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Convergence of C^2 Deficient Quartic Spline Interpolation

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Abstract

In this paper we have obtained a precise error estimate of spline interpolant matching the given function values at the mesh points and its derivative at midpoint and also boundary points.

Keywords: Best error bounds, Interpolation, Deficient Quartic Splines.

1. INTRODUCTION

It has been observe that for efficient approximation of functions cubic and higher degree splines are still popular (see Deboor [1]). As in the study of lower degree spline i.e. linear, we get corner's at joints of two linear pieces and therefore more date than higher order method are usually required to achieve a prescribed accuracy, thus for a smooth and more efficient approximation one has to go higher degree splines. Various aspects of cubic interpolatory splines have been extended number of authors Dubean [3], Rana and Dubey [10], Morken and Reimers [8]. Kopotun [7] has shown the equivalence of moduli of smoothness and applications of univariate splines . Howell and Verma [6] give an interesting application of the theorem of Birkhoff and Deboor [4,5] concerning optimal bounds. In the present paper we consider a related problem of deficient quartic spline interpolation matching the given function values at inerpolatory and its derivative at mid point and boundary points also.

2. EXISTENCE AND UNIQUENESS.

Consider a mesh P of [0, 1] given by $0 = x_0 < x_1, \ldots, x_n = 11$ such that $x_{i+1} - x_i = h_i$, $i = 0,1,\ldots,k-1$. For a positive integer m, let π_m $[0,1]$ denote the set of all real algebraic polynomials of degree not grater than m. For a function defined over P, we denote the restriction of s over $[x_i, x_{i+1}]$ by s_i, the class of $s(4,2,P)$ of deficient quartic spline functions over P is given by

S(4,2, P) = { s: s∈C^2 [0,1], $s_i \in \pi_4$: i = 0,1,2, ...,k-1},

where in S^* (4,2, P) denotes the class of all deficient quartic splines S(4,2, P) which satisfies the boundary conditions.

$$s'(x_0) = f'(x_0), \qquad s'(x_k) = f'(x_k). \tag{2.1}$$

We introduce the following interpolatory conditions

$$s(x_i) = f(x_i), \qquad i = 0, \ldots\ldots, k-1, k \tag{2.2}$$

$$s(x_i) = f(x_i), \qquad i = 0, \ldots\ldots, k-1 \tag{2.2}$$

$$s'(z_i) = f'(z_i), \qquad i = 0, \ldots\ldots, k-1 \tag{2.3}$$

where $z_i = \dfrac{x_i + x_{i-1}}{2}$

Infact, we shall prove the following,

THEOREM 2.1: Let f' exist, then there exists a unique deficient quartic splines in $S^*(4,2,P)$ which satisfies the interpolatory condition (2.1) - (2.3)

Let $q(z)$ is a quartic Polynomial defined on $[0,1]$. It can easily verified that

$$q(z) = q(0)P_1(z) + q'\left(\frac{1}{2}\right) P_2(z) + q(1)P_3(z) + q'(0)P_4(z) + q'(1)P_5(z) \tag{2.4}$$

where

$$P_1(z) = \frac{(1-z)^2}{2}[2 + 4z - z^2]$$

$$P_2(z) = -z^2(z-1)^2$$

$$P_3(z) = \frac{z^2}{2}(7 - 6z + z^2)$$

$$P_4(z) = z(1-z)^2\left(1 + \frac{z}{4}\right)$$

$P_5(z) = \dfrac{z^2(z-1)(3+z)}{4}$

Now writing $x = x_i + th_i \quad 0 \le t \le 1.$

(2.4) may be expressed in restriction $S_i(x)$ of S on $[x_i, x_{i+1}]$, as follows

$s_i(x) = f(x_i)P_1(t) + f'(Z_{i+1})P_2(t) + f(x_{i+1})P_3(t) + h_i s_i'(x_i)P_4(t) + h_i s_i'(x_{i+1})P_5(t)$ (2.5)

Which is clearly satisfy the condition (2.1) - (2.3) and $s_i(x)$ is a quartic in $[x_i, x_{i+1}]$ for $i = 0,1,\dots,k-1$.

Now observing that $s(x) \in C^2[0,1]$, therefore applying continuity condition

$$s''(x_i+) = s''(x_i-), \quad i = 0,1,2,\dots,k-1. \tag{2.6}$$

we get

$$5h_i s'(x_{i-1}) + (9h_i + 7h_{i-1})s'(x_i) + 3h_{i-1}s'(x_{i+1}) = \frac{14h_{i-1}}{h_i}\Delta f(x_i) + \frac{10h_i}{h_{i-1}}\Delta f(x_{i-1}) + 4\left(\frac{h_i}{h_{i-1}}f'(Z_i) - \frac{h_{i-1}}{h_i}f'(Z_{i+1})\right) \tag{2.7}$$

It may be seen easily that the coefficient matrix of the system of equation (2.7) is diagonally dominant and hence invertible.

