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# Strong Result for Real Zeros of Random Algebraic Polynomials (II)

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**Abstract**

We estimate a lower bound for the number of real roots of a random algebraic equation whose coefficients are dependent Gaussian variables.
**Keywords:** Random polynomial, Dependent normal distribution, Real roots

## 1. INTRODUCTION

Let $N_n(\mathcal{R},\omega)$ be the number of real roots of the random algebraic equation

$$F_n(x,\omega)=\sum_{\nu=0}^{n} a_\nu(\omega)x^\nu = 0 \tag{1}$$

where the $a_\nu(\omega), \nu = 0, 1, \dots, n$ are random variables defined on a fixed probability space $(\Omega, \mathcal{A}, \Pr)$ assuming real values only.

Littlewood and Offord[1] initiated the estimation for the lower bound of $N_n(\mathcal{R},\omega)$. They considered the case when the coefficients are independent and identically distributed where they are the standard Gaussian variables, the continuously uniform variables in $[-1, 1]$ or the discretely uniform variables taking $+1$ or $-1$. The result of Littlewood and Offord[1] is of the form

$$\Pr\left(\frac{N_n(\mathcal{R},\omega)}{\frac{\log n}{\log\log\log n}} > C\right) \geq 1 - \frac{C'}{\log n} \tag{2}$$

where $C$ and $C'$ are the absolute constants. In this case, the exceptional set depends on the degree $n$ of the equation (1).

Another interesting study is due to Evans[2]. Under the assumption that the coefficients follow the independent standard Gaussian distribution, he proved the following result of the form

$$\Pr\left(\inf_{n>n_0} \frac{N_n(\mathcal{R},\omega)}{\frac{\log n}{\log\log n}} > C\right) \geq 1 - \frac{C'\log\log n_0}{\log n_0} \tag{3}$$

where $C$ and $C'$ are the absolute constants. In this case, the exceptional set is independ of the degree $n$ of the equation. The result of Evans[2] can be regarded as 'strong' version of the work of Littlewood and Offord[1].

Since the works of Littlewood and Offord[1], and Evans[2] appeared, there have been the various streams of papers by many researchers. We remark the main streams originated from the following four studies on the estimation for the lower bound of $N_n(\mathcal{R},\omega)$.

i. Samal[3]: the case of the extension of the distribution of coefficients that are independent and identically distributed.

ii. Samal and Mishra[4]: the case of the nonidentically distributed coefficients under the following type of the random algebraic equation

$$f_n(x,\omega) = \sum_{\nu=0}^{n} a_\nu(\omega) b_\nu x^\nu = 0 \tag{4}$$

where the $a_\nu(\omega)$'s have symmetric distribution and the $b_\nu$'s are non-zero real numbers.

iii. Renganathan and Sambandham[5]: the case of the generalization from the independence to the dependency of Gaussian coefficients.

iv. Samal and Pratihari[6]: the generalized estimate with $\varepsilon_n$ imposed a give condition to improve the accuracy of the lower bound and the measure of the exceptional set.

Uno[7] obtained the result combining the directions ii., iii. and iv., when the measure of the exceptional set depends on the degree $n$.

The object of this paper is to show the 'strong' result for the lower bound of $N_n(\mathcal{R},\omega)$ in the case of $(4)$ when the coefficients are nonidentically distributed dependent Gaussian random variables. We suppose that the $a_\nu(\omega),\ \nu = 0, 1, \ldots, n$ have mean zero and joint density function

$$|M|^{1/2}(2\pi)^{-(n+1)/2}\exp\left(-(1/2)\boldsymbol{a}'M\boldsymbol{a}\right) \tag{5}$$

where $M^{-1}$ is the moment matrix with

$$\rho_{ij} = \begin{cases} 1 & (i = j) \\ \rho_{|i-j|} & (1 \le |i-j| \le m) \\ 0 & (|i-j| > m), \quad i, j = 0, 1, \ldots, n \end{cases} \tag{6}$$

for a positive integer $m$, where $0 \le \rho_j < 1, j = 1, 2, \ldots, m$ in (6). That is to say we assume the $a_\nu(\omega)'s$ to be $m$-dependent stationary Gaussian random variables. With Yoshihara ([8], p.29]), we see that this assumption is equivalent to the following two statements for a stationary Gaussian sequence:

1. $\{a_\nu\}$ is * - mixing
2. $\{a_\nu\}$ is $\phi$ - mixing.

Our theorem is the general version for the accurate estimation of 'strong' result by Uno[9].

