International Journal of Applied Mathematical Sciences.
ISSN 0973-0176 Volume 19, Number 2 (2026), pp. 165-179
©Research India Publications
http://www.ripublication.com/jams.htm

Algebraic Spectral Signatures of Planar Multi-Delay Systems: Invariant Reduction and Delay-Sum Aggregation

Mohamed Taki ABDOUL-KARIM

Department of Statistics, University Institute of Technology (IUT)
LMSIA, University of Comoros (UDC) Moroni, Comoros
Corresponding author: takimed2002@gmail.com
ORCID (corresponding author): 0009-0002-9349-4160

Abstract

We investigate the matrix information retained by the characteristic quasipolynomial of planar linear systems with multiple discrete delays. The quadratic determinant is expressed through traces, determinants and pairwise polarized determinant invariants. Decomposing each coefficient matrix into scalar and traceless parts identifies the universal spectral signature with scalar traces and a Gram matrix on the traceless matrix space. Substituting numerical delays induces a linear aggregation map governed by coincidences between delay sums. We prove that its formal rank equals the number of distinct delays plus the number of distinct pairwise delay sums. For two positive delays, nine generically independent invariants are recoverable away from the ratios 1, 2 and $1/2$. The recoverable dimension drops to eight on the $1:2$ and $2:1$ resonance lines and to five when the delays coincide. Explicit kernels and real matrix examples establish sharpness. The framework separates algebraic information loss from arithmetic aggregation and supplies invariant coordinates for subsequent spectral analysis. The analysis is carried out at the level of the exact determinant, so the aggregation mechanism can be examined independently of numerical root computations. We distinguish the formal linear rank of the aggregation operator from the realizability constraints imposed by actual matrix tuples. This distinction clarifies which coefficient combinations are merged by resonant delays and which limitations are already present in the universal invariant representation.

Keywords: delay differential equation, matrix invariant, trace, determinant, quasipolynomial, Gram matrix, spectral recoverability, delay resonance, Sidon set

1. INTRODUCTION

Linear delay differential equations arise in control, biological dynamics, epidemiology, networked systems, and engineering whenever present rates of change depend on past states. Even for linear systems, the introduction of constant delays replaces a finite-dimensional eigenvalue problem by an infinite spectral problem governed by the roots of a characteristic quasipolynomial; see, for example, [1, 2].

For systems with two independent delays, an extensive literature describes imaginary characteristic roots, stability crossing curves, crossing directions, and stability regions in the delay plane [3, 4]. These approaches are primarily concerned with the position and motion of characteristic roots. The present work asks a different structural question: *which algebraic information about the coefficient matrices is actually observable through the exact characteristic quasipolynomial?*

Suppose that all coefficient matrices are simultaneously conjugated,

$$A_j \longmapsto P^{-1}A_jP, \qquad P \in GL_2.$$

The state coordinates change, whereas the characteristic equation does not. Thus individual matrix entries are not intrinsic spectral parameters. Classical invariant theory provides the natural language for this issue. The simultaneous-conjugation invariants of tuples of 2×2 matrices were studied by Sibirskii and, in a broad framework, by Procesi; later accounts include Drensky [5, 6, 7]. For three or more matrices the full invariant algebra contains cubic trace information. Reconstruction from traces of words of lengths one and two is therefore generically finite-to-one rather than injective [8]. These classical results are background here; we do not claim a new generating theory for matrix invariants.

The characteristic determinant of a planar delay system has lower algebraic complexity. Because the determinant on M_2 is quadratic, the universal characteristic polynomial sees only traces, determinants, and pairwise mixed quadratic invariants. This motivates our first reduction:

$$\text{matrix tuple} \longrightarrow \text{universal algebraic spectral signature.}$$

A second reduction occurs after the numerical delays are fixed. Substitution of $z_j = e^{-\lambda\tau_j}$ may force different universal monomials to acquire the same exponential factor. Their invariant coefficients are then aggregated. We study this second map exactly:

$$\text{universal spectral signature} \longrightarrow \text{fixed-delay quasipolynomial.}$$

This viewpoint is deliberately different from structural identifiability in the input–output sense. Time-delay identifiability has been investigated, for example, through generalized

frequency-response functions [9]. We instead assume the characteristic quasipolynomial itself is known exactly and ask which invariant coefficients can be recovered from it. We therefore use the terms *spectral recoverability* and *quasipolynomial recoverability*.