This proof of theorem 2.1.

3. ERROR BOUNDS.

In this section of the paper, we shall estimate the bounds of error function and of its derivative

i.e $e^r(x) = f^r(x) - s^r(x), \; r = 0,1.$

for the spline interpolant of theorem 2.1 which are best possible. Let $s(x)$ be the twice continuously differentiable quartic spline function satisfying the condition of theorem 2.1. Now considering $f \in C^{(5)}$ [0,1] $f \in C^5$ [0,1] and denoting the unique polynomial $L_i\,[f,x]$ for which agrees with given functional values and derivative i.e. $f(x_i)$, $f\,(x_{i+1})$, $f'(Z_i), f'(x_i)$ and $f'(x_{i+1})$, we see that for $x \in [x_i, x_{i+1}]$,

$$|s(x) - f(x)| \cong |s_i(x) - f(x)| \le |s_i(x) - L_i[f,x]| + |L_i\,[f,x] - f(x)| \tag{3.1}$$

We now proceed to get pointwise bounds of both the terms on the right hand side of (3.1). The error estimate of the second terms can be obtained by following a well known reminders theorem for polynomial (see Cauchy [9]).

$$\Rightarrow |f(x) - \mathcal{L}[f,x]| \leq \frac{h_i^5}{5!}\left|t^2\left(\frac{1}{2}-t\right)(1-t)^2\right| U$$

$$\text{where } t = \frac{x - x_i}{h_i}, \quad U = \max_{0 \leq X \leq 1} |f^5(x)|$$

Now from (2.4)

$$\mathcal{L}[f,x] - s_i(x) = h_i[f'(x_i) - s_i'(x_i)]P_4(t) + h_i[f'(x_{i+1}) - s'(x_{i+1})]P_5(t) \tag{3.3}$$

We set $e'(x_i) = f'(x_i) - s_i'(x_i)$

and using equation from (2.4)

$$|\mathcal{L}_i[f,x] - s_i(x)| \leq h_i|e'(x_i)|\ |P_4(t)| + h_i|e'(x_{i+1})|\ |P_5(t)| \tag{3.4}$$

As $P_4(t) = t(1-t)^2\left(1+\frac{t}{4}\right)$, $\quad P_5(t) = \frac{t^2(t-1)(z+t)}{4}$

Noted that $P_4(t) \geq 0$, when $0 \leq t \leq 1$. $P_5(t) \leq 0$.

$$\Rightarrow \quad |P_4(t)| + |P_5(t)| = |P_4(t) + P_5(t)| = |t(1-t)| \tag{3.5}$$

By using (3.4) and (3.5) it follows that

$$|\mathcal{L}_i[f,x] - s(x)| \leq h_i \max\{|e'(x_i),|e'(x_{i+1})|\}\ t\,(1-t) \tag{3.6}$$

Next, we set, $|e'(x_i)| = \max_{i=1,2,\ldots,k-1.} |e'(x_i)\,|$ and

$h = \max h_i$, $\qquad i = 0,1,\ldots,k-1$.

Then we may express (3.6) as

$$|\mathcal{L}_i[f,x] - s(x)| \leq h|e'(x_i)|\ \ t(1-t) \tag{3.7}$$

Next, we find upper bound for $e'(x_i)$. Using equation (2.7), it follows that

$$5h_i\, e'(x_{i-1}) + (9h_i + 7h_{i-1})e'(x_i) + 3h_{i-1}e'(x_{i+1}) = B_0(f) \tag{3.8}$$

where

$$B_0(t) = 5h_i f'(x_{i-1}) + (9h_i + 7h_{i-1})f'(x_i) + 3h_{i-1}f'(x_{i+1}) - \frac{14h_{i-1}}{h_i}\Delta f(x_i)$$

$$-\frac{10h_i}{h_{i-1}}\Delta f(x_{i-1}) + 4\left(-\frac{h_i}{h_{i-1}} f'(z_i) + \frac{h_{i-1}}{h_i} f'(z_{i+1})\right).$$

Noted that $B_0(f)$ is a linear functional which is zero for polynomials of degree 4 or less. We can apply Piano theorem [9] to obtain

$$B_0(f) = \int_{x_{i-1}}^{x_{i+1}} \frac{f^5(y)}{4!} B_0[(x-y)_+^4]\ dy \tag{3.9}$$

so
$$|B_0(f)| \leq \frac{U}{4!} \int_{x_{i-1}}^{x_{i+1}} |B_0[(x-y)_+^4]|\ dy\,, \tag{3.10}$$

where $U = \max_{0<x<1} |f^5(x_i)|$. From (3.8) that for $x_{i-1} \leq y \leq x_{i+1}$.