Throughout the paper, we suppose $n$ is sufficiently large. We will follow the line of proof of Samal and Mishra[10].

THEOREM 1.1 *Let*

$$f_n(x, \omega) = \sum_{\nu=0}^{n} a_\nu(\omega) b_\nu x^\nu = 0$$

*be a random algebraic equation of degree* $n$*, where the* $a_\nu(\omega)'s$ *are dependent normally distributed with mean zero, joint density function* (5) *where* $M^{-1}$ *the moment matrix given by* (6). *Let* $b_\nu$, $\nu = 0, 1, \ldots, n$ *be positive numbers, where* $k_n = \max_{0 \le \nu \le n} b_\nu$, $t_n = \min_{0 \le \nu \le n} b_\nu$. *Take* $\{\varepsilon_n\}$ *to be a sequence tending to zero such that* $\varepsilon_n^2 \log n$ *tends to infinity as* $n$ *tends to infinity.*

*Then there exists an integer* $n_0$ *such that for each* $n > n_0$*, the number of real roots of most of the equations* $f_n(x, \omega) = 0$ *is at least* $\varepsilon_n \log n$ *outside a set of measure at most* $\frac{C}{\varepsilon_{n_0} \log n_0}$*, where* $C$ *is a positive constant.*

## 2. PROOF OF THEOREM

Take

$$\beta_n = \frac{t_n}{k_n} \exp\left(\frac{C_1}{\varepsilon_n^2 \log n}\right)$$

where $C_1$ is a constant to be chosen later. Let

$$\lambda_l = l\beta_n \text{ and } M_n = \left[\alpha^2 \beta_n^2 \left(\frac{k_n}{t_n}\right)^2\right] + 1 \tag{7}$$

where $\alpha$ is a positive constant. So

$$C_2\left(\frac{k_n}{t_n}\right)^2\beta_n^2 \le M_n \le C_3\left(\frac{k_n}{t_n}\right)^2\beta_n^2$$

where $C_2$ and $C_3$ are constants. We define

$$\phi(x)=x^{[\log x]+x}$$

provided that $[x]$ denotes the greatest integer not exceeding $x$. Let $k$ be the integer determined by

$$\phi(8k+7)M_n^{8k+7}\le n<\phi(8k+11)M_n^{8k+11}.$$

The second inequality gives

$$C_4\varepsilon_n\log n<k \tag{8}$$

where $C_4$ is a constant.

We consider $f_n(x_l,\omega)=U_l(\omega)+R_l(\omega)$ at the points

$$x_l=\left\{1-\frac{1}{\phi(4l+1)M_n^{4l}}\right\}^{\frac{1}{2}}$$

for $l=\left[\frac{k}{2}\right]+1,\left[\frac{k}{2}\right]+2,\ldots,k$, where

$$\begin{aligned} U_l(\omega) &= \sum_1 a_\nu(\omega)b_\nu x_l^\nu \\ R_l(\omega) &= \left(\sum_2+\sum_3\right)a_\nu(\omega)b_\nu x_l^\nu \end{aligned}$$

the index $\nu$ ranging from $\phi(4l-1)M_n^{4l-1}+1$ to $\phi(4l+3)M_n^{4l+3}$ in $\sum_1$. from $0$ to $\phi(4l-1)M_n^{4l-1}$ in $\sum_2$, from $\phi(4l+3)M_n^{4l+3}+1$ to $n$ in $\sum_3$.

The following lemmas are necessary for the proof of the theorem.

Lemma 2.1 *For* $\alpha_1>0$,

$$\sigma_l>\alpha_1 t_n\sqrt{\phi(4l+1)}M_n^{2l}$$

*where*

$$\sigma_l^2=\sum_{i=\phi(4l-1)M_n^{4l-1}+1}^{\phi(4l+3)M_n^{4l+3}} b_i^2x_l^{2i}+2\sum_{i=\phi(4l-1)M_n^{4l-1}+1}^{\phi(4l+3)M_n^{4l+3}-1}\ \sum_{j=i+1}^{\phi(4l+3)M_n^{4l+3}} b_ib_jx_l^{i+j}\rho_{j-i}.$$