Additive relations among delays are also central in recent delay-system literature. Dubon [10] studies frequency collisions in neutral functional differential equations using determinant cycle covers, torus lifting, and graph structure. Our mechanism is different: matrix invariants are first defined at the universal-polynomial level and only then aggregated by equal scalar delay sums. The two mechanisms occur at different stages of the determinant construction.

The exceptional sets of delays are related to classical additive combinatorics. Conditions requiring pair sums (or, equivalently after translation, pair differences) to be distinct are associated with Sidon sets and Golomb rulers [11]. Related hyperplane arrangements appear in the mid-hyperplane literature [12]. We do not claim the underlying combinatorial arrangements as new. Their role here is spectral: they determine the rank loss of the fixed-delay aggregation map.

Our main contributions are as follows.

(i) We derive the exact universal characteristic polynomial of a planar multi-delay system in simultaneous-conjugation invariant coordinates.

(ii) We identify the resulting universal spectral signature with scalar traces and a Gram matrix of traceless parts, clarifying the algebraic constraints for arbitrary numbers of delays.

(iii) We define the fixed-delay aggregation map and prove the exact rank formula

$$\operatorname{rank} \mathcal{C}_{\boldsymbol{\tau}} = |D_{\boldsymbol{\tau}}| + |\Sigma_{\boldsymbol{\tau}}|.$$

(iv) For two positive delays, where the nine universal invariants are generically independent, we obtain the complete recoverability stratification $9/8/5$, including explicit kernels and real examples showing sharpness.

2. UNIVERSAL CHARACTERISTIC POLYNOMIAL AND QUADRATIC INVARIANTS

Consider

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^{m} A_j x(t-\tau_j), \qquad A_j \in M_2(\mathbb{K}), \tag{1}$$

where initially $\mathbb{K}$ is an algebraically closed field of characteristic zero. Set $\tau_0 = 0$ and introduce independent formal variables $z_1, \ldots, z_m$, with $z_0 = 1$.

Definition 2.1 (Universal characteristic polynomial)**.** For $\mathbf{A} = (A_0, \dots, A_m)$, define

$$P_{\mathbf{A}}(\lambda, \mathbf{z}) = \det\left(\lambda I - \sum_{i=0}^{m} z_i A_i\right). \tag{2}$$

The characteristic quasipolynomial at the prescribed delays is obtained by the substitution

$$z_j = e^{-\lambda \tau_j}.$$

For each i, put

$$t_i = \operatorname{tr} A_i, \qquad d_i = \det A_i,$$

and for $i < j$ define

$$p_{ij} = t_i t_j - \operatorname{tr}(A_i A_j).$$

Lemma 2.2 (Polarized determinant identity)**.** *For $M, N \in M_2(\mathbb{K})$,*

$$\det(M + N) = \det M + \det N + \operatorname{tr}(M)\operatorname{tr}(N) - \operatorname{tr}(MN). \tag{3}$$

Proof. For every 2×2 matrix M,

$$\det M = \frac{1}{2}\left[(\operatorname{tr} M)^2 - \operatorname{tr}(M^2)\right].$$

Apply the identity to $M + N$, expand, and collect the mixed terms. □

Theorem 2.3 (Exact invariant representation)**.** *The universal characteristic polynomial of* (1) *is*

$$\boxed{P_{\mathbf{A}}(\lambda, \mathbf{z}) = \lambda^2 - \lambda \sum_{i=0}^{m} t_i z_i + \sum_{i=0}^{m} d_i z_i^2 + \sum_{0 \le i < j \le m} p_{ij} z_i z_j.} \tag{4}$$

Proof. Let $M(\mathbf{z}) = \sum_i z_i A_i$. Since $M(\mathbf{z}) \in M_2$,

$$\det(\lambda I - M) = \lambda^2 - \lambda \operatorname{tr} M + \det M.$$

The trace term is linear:

$$\operatorname{tr} M = \sum_i t_i z_i.$$

Repeated polarization of the quadratic form $\det$ yields

$$\det M = \sum_i d_i z_i^2 + \sum_{i<j} \left[t_i t_j - \operatorname{tr}(A_i A_j)\right] z_i z_j.$$

Substitution gives (4). □

Definition 2.4 (Universal spectral signature)**.** The tuple

$$\mathscr{S}(\mathbf{A}) = (\{t_i\}, \{d_i\}, \{p_{ij}\})$$

is called the universal algebraic spectral signature of $\mathbf{A}$. Two matrix tuples are *universally spectrally equivalent* if their universal characteristic polynomials coincide.

