \beta\,\rho^{r-1}\,. \qquad (8)$$

Where β, δ and r are positive constants.

3.1 Eckart Theory:

In Eckart theory transport equation take the form $\Pi\ = -3\,\zeta\,H$. Using equations (7) and (8), Eckart's theory yields,

$$\dot{H} = \frac{3^{\delta+1}\beta H^{2\delta+1}}{2}\left(1+\frac{k}{a^2H^2}\right)^{\delta} - \frac{3H^2}{2}\left[1+A+\frac{k+3A}{3a^2H^2}+3BH^2\left(1+\frac{k}{a^2H^2}\right)^2\right]. \qquad (9)$$

To find analytic solution in Eckart theory, we consider the following special cases:

(1) When $\delta = 0, B = 0$ and $k = 0$: The scale factor yields $a(t) = a_i\left(a_1 + e^{\frac{3\beta t}{2}}\right)^{\frac{2}{3}(1+A)}$, where a_i and a_1 are constants. It shows important emergent universe scenario.

(2) When $\delta = \frac{1}{2}$, $k = 0$ and $\beta = \frac{1+A}{\sqrt{3}}$: In this case the scale factor varies exponentially with cosmic time $a(t) = a_0 \exp\left(\frac{1}{\sqrt[3]{4B}} t^{\frac{2}{3}}\right)$. The expansion of evolution is higher for smaller value of B.

(3) When $\delta = \frac{1}{2}, B = 0, k = 0$ and $\beta \neq \frac{1+A}{\sqrt{3}}$: In this special case the scale factor shows power law expansion as $a(t) = a_0\, t^{\frac{2}{3(1+A-\beta\sqrt{3})}}$. It follows accelerated expansion ($q < 0$) of the universe for $\beta > \frac{1+3A}{3\sqrt{3}}$.

(4) When $\delta = \frac{3}{2}$, $k = 0$ and $\beta = \frac{B}{\sqrt{3}}$: The universe evolve as power law type expansion, $a(t) = a_0\, t^{\frac{2}{3(1+A)}}$, where a_0 is a constant. One recovers the standard cosmological solution even in the presence of quadratic EoS and bulk viscosity.

In Eckart theory the expression of deceleration parameter yields

$$q = \frac{1+3A}{2} + \frac{k(1+3A)}{2a^2H^2} + \frac{9BH^2}{2}\left(1+\frac{k}{a^2H^2}\right)^2 - \frac{1}{2}\beta 3^{\delta+1}H^{2\delta-1}\left(1+\frac{k}{a^2H^2}\right)^{\delta}. \qquad (10)$$

We plot q vs H for a given set of other parameters to obtain numerical solution. For flat universe ($k = 0$) cosmic time is inversely proportional to Hubble parameter $\left(i.e., t \sim \frac{1}{H}\right)$. Three phase of evolution of the universe i.e., present acceleration, early inflation and intermediate deceleration phase are obtained in the presence of viscosity and quadratic parameter as shown in figure (1).

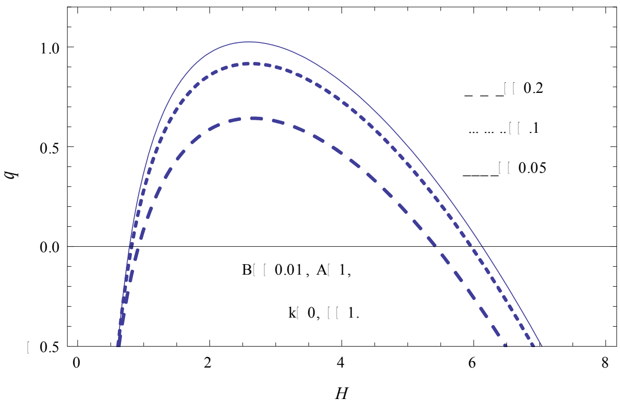

Figure 1: Shows the variation $q\ vs\ H$ for different values δ for a given value of other parameters $B = -0.01,\ A = 1, \beta = 1$ and $k = 0$ in Eckart theory.

3.2 Truncated Israel and Stewart (TIS) Theory:

Using Eqs. (7) and (8) in TIS transport equation (4), for $k = 0$ we obtain [15]

$$\ddot{H} + b_1(H)\dot{H} + b_2(H) = 0\ . \qquad (11)$$

Where $b_1(H) = 3\left[(1+A)H + 6BH^3 + \frac{3^{-r}}{\beta H^{2r-2}}\right]$, $b_2(H) = \frac{9}{2}\left[\frac{1+A+3BH^2}{3^r\beta H^{2r-4}} - 3^{\delta-r}\ H^{3+2\delta-2r}\right]$. The above second order differential equation has two different stationary trivial solutions $H = H_1 = const.$ and $H = 0$. The first implies inflationary expansion with a constant rate given by H_1. One can study the behavior of H near the steady solution analytically. Setting $H = H_1 + \chi$, with $|\chi << H_1|$, after linearization Eq. (11) yields,

$$\alpha\,\ddot{\chi} + \alpha_1\,\dot{\chi} + \alpha_2\,\chi = 0\,. \qquad (12)$$