$$B_0[(x-y)_+^4 = [4(9h_i + 7h_{i-1})(x_i - y)_+^3 + 12h_{i-1}(x_{i+1} - y)^3$$
$$-14\frac{h_{i-1}}{h_i}[(x_{i+1} - y)_+^4 - (x_i - y)] - \frac{10h_i}{h_{i-1}}(x_i - y)_+^4$$
$$+16\left\{-\frac{h_i}{h_{i-1}}(Z_i - y)_+^3 + \frac{h_{i-1}}{h_i}(Z_{i+1} - y)_+^3\right\}$$

In order to estimate the interval of r.h.s. of (3.10), we rewrite the above expression in the symmetric form about x_i to get

$$B_0[(x-y)_+^4 = \frac{2h_i}{h_{i-1}}\ [h_{i-1} - 7(x_i - y)](x_i - y - h_{i-1})^3, \qquad x_{i-1} \leq y \leq Z_i$$
$$= \frac{2h_i}{h_{i-1}}[-7(x_i - y)^4 + 2(x_i - y)^3(11h_{i-1} + 4) - 12(x_i - y)^2$$
$$h_{i-1}(2h_{i-1} + 1) + 2(x_i - y)\{(5h_{j-1} + z)h_{i-1}^2 - h_{i-1}^3(1 + h_{i-1})]$$
$$Z_i \leq y \leq x_i.$$
$$= \frac{2h_{i-1}}{h_i}[-7(x_i - y)^4 + 2(x_i - y)^3(-11h_i + 4) + 12(x_i - y)^2$$
$$h_i(-2h_i + 1) + 2(x_i - y)\{(-5h_j + z)\}h_i^2 + h_i^3(1 - h_i)]$$
$$x_i \leq y \leq Z_{i+1}$$
$$= \frac{2h_{i-1}}{h_i}\ (x_i - y + h_i)^3[-h_i - 7(x_i - y)]$$
$$Z_{i+1} \leq y \leq x_{i+1}$$

From, the above expression, its follows that

$$\int_{x_{i-1}}^{x_{i+1}} |B_0[(x-y)_+^4]| dy \leq \frac{2h_{i-1}h_i\{h_i^3 + h_{i-1}^3\}}{5} \tag{3.11}$$

Using equation (3.10) and (3.11), we have

$$|B_0(f)| \leq \frac{2}{5!} U h_{i-1} h_i \{h_j^3 + h_{i-1}^3\} \tag{3.12}$$

From (3.7), (3.8) and (3.12) we have

$$\max_{i=1,2,\ldots,k-1} |e'(x_i)| = |e_i'| \leq \frac{U}{6!} h^4 \tag{3.13}$$

On combining (3.2), (3.7), (3.13) we have,

$$|f(x) - s(x)| \leq |f(x) - \mathcal{L}[f,x]| + |\mathcal{L}[f,x] - f(x)|$$

$$\leq \frac{h^5}{5!}\left|t^2\left(\frac{1}{2} - t\right)(1-t)^2\right| U + \frac{h^5}{6!} U |t(1-t)|$$

$$\leq \frac{h^5}{5!} U |C(t)| \tag{3.14}$$

$$c(t) = \left\{t^2\left(\frac{1}{2} - t\right)(1-t) + t(1-t)\right\}$$

$$\text{where} \quad C(t) = \left\{t^2\left(\frac{1}{2} - t\right)(1-t) + \frac{t(1-t)}{6}\right\} \tag{3.15}$$

Thus, we prove the following

THEOREM 3.1: Let $s(x)$ be the quartic spline interpolant of theorem 2.1. interpolating a given function and $f \in C^5[0,1]$, Then

$$|e(x) \leq K \frac{h^5}{5!} \max\left|f^{(5)}(x)\right|, \quad \text{where} \quad K = \max_{0 \leq x \leq 1} |C(t)| \tag{3.16}$$

defined by (3.15), also, we have

$$|e'(x)| \leq \frac{h^4}{6!} \max_{0 \leq x \leq 1} \left|f^{(5)}(x)\right| \tag{3.17}$$

Furthermore, it can be seen easily that K in (3.16) be improved for an equally spaced partition. Inequality (3.17) is also best possible. Also we have

$$|e'(x)| \leq K_1 \left(\frac{h^4}{6!}\right) \max_{0 \leq x \leq 1} |f^{(5)}(x)| \tag{3.18}$$

where K_1 is positive constant. Equation (3.14) prove (3.16) where inequality (3.17) is a direct consequence of (3.13).