Proposition 2.5. *Two tuples are universally spectrally equivalent if and only if all their coordinates t_i, d_i, p_{ij} agree.*

Proof. In the polynomial ring $\mathbb{K}[\lambda, z_1, \ldots, z_m]$, the monomials λz_i, z_i^2, and $z_i z_j$ are distinct. Equality of the universal polynomials is therefore equivalent to equality of their coefficients. □

3. TRACE–GRAM REDUCTION

Write

$$A_i = \frac{t_i}{2} I + X_i, \qquad \operatorname{tr} X_i = 0.$$

Hence $X_i \in \mathfrak{sl}_2(\mathbb{K})$, a three-dimensional vector space. Define the symmetric Gram matrix

$$G = (G_{ij}), \qquad G_{ij} = \operatorname{tr}(X_i X_j).$$

A direct calculation gives

$$G_{ii} = \frac{t_i^2}{2} - 2d_i, \tag{5}$$

$$G_{ij} = \frac{t_i t_j}{2} - p_{ij}, \qquad i \neq j. \tag{6}$$

Conversely,

$$d_i = \frac{t_i^2}{4} - \frac{1}{2} G_{ii}, \tag{7}$$

$$p_{ij} = \frac{t_i t_j}{2} - G_{ij}. \tag{8}$$

Thus the spectral coordinates (t_i, d_i, p_{ij}) and $(\mathbf{t}, G)$ are polynomially equivalent.

Theorem 3.1 (Gram description of the universal spectral space)**.** *Let $r = m + 1$. Over an algebraically closed field of characteristic different from two, the universal spectral-signature space is naturally identified with*

$$\mathbb{K}^r \times \{G \in \operatorname{Sym}_r(\mathbb{K}) : \operatorname{rank} G \leq 3\}.$$

For $r \geq 2$, its generic dimension is

$$r + (3r - 3) = 4r - 3.$$

For $r \geq 4$, the Gram component is cut out by the 4×4 minors of G.

Proof. Necessity follows because all X_i lie in the three-dimensional nondegenerate bilinear space $(\mathfrak{sl}_2,\ (X,Y) \mapsto \operatorname{tr}(XY))$. Conversely, over an algebraically closed field every symmetric matrix of rank at most three admits a Gram realization in a fixed nondegenerate three-dimensional symmetric bilinear space. The scalar traces are arbitrary. The dimension formula for symmetric matrices of rank at most three is $3r - 3$. □

Corollary 3.2 (The two-delay case)**.** *For two delays, $r = 3$, the nine invariants*

$$t_A, t_B, t_C, d_A, d_B, d_C, p_{AB}, p_{AC}, p_{BC}$$

are algebraically independent. In particular, the universal spectral signature fills a nine-dimensional affine space.

Remark 3.3
(Information beyond the linear delay spectrum). The full simultaneous-conjugation invariant algebra of a generic triple of 2×2 matrices contains cubic trace information not present in (4); see [5, 6, 7, 8]. For traceless parts X_A, X_B, X_C, the classical cubic orientation quantity

$$\Omega = \operatorname{tr}(X_A X_B X_C)$$

satisfies

$$\Omega^2 = -\frac{1}{2} \det G.$$

This classical invariant-theoretic relation explains why the characteristic quasipolynomial observes the quadratic Gram data but not all information in the full conjugacy quotient. We use this only as interpretation, not as a novel invariant-theory result.

4. REAL REALIZABILITY

For applications the coefficient matrices are real. In the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

the bilinear form $(X,Y) \mapsto \operatorname{tr}(XY)$ on $\mathfrak{sl}_2(\mathbb{R})$ has matrix

$$\operatorname{diag}(2, 2, -2),$$

hence signature $(2, 1)$.