*Proof.* We get

$$\sum_{i=\phi(4l-1)M_n^{4l-1}+1}^{\phi(4l+3)M_n^{4l+3}} b_i^2 x_l^{2i} \geq t_n^2 \sum_{i=\phi(4l-1)M_n^{4l-1}+1}^{\phi(4l+1)M_n^{4l}} x_l^{2i} > t_n^2 \phi(4l+1)M_n^{4l} \left(\frac{B}{A}\right) e^{-1}$$

where $A$ and $B$ are positive constants such that $A > 1$ and $0 < B < 1$. Also we have

$$\begin{aligned} 2 & \sum_{i=\phi(4l-1)M_n^{4l-1}+1}^{\phi(4l+3)M_n^{4l+3}-1} \sum_{j=i+1}^{\phi(4l+3)M_n^{4l+3}} b_i b_j x_l^{i+j} \rho_{j-i} \geq 2t_n^2 \sum_{i=\phi(4l-1)M_n^{4l-1}+1}^{\phi(4l+3)M_n^{4l+3}-1} \sum_{j=i+1}^{\phi(4l+3)M_n^{4l+3}} x_l^{i+j} \rho_{j-i} \\ &= 2t_n^2 \frac{x_l^{2\left\{\phi(4l-1)M_n^{4l-1}+1\right\}}}{1-x_l^2} \left\{ \sum_{i=1}^{m} \rho_i x_l^i - \sum_{i=1}^{m} \rho_i x_l^{2\left\{\phi(4l+3)M_n^{4l+3}-\phi(4l-1)M_n^{4l-1}\right\}-i} \right\} \\ &\geq \frac{B'}{A'} t_n^2 \phi(4l+1)M_n^{4l} \end{aligned}$$

where $\rho_0 = \sum_{j=1}^{m} \rho_j$ and $A'$ and $B'$ are positive constants satisfying $A' > 1$ and $0 < B' < 1$. So we get

$$\sigma_l^2 > \alpha_1^2 t_n^2 \phi(4l+1)M_n^{4l}$$

where $\alpha_1$ is a positive constant, as required.

LEMMA 2.2

$$\Pr\left(\left\{ \omega\,; \left|\sum_2 a_\nu(\omega) b_\nu x_l^\nu\right| > \lambda_l \tilde{\sigma}_l \right\}\right) < \sqrt{\frac{2}{\pi}} \frac{1}{\lambda_l} e^{-\lambda_l^2/2}$$

*where*

$$\tilde{\sigma}_l^2 = \sum_{i=0}^{\phi(4l-1)M_n^{4l-1}} b_i^2 x_l^{2i} + 2 \sum_{i=0}^{\phi(4l-1)M_n^{4l-1}-1} \sum_{j=i+1}^{\phi(4l-1)M_n^{4l-1}} b_i b_j x_l^{i+j} \rho_{j-i}.$$

*Proof.* We get

$$\Pr\left(\left\{ \omega\,; \left|\sum_2 a_\nu(\omega) b_\nu x_l^\nu\right| > \lambda_l \tilde{\sigma}_l \right\}\right) = \sqrt{\frac{2}{\pi}} \int_{\lambda_l}^{\infty} e^{-u^2/2} du < \sqrt{\frac{2}{\pi}} \frac{1}{\lambda_l} e^{-\lambda_l^2/2}$$

by Feller's inequality.

LEMMA 2.3

$$\Pr\left(\left\{ \omega\,; \left|\sum_3 a_\nu(\omega) b_\nu x_l^\nu\right| > \lambda_l \tilde{\tilde{\sigma}}_l \right\}\right) < \sqrt{\frac{2}{\pi}} \frac{1}{\lambda_l} e^{-\lambda_l^2/2}$$

*where*

$$\tilde{\tilde{\sigma}}_l^2 = \sum_{i=\phi(4l+3)M_n^{4l+3}+1}^{n} b_i^2 x_l^{2i} + 2 \sum_{i=\phi(4l+3)M_n^{4l+3}+1}^{n-1} \sum_{j=i+1}^{n} b_i b_j x_l^{i+j} \rho_{j-i}.$$

*Proof.* The proof is similar to Lemma 2.2.

LEMMA 2.4 *For a fixed $l$,*

$$\Pr\left(\{\,\omega\,;|R_l(\omega)|<\sigma_l\}\right)>1-2\sqrt{\frac{2}{\pi}}\frac{1}{\lambda_l}e^{-\lambda_l^2/2}.$$