Now, we turn to see that the inequality is best possible in the limit. Considering $f(x) = \frac{x^5}{5!}$ and using the Cauchy formula [9], we have

$$\mathcal{L}_i \frac{[t^5, x]}{5!} - \frac{x^5}{5!} = \frac{h^5}{(5!)}(1-t)^2\left(t-\frac{1}{2}\right)t^2 \tag{3.19}$$

Further, for $x_{i+1} - x_i = h, \quad \forall\, i = 1,2,\dots,k-1.$

$$B_0\left[\frac{x^5}{5!}\right] = 5e'(x_{i-1}) + 16\, e'(x_i) + 3e'(x_{i+1})$$

$$= \frac{4h^4}{5!} \tag{3.20}$$

Suppose for a moment that

$$e'(x_i) = \frac{h^4}{(6!)} = e'(x_{i+1}) = e'(x_{i-1}) \tag{3.21}$$

Then, on using (3.4) we have

$$s(x) - \mathcal{L}_i[f,x] = \frac{h^5}{(6!)}\left(P_4(t) + P_5(t)\right) \;= \frac{h^5}{(6!)}\left(t(1-t)\right) \tag{3.22}$$

On combining (3.19) and (3.22) we have,

$$f(x) - s(x) = \frac{h^5}{5!}\left[\left\{\frac{1}{6}t(1-t) + (1-t)^2\left(t-\frac{1}{2}\right)t^2\right\}\right], \quad x_i \le x \le x_{i+1} \tag{3.23}$$

From (3.23) follows that (3.16) is best possible provided we could prove that

$$e'(x_{i-1}) = e'(x_{i+1}) = e'(x_i) = \frac{h^4}{6!} \quad \text{for} \quad i = 1,2,\dots,k-1. \tag{3.24}$$

In fact (3.24) is attained only in the limit. The difficulty will appear in the case of boundary conditions i.e. $e'(x_0) = e'(x_k) = 0$.

However, it can be shown that as we move many subintervals away from the boundaries $e'(x_i) \longrightarrow \frac{h^4}{6!}$ for that we shall apply (3.20) inductively, to move away from the end condition $e'(x_0) = e'(x_k) = 0$.

The first step in the direction is to establish that $e'(x) \ge 0$ for $i = 0,\dots,k$, which can be shown by a contradictory results. Let $e'(x_i) < 0$ for some $i,\; i = 1,\dots,k-1$. Now making use of (3.17) we get

$$\frac{h^4}{(6!)} \ge \max|e'(x_i)| \ge 5e'(x_{i-1}) + 16e'(x_i) + 3(x_{i+1})$$

$\because e'(x_i) < 0,\; i = 1,2,\dots,k-1$ $\quad \frac{1}{12}\frac{h^5}{5!} > \frac{h^4}{5!}$

$\frac{h^4}{(6!)} > \frac{4h^4}{5!}$ when we appeal to (3.20)

Thus, we have a contradiction so, $e'(x_i) \geq 0$ for $i = 0, \dots, k$.

Now from (3.20), we can write

$$16e'(x_i) = \frac{4h^4}{5!} - [5e'(x_{i-1}) + 3e'(x_{i+1})]$$

Since $e'(x_i) \geq 0$, we have

$$e'(x_i) \leq \frac{1}{4}\frac{h^4}{5!}, \qquad i = 1,2, \dots, k-1. \tag{3.25}$$

Again using (3.25) in (3.20), we have

$$e'(x_i) \leq \frac{1}{4}\frac{h^4}{5!}\left(1 - \frac{1}{2}\right), \qquad i = 1,2, \dots, k-1. \tag{3.25}$$

Thus , it clear that repeated use of (3.20) leads us to

$$e'(x_i) \leq \frac{1}{4}\frac{h^4}{5!}\left[1 - \frac{1}{2} + \frac{1}{2^2} - \cdots\right] \tag{3.26}$$

How, it can be seen that easily that r.h.s of (3.26) $\longrightarrow \frac{h^4}{6!} \longrightarrow \frac{1}{12}\frac{h^4}{5!}$ and hence in limit case

$$e'(x_i) \leq \frac{h^4}{6!} \tag{3.27}$$

which is verifies proof of (3.17). Thus corresponding to the function $f(x) = \frac{x^5}{5!}$, (3.26) and (3.27) imply $e'(x_i) \longrightarrow \frac{h^4}{6!}$ $e'(x_i) \rightarrow \frac{3}{2}\frac{h^5}{5!}$ in the limiting case for equally spaced knots. This complete the proof of theorem 3.1.

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