Theorem 4.1 (Real spectral realizability)**.** *A pair*

$$(\mathbf{t}, G) \in \mathbb{R}^r \times \operatorname{Sym}_r(\mathbb{R})$$

is realized by real matrices $A_0, \ldots, A_m \in M_2(\mathbb{R})$ if and only if

$$\operatorname{rank} G \le 3, \qquad n_+(G) \le 2, \qquad n_-(G) \le 1.$$

Proof. Necessity follows from the inertia of the trace form on $\mathfrak{sl}_2(\mathbb{R})$. Conversely, Sylvester's law of inertia gives a factorization $G = V^T J V$, where $J = \mathrm{diag}(2, 2, -2)$ after deletion of unused directions. The columns of V define traceless matrices X_i, and $A_i = X_i + (t_i/2)I$ has the prescribed signature. □

5. THE FIXED-DELAY AGGREGATION MAP

The previous sections concern the universal polynomial, in which the delay variables are independent. We now fix the delay values. Define

$$q_{ii} = d_i, \qquad q_{ij} = p_{ij} \quad (i < j).$$

Then

$$\Delta_{\boldsymbol{\tau}}(\lambda) = \lambda^2 - \lambda \sum_{i=0}^{m} t_i e^{-\tau_i \lambda} + \sum_{0 \le i \le j \le m} q_{ij} e^{-(\tau_i + \tau_j)\lambda}. \tag{9}$$

Let

$$V_m = \mathbb{K}^{m+1} \oplus \mathbb{K}^{(m+1)(m+2)/2}$$

be the *formal* spectral-coefficient space with coordinates (t_i, q_{ij}). We emphasize that for $m \ge 3$, the realizable matrix signatures form a proper algebraic subset of this formal space due to the Gram rank constraints.

Definition 5.1 (Delay-sum aggregation map)**.** For fixed delays $\boldsymbol{\tau}$, define the linear map

$$\mathcal{C}_{\boldsymbol{\tau}} : V_m \to \mathcal{E}_{\boldsymbol{\tau}}$$

by

$$\mathcal{C}_{\boldsymbol{\tau}}(t, q) = -\lambda \sum_{i} t_i e^{-\tau_i \lambda} + \sum_{i \le j} q_{ij} e^{-(\tau_i + \tau_j)\lambda},$$

where $\mathcal{E}_{\boldsymbol{\tau}}$ is the vector space spanned by the exponential-polynomial functions that occur in the expression.

Define

$$D_{\boldsymbol{\tau}} = \{\tau_i : 0 \le i \le m\}$$

and

$$\Sigma_{\boldsymbol{\tau}} = \{\tau_i + \tau_j : 0 \le i \le j \le m\}.$$

Lemma 5.2 (Independence of exponential polynomials)**.** *Let $\alpha_1, \dots, \alpha_s$ be distinct real numbers and let $P_k(\lambda)$ be polynomials. If*

$$\sum_{k=1}^{s} P_k(\lambda) e^{-\alpha_k \lambda} \equiv 0,$$

then $P_k \equiv 0$ for every k.

Proof. Order the exponents increasingly. Multiply by the exponential corresponding to the smallest exponent and let real $\lambda \to +\infty$. The first polynomial must vanish identically; iterate. □

Theorem 5.3 (Exact rank of delay-sum aggregation)**.** *For the formal fixed-delay aggregation map,*

$$\boxed{\operatorname{rank} \mathcal{C}_{\boldsymbol{\tau}} = |D_{\boldsymbol{\tau}}| + |\Sigma_{\boldsymbol{\tau}}|.} \tag{10}$$

Moreover,

$$\ker \mathcal{C}_{\boldsymbol{\tau}} = \left\{ (\delta t, \delta q) : \sum_{\tau_i = \alpha} \delta t_i = 0 \; \forall \alpha, \quad \sum_{\tau_i + \tau_j = \beta} \delta q_{ij} = 0 \; \forall \beta \right\}.$$

Proof. For every $\alpha \in D_{\boldsymbol{\tau}}$, define

$$T_\alpha = \sum_{\tau_i = \alpha} t_i,$$

and for every $\beta \in \Sigma_{\boldsymbol{\tau}}$, define

$$Q_\beta = \sum_{\substack{i \le j \\ \tau_i + \tau_j = \beta}} q_{ij}.$$

Equation (9) becomes

$$-\lambda \sum_{\alpha \in D_{\boldsymbol{\tau}}} T_\alpha e^{-\alpha\lambda} + \sum_{\beta \in \Sigma_{\boldsymbol{\tau}}} Q_\beta e^{-\beta\lambda},$$

apart from the fixed leading term λ^2.