*Proof.* By Lemma 2.2 and 2.3, we get for a given $l$,

$$|R_l(\omega)|<\lambda_l\left(\tilde{\sigma}_l+\tilde{\tilde{\sigma}}_l\right)$$

outside a set of measure at most $2\sqrt{\frac{2}{\pi}}\frac{1}{\lambda_l}e^{-\lambda_l^2/2}$. Again we have

$$\sum_{i=0}^{\phi(4l-1)M_n^{4l-1}} b_i^2x_l^{2i}\le 2k_n^2\phi(4l-1)M_n^{4l-1},$$

$$\begin{aligned}\sum_{i=0}^{\phi(4l-1)M_n^{4l-1}-1}\ \sum_{j=i+1}^{\phi(4l-1)M_n^{4l-1}} b_ib_jx_l^{i+j}\rho_{j-i} &\le k_n^2\sum_{i=1}^{m}\rho_i\sum_{j=1}^{\phi(4l-1)M_n^{4l-1}-i+1} x_l^{2j+i-2}\\ &\le k_n^2\phi(4l-1)M_n^{4l-1}\rho_0.\end{aligned}$$

Hence we get for a positive constant $\alpha_2$,

$$\tilde{\sigma}_l^2\le\alpha_2^2k_n^2\phi(4l-1)M_n^{4l-1}.$$

Since $\phi(4l-1)<\frac{\phi(4l+1)}{16l^2}$, we obtain

$$\tilde{\sigma}_l^2\le\alpha_2^2k_n^2\frac{\phi(4l+1)}{l^2}M_n^{4l-1}.$$

Similarly, we have

$$\tilde{\tilde{\sigma}}_l^2\le\alpha_3^2k_n^2\frac{\phi(4l+1)}{l^2}M_n^{4l-2}$$

for a positive constant $\alpha_3$. Therefore we obtain outside the exceptional set,

$$|R_l(\omega)|\le\left(\frac{\alpha_2+\alpha_3}{\alpha_1}\frac{k_n}{t_n}\right)\frac{\lambda_l}{l}\frac{\sigma_l}{M_n^{\frac{1}{2}}}\le\sigma_l$$

by Lemma 2.1 and (7).

Let us define random events $E_p$ , $F_p$ and $G_p$ by

$$\begin{aligned}E_p &= \{\omega\,;\,U_{3p}(\omega)\ge\sigma_{3p},\ U_{3p+1}(\omega)<-\sigma_{3p+1}\}\,,\\ F_p &= \{\omega\,;\,U_{3p}(\omega)<-\sigma_{3p},\ U_{3p+1}(\omega)\ge\sigma_{3p+1}\}\\ &\text{and}\\ G_p &= \{\omega\,;\,|R_{3p}(\omega)|<\sigma_{3p},\ |R_{3p+1}(\omega)|<\sigma_{3p+1}\}\end{aligned}$$

for $(3p, 3p+1)$ such that $\left[\frac{k}{2}\right] + 1 \le 3p < 3p+1 \le k$. It can be easily seen that

$$\Pr\left(E_p \cup F_p\right) = \delta_p \ \ (say) > \delta$$

where $\delta > 0$ is a certain constant. And we define random variables $\eta_p, \zeta_p$ and $\xi_p$ such that

$$\eta_p = \begin{cases} 1 & \text{on } E_p \cup F_p \\ 0 & \text{elsewhere ,} \end{cases}$$

$$\zeta_p = \begin{cases} 0 & \text{on } G_p \\ 1 & \text{elsewhere} \end{cases}$$

and

$$\xi_p = \eta_p - \eta_p \zeta_p.$$

If $\xi_p = 1$, there is a root of the polynomial in the interval $(x_{3p}, x_{3p+1})$. Let $p_{\min}$ and $p_{\max}$ be the integers such that

$$\begin{aligned} p_{\min} &= \min\left\{p \in \mathcal{N} \,\middle|\, \left[\frac{k}{2}\right] + 1 \le 3p < 3p+1 \le k\right\} \\ p_{\max} &= \max\left\{p \in \mathcal{N} \,\middle|\, \left[\frac{k}{2}\right] + 1 \le 3p < 3p+1 \le k\right\}. \end{aligned}$$

Then the number of roots in the $(x_{\left[\frac{k}{2}\right]+1}, x_k)$ must exceed $\sum_{p=p_{\min}}^{p_{\max}} \xi_p$.

We shall need the strong low of large numbers in the following form.