Where $\beta\, 3^{\delta}\, H_1^{2\delta-1} - 3BH_1^2 = 1 + A,\ \alpha = 2\beta 3^{r-1}H_1^{2r},\ \alpha_1 = 2\beta 3^r H_1^{2r+1}(1 + A + 6BH_1^2) + 2H_1^2$ and $\alpha_2 = 3H_1^3[(1-2\delta)(1+A) + (3-2\delta)3BH_1^2]$. The solution of equation (12) yields

$$\chi(t) = \chi_1\, exp(\delta_+ t) + \chi_2\, exp(\delta_- t), \qquad (13)$$

where χ_1 and χ_2 are constants which depend on initial condition, while δ_+ and δ_- are the roots $\delta_\mp = \frac{\alpha_1}{\alpha}\left[-1 \mp \sqrt{1 - \frac{4\alpha\alpha_2}{\alpha_1^2}}\right]$. If the quantity under the square root is negative then H shows an oscillatory damped behavior around $H = H_1$ with a frequency $\omega = \frac{3\sqrt{2}}{2(1+A)}\sqrt{3H_1(1 + A + 3BH_1^2)\big((3-8\delta)(1+A) + 6BH_1^2(5-4\delta)\big) - 2}$. For linear EoS ($B = 0$) and $\delta = r$ a damped oscillatory behavior may be obtained for

$\delta < \frac{3}{8} - \frac{1}{4(1+A)^2}$. This special case are studied by Ref [18], and obtained a damped oscillatory behavior for $\delta < \frac{1}{2}$.

We express the equation (11) in terms of deceleration parameter (q) to get non-stationary solutions via numerical integration. Using equation (6), equation (11) yields

$$q' + \frac{2(q+1)}{H} - \frac{b_1(H)}{H^2} + \frac{b_2(H)}{(q+1)H^4} = 0 \; . \tag{14}$$

Where prime $(\,'\,)$ represent the derivative with respect to Hubble parameter (H). To find a numerical solution one can plot $q\ vs\ H$ with an initial condition $(q[1])$ and for a given set of other parameters. Figure (2) shows three phase of evolution of the universe, two accelerating phase (early inflation and present acceleration) and one intermediate decelerating phase in the presence of dissipative effect and quadratic parameter (B) in TIS theory. If we choose at present value of age is $H = 1$, then present value of deceleration parameter is $q[1] = -0.50$, which is compatible with observational data [19].

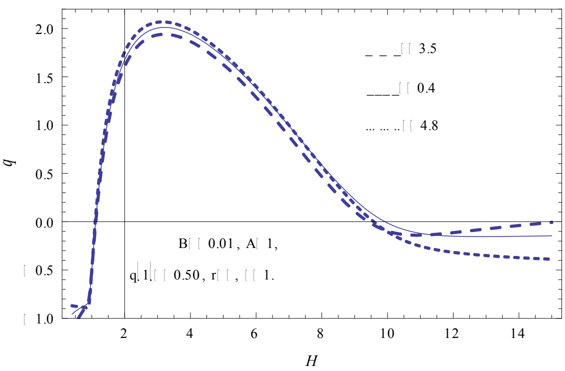

Figure 2: Shows the variation $q\ vs\ H$ for different values of δ with a given set of other parameters $B = -0.01,\ A = 1, \beta = 1$ and $q[1] = -0.5$ in TIS theory.

4. STABILITY ANALYSIS:

Introducing the quantity $h = \frac{H}{H_0}$ and rescaling the time as $t^* = H_0 t$ equation (9) can be written as

$$\dot{h} = \frac{9}{2}\beta_1 h^{2\delta+1} - \frac{3}{2}(1+A)h^2 - \frac{9}{2}B_1 h^4 = f(h). \tag{15}$$

Where $\beta_1 = \beta 3^{s-1} H_0^{2\delta-1}, B_1 = BH_0^2$ and dot(.) represent derivative with respect to time(t^*).

The fixed $(f(h^*) = 0)$ point represents equilibrium or steady solution. The stability of the fixed points are determined by a local stability analysis. We consider the following special case of fixed point and determine their stability by linear stability analysis method [20] as summarized in Table 1. In TIS theory one can rewrite equation (11) in term of reduce Hubble parameter (h) similar to Eckart Theory. To study stability analysis of the critical point we consider a combination of two following first order differential equations; $\dot{h} = y$,

$$\dot{y} = P(h,y) = \frac{9}{2}\left[3^{\delta-r}\, H^{3+2\delta-2r} - \frac{1+A+3BH^2}{3^{1+r-\delta}\beta_1 H^{2r-4}}\right] - 3\left[(1+A)H + 6BH^3 + \frac{3^{\delta-r-1}}{\beta_1 H^{2r-2}}\right]. (16)$$

The fixed or equilibrium point h_1 is determined by $P(0,h_1) = 0$. Using Taylor's expansion of equation (16) around critical point can be written as

$$\dot{y} = \frac{\partial P(h,y)}{\partial h}|_{h_1} h + \frac{\partial P(h,y)}{\partial y}|_{h_1} y + Q(h,y). \qquad (17)$$

Where $Q(h,y)$ are the higher order function of h and y. Using linear approximation in the neighborhoods of the equilibrium points, we can write equation (17) as

$$\dot{y} = b\,h + \,c\,y\,. \qquad (18)$$

Where b and c are constants. The stability of the fixed points is determined by a local stability analysis [21]. We consider the following special case of fixed point and determine their stability by linear stability analysis method as summarized in Table 1.

