Advances in Computational Sciences and Technology (ACST).
ISSN 0973-6107 Volume 10, Number 5 (2017), pp. 1273–1285

http://www.ripublication.com/acst.htm

Semilocal Convergence Analysis of a Fourth-order Method in Banach Spaces and its Dynamics

Gagandeep
Department of Mathematics,
Hans Raj Mahila Mahavidyalaya,
Jalandhar-144008, India.
E-mail: gagan.hmv@gmail.com

Rajni Sharma[1]
Department of Applied Sciences,
DAV Institute of Engineering and Technology,
Jalandhar-144008, India.
E-mail: rajni_daviet@yahoo.com

Abstract

In this paper, we study the semilocal convergence for a fourth-order method for solving nonlinear equations in Banach spaces by using recurrence relations. The recurrence relations for the method are derived and then an existence-uniqueness theorem has been proved. The basins of attraction of existing methods and the presented method are given to demonstrate their performance.

AMS subject classification: 65D99.
Keywords: Nonlinear equations in Banach spaces, Recurrence relations, Semilocal convergence, Iterative methods, Convergence domain, Basins of attraction.

1. Introduction

The solution of nonlinear equations in Banach spaces is one of the most important problems in numerical analysis. Such equations are given by

$$F(x) = 0,$$

[1]Corresponding author.

where $F : \Omega \subseteq X \to Y$ is a nonlinear operator on an open convex subset Ω of a Banach space X with values in a Banach space Y. To solve nonlinear equations, we have to use iterative methods. The well known Newton's method and its variants are used to solve such nonlinear equations. The convergence of Newton's method in Banach spaces was established by Kantorovich in [1] This technique has been used by many authors in order to establish the order of convergence. The convergence of the sequence obtained by iterative expression is derived from the convergence of majorizing sequences [2].

Rall in [3] suggested a new approach for the convergence of these methods by recurrence relations. Candela and Marquina [4, 5], Hernández [6, 7, 8], Ezquerro and Hernández [9, 10, 11], Gutiérrez and Hernández, [12, 13], Parida and Gupta [14], Wang *et al.* [15], Chun *et al.* [16], Zheng and Gu [17], Cordero *et al.* [18] etc. used this idea to prove semilocal convergence for several methods of different orders.

In this paper, we use the technique of recurrence relations to establish the semilocal convergence of fourth-order method in [19] for solving system of nonlinear equations. This technique consists of generating a sequence of positive real numbers that guarantees the convergence of iterative method in Banach spaces, providing a suitable convergence domain.

The paper is organized as follows. In section 2, we derive recurrence relations for the fourth order method. In section 3, the convergence analysis of proposed method based on these recurrence relations is given. Section 4 contains concluding remarks.

2. Recurrence relations

Let X, Y be Banach spaces and $F : \Omega \subseteq X \to Y$ be a nonlinear twice Fréchet differentiable operator in an open convex domain $\Omega_0 \subseteq \Omega$. In this paper, we consider the semilocal convergence for the fourth-order method proposed in [19]. We first extend this method to Banach spaces and write it as

$$\begin{aligned} y_n &= x_n - \Gamma_n F(x_n), \\ z_n &= x_n + \frac{2}{3}(y_n - x_n), \\ x_{n+1} &= x_n + \frac{1}{2}(I - G(x_n, z_n))\Gamma_n F(x_n), \end{aligned} \tag{2.1}$$

where $\Gamma_n = [F'(x_n)]^{-1}$ and $G(x_n, z_n) = \frac{9}{4}F'(z_n)^{-1}F'(x_n) + \frac{3}{4}F'(x_n)^{-1}F'(z_n)$, for $n \in \mathbb{N}$.

Let us assume that $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists at some $x_0 \in \Omega_0$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y into X. In the following, we will assume that $y_0, z_0 \in \Omega_0$ and

(i) $\|F''(x)\| \leq M, x \in \Omega_0,$

(ii) $\|\Gamma_0\| \leq \beta,$

(iii) $\|\Gamma_0 F(x_0)\| \leq \eta$.