THEOREM *If $\eta_2, \eta_3, \ldots$ are independent random variables with $V(\eta_i) < 1$ for all $i$, then for given any $\varepsilon > 0$, we have*

$$\Pr\left(\left\{\sup_{p_{\max}-p_{\min}+1 \ge k_0} \left|\frac{1}{p_{\max}-p_{\min}+1} \sum_{p=p_{\min}}^{p_{\max}} \left(\eta_p - E(\eta_p)\right)\right| \ge \varepsilon\right\}\right) \le \frac{D}{\varepsilon^2 k_0}$$

*where $D$ is a positive constant.*

Here we get

$$\left|\sum_{p=p_{\min}}^{p_{\max}} \left(\xi_p - E\left(\eta_p\right)\right)\right| \le \left|\sum_{p=p_{\min}}^{p_{\max}} \left(\eta_p - E\left(\eta_p\right)\right)\right| + \sum_{p=p_{\min}}^{p_{\max}} \zeta_p.$$

Since

$$E\left(\eta_p\right) \le 4\sqrt{\frac{2}{\pi}} \frac{1}{\lambda_{3p}} e^{-\lambda_{3p}^2/2}$$

from Lemma 2.4, we have

$$\sum_{p=p_{\min}}^{p_{\max}} \zeta_p < (p_{\max} - p_{\min} + 1)\,\varepsilon_1$$

outside an exceptional set of measure at most

$$4\sqrt{\frac{2}{\pi}} \sum_{p=p_{\min}}^{p_{\max}} \frac{1}{(p_{\max} - p_{\min} + 1)\,\varepsilon_1} \frac{1}{\lambda_{3p}} e^{-\lambda_{3p}^2/2} < C_5 \frac{1}{\lambda_{3p_{\min}}} e^{-\lambda_{3p_{\min}}^2/2}.$$

Thus we obtain

$$\sup_{p_{\max} - p_{\min} + 1 \geq k_0} \frac{1}{p_{\max} - p_{\min} + 1} \sum_{p=p_{\min}}^{p_{\max}} \zeta_p < \varepsilon_1$$

outside an exceptional set of measure at most

$$C_5 \sum_{p_{\max} - p_{\min} + 1 \geq k_0} \frac{1}{\lambda_{3p_{\min}}} e^{-\lambda_{3p_{\min}}^2/2}.$$

By using the strong law of large numbers since the $\eta_p$'s are independent for sufficiently large $n$, we have

$$\sup_{p_{\max} - p_{\min} + 1 \geq k_0} \frac{1}{p_{\max} - p_{\min} + 1} \left| \sum_{p=p_{\min}}^{p_{\max}} (\xi_p - E(\eta_p)) \right| < \varepsilon$$

outside an exceptional set of measure at most

$$C_5 \sum_{p_{\max} - p_{\min} + 1 \geq k_0} \frac{1}{\lambda_{3p_{\min}}} e^{-\lambda_{3p_{\min}}^2/2} + \frac{C_6}{k_0}$$

where $C_6$ is a constant. A simple calculation shows that

$$p_{\max} = \left[\frac{k+2}{3}\right] - 1 \text{ and } p_{\min} = \left[\frac{k}{6}\right] + 1.$$

Hence we obtain

$$\frac{1}{p_{\max} - p_{\min} + 1} \sum_{p=p_{\min}}^{p_{\max}} \xi_p > \frac{1}{p_{\max} - p_{\min} + 1} \sum_{p=p_{\min}}^{p_{\max}} E(\eta_p) - \varepsilon$$

for all $k$ such that $p_{\max} - p_{\min} + 1 \geq k_0$ outside an exceptional set. Applying $E(\eta_p) > \delta$ and using (8),

$$N_n > \sum_{p=p_{\max}}^{p_{\min}} \xi_p > (p_{\max} - p_{\min} + 1)(\delta - \varepsilon) > C_7 k > C_8 \varepsilon_n \log n$$

for all $k$ such that $p_{\max} - p_{\min} + 1 \geq k_0$ outside an exceptional set, where $C_7$ and $C_8$ are constants.

It can be seen that the set $\{k \in \mathcal{N} \, | p_{\max} - p_{\min} + 1 \geq k_0\}$ are included by the set $\{k \in \mathcal{N} \, | k \geq 6k_0 - 2\}$. If $n = n_0$ corresponds to $k = 6k_0 - 2$, then all $n > n_0$ will correspond to $k > 6k_0 - 2$. Therefore we have for all $n > n_0$,

$$N_n > \varepsilon_n \log n$$

outside a set of measure at most

$$C_5 \sum_{p_{\max} - p_{\min} + 1 \geq k_0} \frac{1}{\lambda_{3p_{\min}}} e^{-\lambda_{3p_{\min}}^2/2} + \frac{C_6}{k_0} \leq C_9 \sum_{q \geq k_0} \frac{1}{\lambda_{3q}} e^{-\lambda_{3q}^2/2} + \frac{C_6}{k_0}$$

where $C_9$ is a constant.

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