After grouping any exponent occurring in both sets, the coefficient of $e^{-\gamma\lambda}$ is a polynomial of degree at most one, $-T_\gamma \lambda + Q_\gamma$. By the preceding lemma, all such coefficient polynomials are independent. Consequently there is one independent observable coordinate for each distinct delay and one for each distinct pair sum, proving (10). The kernel description is the corresponding system of aggregation equations. □

Remark 5.4. Theorem 5.3 concerns the formal coefficient map. For three or more delays, the dimension of the image of the *realizable matrix-signature variety* under $\mathcal{C}_{\boldsymbol{\tau}}$ may be smaller and is a separate algebraic-geometric problem. For two delays, the nine universal coordinates are algebraically independent, so the formal rank equals the generic recoverable dimension.

6. COMPLETE TWO-DELAY STRATIFICATION

Consider

$$\dot{x}(t) = Ax(t) + Bx(t-\tau_1) + Cx(t-\tau_2), \qquad \tau_1, \tau_2 > 0. \tag{11}$$

Its nine universal coordinates are ordered as

$$(t_A, t_B, t_C, d_A, d_B, d_C, p_{AB}, p_{AC}, p_{BC}).$$

Theorem 6.1 (Recoverability stratification for two delays)**.** *For* (11)*,*

$$\boxed{\operatorname{rank} \mathcal{C}_{\tau_1,\tau_2} = \begin{cases} 9, & \tau_1/\tau_2 \notin \{1, 2, 1/2\}, \\ 8, & \tau_1/\tau_2 \in \{2, 1/2\}, \\ 5, & \tau_1 = \tau_2. \end{cases}}$$

More precisely,

$$\ker \mathcal{C}_{\tau,2\tau} = \operatorname{span}\{e_{d_B} - e_{p_{AC}}\},$$

$$\ker \mathcal{C}_{2\tau,\tau} = \operatorname{span}\{e_{d_C} - e_{p_{AB}}\},$$

whereas for equal delays the kernel has dimension four and is determined by

$$\delta t_B + \delta t_C = 0, \qquad \delta p_{AB} + \delta p_{AC} = 0,$$

$$\delta d_B + \delta d_C + \delta p_{BC} = 0, \qquad \delta t_A = \delta d_A = 0.$$

Proof. The relevant pair sums are

$$0,\ \tau_1,\ \tau_2,\ 2\tau_1,\ \tau_1 + \tau_2,\ 2\tau_2.$$

For positive delays, two different entries in this list coincide exactly when $\tau_1 = \tau_2$, $\tau_2 = 2\tau_1$, or $\tau_1 = 2\tau_2$. Apply Theorem 5.3. The explicit kernels follow by recording the colliding coefficients. When the delays coincide, the system reduces to

$$\dot{x}(t) = Ax(t) + (B + C)x(t-\tau),$$

so only the five combinations

$$t_A, \quad d_A, \quad t_B + t_C, \quad p_{AB} + p_{AC}, \quad d_B + d_C + p_{BC}$$

remain. □

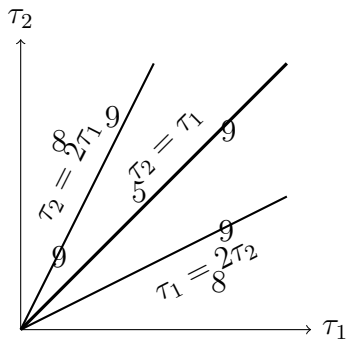


Figure 1: Generic recoverable dimension of the nine-dimensional universal spectral signature in the positive two-delay plane. The generic regions have dimension 9; the $1:2$ and $2:1$ resonance rays have dimension 8; coincident delays have dimension 5.

7. SIDON-TYPE INTERPRETATION

The full formal recovery of the quadratic coefficients occurs precisely when the map

$$\{i, j\} \longmapsto \tau_i + \tau_j$$

is injective on unordered pairs with repetition. This is the pair-sum uniqueness property underlying Sidon or B_2 sets. In our setting $\tau_0 = 0$, and the exceptional loci are hyperplanes

$$\tau_i + \tau_j = \tau_k + \tau_\ell.$$

Related arrangements are classical in additive combinatorics and ranking geometry [11, 12]. Our use of them is restricted to their role as the exceptional set of quasipolynomial recoverability.

Corollary 7.1 (Sidon-type sufficient condition)**.** *If the delay set*

$$\{0, \tau_1, \ldots, \tau_m\}$$

has distinct unordered pair sums, then no quadratic invariant coefficient is aggregated by the fixed-delay substitution. If, in addition, all delays are distinct, the formal aggregation map is injective.