Let us denote $a_0 = M\beta\eta$ and define the sequence $a_{n+1} = a_n f(a_n)^2 g(a_n)$, where

$$f(x) = \frac{2(3-2x)}{x^3 - 2x^2 - 10x + 6},$$

$$g(x) = \frac{x(108 - 36x - 4x^3 + x^4)}{8(3-2x)^2}.$$

Notice that

$$M\|\Gamma_0\|\|\Gamma_0 F(x_0)\| \leq a_0.$$

Assuming that

$$\frac{2}{3}a_0 < 1 \quad \text{and} \quad \frac{a_0(6 + 2a_0 - a_0^2)}{2(3-2a_0)} < 1,$$

We have

$$\begin{aligned}\|I - \Gamma_0 F'(z_0)\| &\leq \|\Gamma_0\|\|F'(x_0) - F'(z_0)\| \\ &\leq M\|\Gamma_0\|\|z_0 - x_0\| \\ &\leq \frac{2}{3}a_0 < 1.\end{aligned}$$

Then, from Banach lemma, $F'(z_0)^{-1}$ exists and

$$\|F'(z_0)^{-1}\| \leq \frac{\|\Gamma_0\|}{1 - \|\Gamma_0\|\|F'(x_0) - F'(z_0)\|} \leq \frac{1}{1 - \frac{2}{3}a_0}\|\Gamma_0\| = \frac{3}{3-2a_0}\|\Gamma_0\|.$$

Now consider

$$\begin{aligned}I - G(x_0, z_0) &= I - \left(\frac{9}{4}F'(z_0)^{-1}F'(x_0) + \frac{3}{4}F'(x_0)^{-1}F'(z_0)\right) \\ &= I + \frac{9}{4}F'(z_0)^{-1}\left[F'(z_0) - F'(x_0) - F'(z_0)\right] \\ &\quad - \frac{3}{4}F'(x_0)^{-1}\left[F'(z_0) - F'(x_0) + F'(x_0)\right] \\ &= -2I + \frac{3}{4}\left(3F'(z_0)^{-1} - F'(x_0)^{-1}\right)\left(F'(z_0) - F'(x_0)\right).\end{aligned}$$

From above step we have

$$\begin{aligned}\|I - G(x_0, z_0)\| &\leq 2 + \frac{3}{4}\left(3\|F'(z_0)^{-1}\| + \|F'(x_0)^{-1}\|\right)\|F'(z_0) - F'(x_0)\| \\ &\leq 2 + \frac{M}{2}\left(\frac{9}{3-2a_0}\|\Gamma_0\| + \|\Gamma_0\|\right)\|y_0 - x_0\| \\ &\leq 2 + \frac{a_0}{2}\left(\frac{9}{3-2a_0} + 1\right) = \frac{6 + 2a_0 - a_0^2}{3 - 2a_0}.\end{aligned}$$

Then from third step of (2.1), we have

$$\begin{aligned}\|x_1 - x_0\| &\le \frac{1}{2}\|I - G(x_0, z_0)\|\|\Gamma_0 F(x_0)\| \\ &\le \frac{6 + 2a_0 - a_0^2}{2(3 - 2a_0)}\|\Gamma_0 F(x_0)\|.\end{aligned}$$

In this situation, we prove the following statements for $n \ge 1$:

$$\begin{aligned}&\text{(I) } \|\Gamma_n\| = \|F'(x_n)^{-1}\| \le f(a_{n-1})\|\Gamma_{n-1}\|,\\ &\text{(II) } \|\Gamma_n F(x_n)\| \le f(a_{n-1})g(a_{n-1})\|\Gamma_{n-1}F(x_{n-1})\|,\\ &\text{(III) } M\|\Gamma_n\|\|\Gamma_n F(x_n)\| \le a_n,\\ &\text{(IV) } \|x_{n+1} - x_n\| \le \frac{6 + 2a_n - a_n^2}{2(3 - 2a_n)}\|\Gamma_n F(x_n)\|.\end{aligned}$$

We have

$$\begin{aligned}\|I - \Gamma_0 F'(x_1)\| &\le \|\Gamma_0\|\|F'(x_0) - F'(x_1)\| \\ &\le M\|\Gamma_0\|\|x_1 - x_0\| \\ &\le \frac{a_0(6 + 2a_0 - a_0^2)}{2(3 - 2a_0)} < 1.\end{aligned}$$

Then from Banach lemma, Γ_1 is defined and

$$\begin{aligned}\|\Gamma_1\| &\le \frac{\|\Gamma_0\|}{1 - \|\Gamma_0\|\|F'(x_0) - F'(x_1)\|} \\ &\le \frac{2(3 - 2a_0)}{6 - 10a_0 - 2a_0^2 + a_0^3}\|\Gamma_0\| = f(a_0)\|\Gamma_0\|. \qquad (2.2)\end{aligned}$$

So (I) is true for $n = 1$. Using Taylor formula, we have

$$F(x_1) = F(x_0) + F'(x_0)(x_1 - x_0) + \int_0^1 F''(x_0 + t(x_1 - x_0))(x_1 - x_0)^2(1 - t)dt.$$