Table1: Stability analysis of fixed point with quadratic EoS in Eckart & TIS theory

Special Case	*Fixed point* h_1	*Type of fixed point*(h_1)*in Eckart theory*	*Type of fixed point*(h_1) *in TIS theory*
$\delta = 0$	$h_1 = \frac{\sqrt[3]{2}(1+A)}{3B_1\Delta} - \frac{\Delta}{3\sqrt[3]{2}}$	*Stable if* $\Delta < \sqrt[3]{2}\sqrt{\frac{1+A}{B_1}}$	*Stable node if* $\Delta < \sqrt[3]{2}\sqrt{\frac{1+A}{B_1}}, \frac{h_1\Theta^2}{1+A+9B_1h_1^2} > \frac{3\beta_1}{2}$. *Stable spiral if* $\Delta < \sqrt[3]{2}\sqrt{\frac{1+A}{B_1}}, \frac{h_1\Theta^2}{1+A+9B_1h_1^2} < \frac{3\beta_1}{2}$.
$\delta = \frac{1}{2}$	$h_1 = \left(\frac{3\beta_1 - 1 - A}{3B_1}\right)^{\frac{1}{2}}$	*Stable if* $\beta_1 > \frac{1+A}{3}$	*Stable node if* $\beta_1 > \frac{1+A}{3}, \frac{\Theta^2}{B_1h_1^2} > \frac{9\beta_1}{2}$.

			$Stable\ spiral\ if$ $\beta_1 > \frac{1+A}{3}, \frac{\Theta^2}{B_1 h_1^2} < \frac{9\beta_1}{2}$
$B_1 = 0$	$h_1 = \left(\frac{1+A}{3B_1}\right)^{\frac{1}{2\delta-1}}$	$Stable\ if\ \delta < \frac{1}{2}$	$Stable\ node\ if\ \delta < \frac{1}{2}$, $\left(\frac{1}{1+A} + 1 + A\right)^2 > \delta - \frac{1}{2}$.
$\delta = 1$	$h_1 = \frac{\beta_1 \pm \sqrt{\beta_1^2 - \frac{4B_1(1+A)}{3}}}{2B_1}$	$Stable\ if$ $h_1 > \frac{2(1+A)}{3\beta_1}$, $\beta_1 > \sqrt{\frac{4B_1(1+A)}{3}}$.	$Unstable\ saddle$
$\delta = \frac{3}{2}$	$h_1 = \left(\frac{1+A}{3\beta_1 - 3B_1}\right)^{\frac{1}{2}}$	$Stable\ if$ $\beta_1 < B_1$	$Unstable\ saddle$

Where

$$\Delta = -\frac{27B_1}{\beta_1} + \sqrt{4\left(\frac{1+A}{B_1}\right)^3 + \left(\frac{27B_1}{\beta_1}\right)^2}, \Theta = 1 + (1 + A + 2B_1h_1^2) \times (1 + A + 3B_1h_1^2).$$

5. DISCUSSION

In this paper, we explore cosmological solution with quadratic EoS in Eckart and TIS theory. In Eckart theory: (i) Exponential expansion are obtained for $k = 0$, $\delta = \frac{1}{2}$, $\beta = \frac{1+A}{\sqrt{3}}$. The expansion of evolution is higher for smaller value of B. (ii) Standard cosmological solutions are recovered with quadratic EoS for $k = 0$, $\delta = \frac{3}{2}$ and $B = \frac{\beta}{\sqrt{3}}$. To get numerical solution we plot $q\ vs\ H$ for a given set of other parameters as shown in figure (1) which show three phase of evolution (early inflation, present acceleration and intermediate deceleration phase) in the presence of quadratic EoS with dissipative effect. In TIS theory exponential inflationary expansion with damped oscillator behavior may be obtained for quadratic EoS. For linear EoS damped oscillatory behavior are obtained for $\delta < \frac{3}{8} - \frac{1}{4(1+A)^2}$. Figure (2) shows three phase of evolution i.e., early inflation, present acceleration and intermediate deceleration phase with quadratic EoS with other set of parameters. The stability analysis of the fixed point studied in both Eckart and TIS theory are tabulated in Table-1 for some special cases. The fixed points represent equilibrium solution which exhibits de Sitter type expansion $(a(t) \sim e^{h_1 t})$ even in the presence of matter. A stable de Sitter type expansion of the universe may be obtained both in Eckart and TIS theory for: $(i)\delta = 0,\ \Delta < \sqrt[3]{2}\sqrt{\frac{1+A}{B_1}},\ \ (ii)\delta = \frac{1}{2},\ \beta_1 > \frac{1+A}{3}$. These solutions are particularly interesting because it give rise to accelerated expansion and comparable with observational data [1].

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