Advances in Algebra
ISSN 0973-6964 Volume 15, Number 1 (2022), pp. 1-4

http://www.ripublication.com

On "An Elementary Proof of Fermat's Last Theorem" and Further Elementary Results

Richard Kaufman*

Office Expander (www.OfficeExpander.com)
North Andover, MA, U.S.A.

Abstract

This brief paper shows that an elementary proof of Fermat's Last Theorem published in this journal is not correct. However, the present paper also provides for further elementary results on Fermat's Last Theorem, where there would be positive integer counterexamples to Fermat's Equation, $a^n + b^n = c^n$, for $n \geq 3$. Specifically, as n increases, then b/c approaches 1, and both b and c must eventually increase. Using previous results for $a = x + y$, $b = y + z$, and $c = x + z$, where x, y, and z are positive integers, then it is established that z must also eventually increase with n.

Mathematics Subject Classification (2020): 11A05; 11C08

Keywords: Fermat's Last Theorem, Elementary Results

1. INTRODUCTION

A quote attributed to Piet Hein states, "*Problems worthy of attack prove their worth by fighting back.*" This quote seems apt to the search for an elementary proof of Fermat's Last Theorem.

This brief paper indicates an issue appearing in the paper [1] titled, "*An elementary Proof of Fermat's Last Theorem*" published by this journal. The paper attempts to use a proof by contradiction. If Fermat's Last Theorem was false, then there would exist positive integers $x, y,$ and z for the Fermat equation $x^n + y^n = z^n$ with integer $n \geq 3$.

Without loss of generality, $x, y,$ and z are assumed to be in lowest terms (no common denominator). The paper aims to show that a contradiction would result, thereby proving Fermat's Last Theorem.

In what follows, we may assume, without loss of generality, that $0 < x < y < z$. So $p = z - x > 1$, as stated in the paper. The paper rewrites the Fermat equation as $(x + p)^n = x^n + y^n$. It then correctly states that $p = z - x$ must divide y^n. This result is correct and was previously known [2].

The paper states that $\gcd(p, y) = u > 1$. However, the paper later asserts that $\frac{y^n}{p}$ would be divisible by u, which is not justified.

To see this, consider that $y = uv$, where u and v are co-prime positive integers, and that $p = u^n$. Clearly $p = u^n$ divides $y^n = u^n v^n$. However, $\frac{y^n}{p} = \frac{u^n v^n}{u^n} = v^n$, which is not divisible by u.

The above example is reminiscent of the 1810 relations of Barlow [3], when y is not divisible by n. For the case when y is divisible by n, then the relation would be similar to $y = n^m uv$, where $p = z - x = n^{nm-1} u^n$, for positive integer $m \geq 2$. Here u and v are co-prime positive integers that are not divisible by n. In this case, $\frac{y^n}{p} = \frac{n^{nm} u^n v^n}{n^{nm-1} u^n} = nv^n$, which is not divisible by u.

It should be noted that the paper does reference Andrew Wiles [4] non-elementary proof of Fermat's Last Theorem. In spite of this proof, it has still never been proven that no elementary proof could exist. It is in that spirit of seeking elementary results that the following is presented.

2. FURTHER ELEMENTARY RESULTS ON FERMAT'S LAST THEOREM

My recent paper [5] considered that elementary results for Fermat's Last Theorem could generate interesting mathematics even though a non-elementary proof was provided by Andrew Wiles. My paper used different letters (*i.e.,* a, b, c, x, y, z) then the paper referenced earlier in this journal. The present paper expands on those recent elementary results and shows that positive integer solutions to Fermat's Equation, $a^n + b^n = c^n$, must eventually increase as integer *n* increases above 2. Further requirements on how the solutions must increase are discussed.

If Fermat's Last Theorem were false, then there would exist positive integers a, b, and c, where $0 < a < b < c$ and $a^n + b^n = c^n$, for some integer $n \geq 3$. In [5], it was shown that $a = x + y$, $b = y + z$, and $c = x + z$, where x, y, and z are positive integers with $0 < y < x < z$.

The following two restrictions were also established: $a/c < 2^{-1/n}$ and $b/c > 2^{-1/n}$. A plot of $f(n) = 2^{-1/n}$ for $n \geq 3$ is shown in Fig. 1. Since $b < c$ and $b/c > 2^{-1/n}$, then the $\lim_{n\to\infty} 2^{-1/n} = 1$ establishes that b/c approaches 1 as n increases above 2.