8. EXAMPLES AND SHARPNESS

8.1. Reference tuple

Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Its universal signature is

$$\mathscr{S}_0 = (2, 4, -2, 0, 3, 2, 4, -2, -4),$$

and the Gram matrix of the traceless parts is

$$G_0 = \operatorname{diag}(2, 2, -2), \qquad \det G_0 = -8.$$

8.2. Generic stratum: rank nine

Take

$$\tau_1 = \tau, \qquad \tau_2 = \sqrt{2}\,\tau.$$

All pair sums are distinct, and

$$\begin{aligned}\Delta(\lambda) = \lambda^2 - 2\lambda - 4\lambda e^{-\tau\lambda} + 2\lambda e^{-\sqrt{2}\tau\lambda} \\ + 4e^{-\tau\lambda} - 2e^{-\sqrt{2}\tau\lambda} + 3e^{-2\tau\lambda} \\ - 4e^{-(1+\sqrt{2})\tau\lambda} + 2e^{-2\sqrt{2}\tau\lambda}.\end{aligned}$$

All nine universal coordinates are separately recoverable.

8.3. $1:2$ stratum: rank eight

Define

$$\widetilde{B} = \begin{pmatrix} 2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 2 \end{pmatrix}, \qquad \widetilde{C} = \begin{pmatrix} -\frac{3}{4} & \frac{\sqrt{17}}{4} \\ -\frac{\sqrt{17}}{4} & -\frac{5}{4} \end{pmatrix},$$

and $\widetilde{A} = A$. Then

$$\widetilde{\mathscr{S}} = \left(2, 4, -2, 0, \frac{7}{2}, 2, 4, -\frac{5}{2}, -4\right) \neq \mathscr{S}_0,$$

but

$$d_B + p_{AC} = 3 - 2 = \frac{7}{2} - \frac{5}{2} = d_{\widetilde{B}} + p_{\widetilde{A}\widetilde{C}}.$$

At

$$(\tau_1, \tau_2) = (\tau, 2\tau), \qquad u = e^{-\tau\lambda},$$

both systems have exactly

$$\Delta(\lambda) = \lambda^2 - \lambda(2 + 4u - 2u^2) + 4u + u^2 - 4u^3 + 2u^4.$$

The second Gram matrix is

$$\widetilde{G} = \begin{pmatrix} 2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -2 \end{pmatrix}, \qquad \det \widetilde{G} = -\frac{17}{4} \neq 0.$$

Thus the rank-eight ambiguity is realized by nondegenerate real tuples.

8.4. Equal-delay stratum: rank five

Set

$$\widehat{B} = B + I, \qquad \widehat{C} = C - I.$$

Then

$$\widehat{B} + \widehat{C} = B + C,$$

whereas

$$\widehat{\mathscr{S}} = (2, 6, -4, 0, 8, 5, 6, -4, -12) \neq \mathscr{S}_0.$$

For $\tau_1 = \tau_2 = \tau$, both systems reduce to

$$\dot{x}(t) = Ax(t) + (B + C)x(t - \tau)$$

and have

$$\Delta(\lambda) = \lambda^2 - \lambda(2 + 2u) + 2u + u^2, \qquad u = e^{-\tau\lambda}.$$

Hence four universal directions are invisible and the rank is five.

9. RELATION WITH FREQUENCY-COLLISION APPROACHES

A structurally related but distinct viewpoint has recently been developed for neutral functional differential equations, where frequency collisions in a characteristic determinant are analyzed through delay-labelled interaction graphs and torus lifting [10]. In that framework, different structural cycle covers may generate identical frequency contributions and affect strong stability.

The mechanism studied here is at a different level. We first reduce a planar retarded matrix system to an already-defined simultaneous-conjugation invariant signature. Numerical delays are introduced only afterwards. Equality of scalar delay sums then aggregates those invariant coefficients. Thus graph-induced frequency degeneracy and delay-sum invariant aggregation are complementary phenomena and should not be conflated.

10. IMPLICATIONS FOR STABILITY AND BIFURCATION

Every property determined solely by the roots of the characteristic quasipolynomial is constant on a fixed-delay spectral-equivalence class. Consequently, stability regions, spectral abscissae, imaginary crossing frequencies, and root-crossing directions can ultimately be expressed in the recoverable invariant coordinates rather than in individual matrix entries.