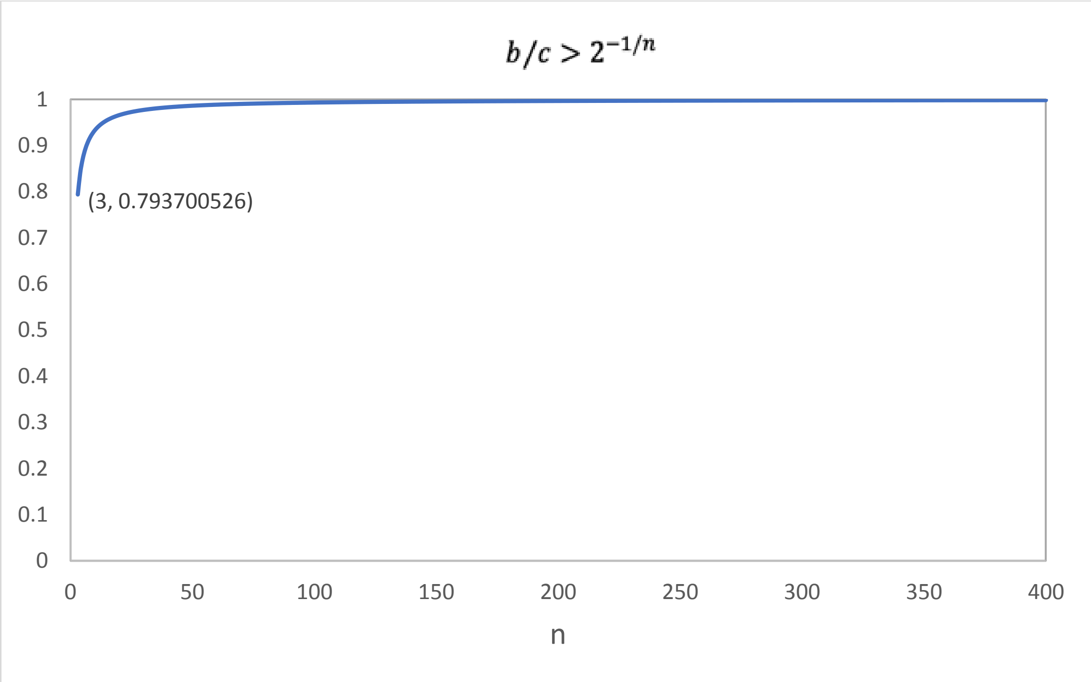

Figure 1

Since b and c are positive integers where $b < c$, then $b + 1 \leq c$, and b cannot get any closer to c then 1. So as n increases and b/c approaches 1, then both $b = y + z$ and $c = x + z$ must *eventually* increase.

Yet, x and y are also positive integers where $y < x$. So $y + 1 \leq x$, and x cannot get any closer to y then 1. Therefore, as n increase, then z must *eventually* increase so that $b/c = (y + z)/(x + z)$ approaches 1.

We have the following as n increases above 2:

- b/c approaches 1
- b and c must eventually increase
- z must eventually increase

These are further restrictions on any counterexamples to Fermat's Last Theorem. As was pointed out in my recent paper [5], Andrew Wiles stated [6, p.180], "The problem with working on Fermat was that you could spend years getting nowhere. It's fine to work on any problem, so long as it generates interesting mathematics along the way—even if you do not solve it at the end of the day."

ACKNOWLEDGEMENT

The author would like to thank Professor (emer.) Steve Pennell of the University of Massachusetts Lowell for his review and corrections to the further elementary results portion of this paper.

REFERENCES

---

[1] Kelvin Muzundu (2021) *An Elementary Proof of Fermat's Last Theorem*, Advances in Algebra, Volume 14, Number 1, pp. 1-2. Online at: https://www.ripublication.com/aa21/aav14n1_01.pdf (accessed Dec. 22, 2021)

[2] Since the Fermat equation can be rewritten as $z^n - x^n = y^n$, and $\frac{(z^n - x^n)}{(z-x)} = \sum_{j=1}^{n} z^{n-j} x^{j-1}$, which is an integer, was previously known. See the proof that "$a^k - b^k$ is always divisible by $a - b$", and that the "quotient is the so-called 'cyclotomic' expression, $a^{k-1} + a^{k-2}b + a^{k-3}b^2 + \cdots + a^2b^{k-3} + ab^{k-2} + b^{k-1}$ of degree $k - 1$" on pages 42-43 of: Ogilvy, Charles Stanley, and John Timothy Anderson. *Excursions in number theory*. Courier Corporation, 1988.

[3] Paulo Ribenboim (1999). *Fermat's Last Theorem* for Amateurs. New York. Springer Verlag. pp. 99-102.

[4] Andrew Wiles (1995). Modular elliptic curves and Fermat's Last Theorem, *Ann. Math.* 141: 443-551.

[5] Richard Kaufman (2020) Limits on Legs of Pythagorean Triples and Fermat's Last Theorem, *The College Mathematics Journal*, 51:1, 53-56, DOI: 10.1080/07468342.2020.1674620

[6] Singh, S. (1998). *Fermat's Last Theorem*. London: Fourth Estate Limited.