For two delays, the generic region of Fig. 1 retains all nine universal coordinates, whereas on the resonance rays some linear spectral information is fundamentally unavailable

from the exact quasipolynomial. This distinction should be accounted for in any invariant formulation of Hopf or double-Hopf conditions.

We do not develop such bifurcation criteria here. They constitute a separate problem and are intentionally deferred in order to keep the present article focused on the algebraic spectral reduction and its recoverability stratification.

11. CONCLUSION

For planar linear systems with multiple discrete delays, the characteristic determinant admits a particularly transparent invariant structure. The quadratic nature of the 2×2 determinant reduces the matrix tuple to traces, determinants, and pairwise polarized determinant invariants. In traceless coordinates this information is equivalent to scalar traces and a rank-at-most-three Gram matrix.

After numerical delay values are fixed, a second and conceptually different information reduction occurs. Equal delay sums aggregate universal invariant coefficients. The resulting linear map has exact formal rank

$$|D_{\boldsymbol{\tau}}| + |\Sigma_{\boldsymbol{\tau}}|.$$

For two positive delays this yields the complete sharp stratification

$$9 \longrightarrow 8 \longrightarrow 5$$

according to whether the delay ratio is generic, $1 : 2$ or $2 : 1$, or $1 : 1$. Explicit real examples show that all three strata correspond to actual matrix systems rather than to formal coefficient artifacts.

The results provide a coordinate-free starting point for subsequent work on stability and bifurcation in planar delay systems. In particular, they make clear which invariant information is available before one attempts to formulate Hopf, double-Hopf, or robust-stability conditions.

DATA AVAILABILITY

No datasets were generated or analyzed in this theoretical study.

DECLARATION OF COMPETING INTEREST

The author declares no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

FUNDING

This research received no specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

REFERENCES

[1] J. K. Hale, S. M. Verduyn Lunel, Introduction to Functional Differential Equations, Vol. 99 of Applied Mathematical Sciences, Springer, New York, 1993.

[2] K. Gu, V. L. Kharitonov, J. Chen, Stability of Time-Delay Systems, Birkhäuser, Boston, 2003.

[3] K. Gu, S.-I. Niculescu, J. Chen, On stability crossing curves for general systems with two delays, Journal of Mathematical Analysis and Applications 311 (1) (2005) 231–253. doi:10.1016/j.jmaa.2005.02.034.

[4] S. Li, X. Song, C. Huang, Further study on the crossing curves in twodelay differential equations with delay-dependent coefficients, Applied Mathematics Letters 158 (2024) 109264. doi:10.1016/j.aml.2024.109264.

[5] K. S. Sibirskii, Algebraic invariants of a system of matrices, Siberian Mathematical Journal 9 (1) (1968) 115–124. doi:10.1007/BF02196663.

[6] C. Procesi, The invariant theory of n $\times$ n matrices, Advances in Mathematics 19 (3) (1976) 306–381. doi:10.1016/0001-8708(76)90027-X.

[7] V. Drensky, Computing with matrix invariants (2006). arXiv:math/ 0506614.

[8] C. A. A. Florentino, Invariants of 2 $\times$ 2 matrices, irreducible SL(2, C) characters and the magnus trace map, Geometriae Dedicata 121 (1) (2006) 167–186. doi:10.1007/s10711-006-9097-5.

[9] G. Szlobodnyik, G. Szederkényi, Structural identifiability analysis of nonlinear time delayed systems with generalized frequency response functions, Kybernetika 57 (6) (2021) 939–957. doi:10.14736/kyb-2021-6-0939.

[10] E. Dubon, Torus lifting and frequency collisions in the strong stability of neutral delay systems, Journal of Dynamics and Differential Equations (2026). doi:10.1007/s10884-026-10530-1.

[11] M. Beck, T. Bogart, T. Pham, Enumeration of golomb rulers and acyclic orientations of mixed graphs, The Electronic Journal of Combinatorics 19 (3) (2012) P42. doi:10.37236/2741.

[12] H. Kamiya, P. Orlik, A. Takemura, H. Terao, Arrangements and ranking patterns, Annals of Combinatorics 10 (2) (2006) 219–235. doi:10.1007/ s00026-006-0284-8.

Biographical Sketch

Mohamed Taki ABDOUL-KARIM is a lecturer and researcher in the Department of Statistics at the University Institute of Technology (IUT), University of Comoros, and is affiliated with LMSIA. His research interests include applied mathematics, delay differential equations, dynamical systems, statistical inference, and mathematical modelling.