Taking into account (2.1), we obtain

$$\begin{aligned}\Gamma_0 F(x_1) = &\frac{3}{8}(3F'(z_0)^{-1} - F'(x_0)^{-1})(F'(z_0) - F'(x_0))\Gamma_0 F(x_0) \\ &+ \Gamma_0 \int_0^1 F''(x_0 + t(x_1 - x_0))(x_1 - x_0)^2(1 - t)dt.\end{aligned}$$

Then, we have

$$\begin{aligned}\|\Gamma_0 F(x_1)\| &\le \frac{M}{4}\left(\frac{9}{3-2a_0}\|\Gamma_0\| + \|\Gamma_0\|\right)\|y_0 - x_0\|\|\Gamma_0 F(x_0)\| \\ &\quad + M\frac{(6+2a_0-a_0^2)^2}{8(3-2a_0)^2}\|\Gamma_0\|\|\Gamma_0 F(x_0)\|^2 \\ &\le \left(\frac{a_0}{4}\left(\frac{9}{3-2a_0}+1\right) + \frac{a_0(6+2a_0-a_0^2)^2}{8(3-2a_0)^2}\right)\|\Gamma_0 F(x_0)\| \\ &= \frac{a_0(108-36a_0-4a_0^3+a_0^4)}{8(3-2a_0)^2}\|\Gamma_0 F(x_0)\| = g(a_0)\|\Gamma_0 F(x_0)\|. \qquad (2.3)\end{aligned}$$

Then from (2.2) and (2.3),

$$\begin{aligned}\|\Gamma_1 F(x_1)\| &\le \|\Gamma_1 F'(x_0)\|\|\Gamma_0 F(x_1)\| \\ &\le f(a_0)g(a_0)\|\Gamma_0 F(x_0)\|. \qquad (2.4)\end{aligned}$$

So (II) is true for $n = 1$. To prove (III) and (IV) for $n = 1$, again from (2.2) and (2.4)

$$\begin{aligned}M\|\Gamma_1\|\|\Gamma_1 F(x_1)\| &\le Mf(a_0)^2 g(a_0)\|\Gamma_0\|\|\Gamma_0 F(x_0)\| \\ &\le a_0 f(a_0)^2 g(a_0) = a_1.\end{aligned}$$

Finally, we deduce easily that

$$\|x_2 - x_1\| \le \frac{6+2a_1-a_1^2}{2(3-2a_1)}\|\Gamma_1 F(x_1)\|.$$

Now following an inductive procedure and assuming that

$$x_n, y_n \in \Omega_0, \frac{2}{3}a_n < 1 \text{ and } \frac{a_n(6+2a_n-a_n^2)}{2(3-2a_n)} < 1, \quad n \in \mathbb{N}, \qquad (2.5)$$

items (I) - (IV) are proved for n arbitrary.

So, we must analyse the real sequence $\{a_n\}$ to study the sequence $\{x_n\}$ defined in a Banach space. To establish the convergence of $\{x_n\}$, we have to prove that this is a Cauchy sequence and the above assumption (2.5).

3. Convergence study

In this section, we study the sequence $\{a_n\}$, defined in section 2, to prove the convergence of the sequence $\{x_n\}$ given by (2.1). First of all,we give a technical lemma including results concerning functions of one variable that we need next.

Lemma 3.1. Under the previous notations, we have that $f(x)$ is increasing and $f(x) > 1$ for $x \in (0, 1.5)$ and $g(x)$ is increasing function of x in $(0, 1.5)$. Moreover, if $\gamma \in (0, 1)$, then $g(\gamma x) \le \gamma g(x)$ for $x \in (0, 1.5)$.

Lemma 3.2. Let $f(x)$ and $g(x)$ be defined as before and $a_0 \in (0, 0.242686\dots)$. Then,

(i) $f(a_0)^2 g(a_0) < 1$,

(ii) $f(a_0) g(a_0) < 1$,

(iii) the sequence $\{a_n\}$ is decreasing and $a_n < 0.242686\ldots$, for $n \geq 0$,

(iv) $\frac{2}{3} a_n < 1$ and $\frac{a_n(6 + 2a_n - a_n^2)}{2(3 - 2a_n)} < 1$ for all $n \geq 0$.

Proof. From definition of functions f and g (i) follows trivially. From (i) and $f(a_0) > 1$, we obtain (ii). We are going to prove (iii) by induction on $n \geq 0$. Firstly, from the definition of a_1 and from (i), we have that

$$a_1 = a_0 f(a_0)^2 g(a_0) < a_0.$$

Now, it is supposed that $a_k < a_{k-1}$, for $k \leq n$. Then,

$$a_{n+1} = a_n f(a_n)^2 g(a_n) < a_{n-1} f(a_{n-1})^2 g(a_{n-1}) = a_n.$$

as f and g are increasing and $f(x) > 1$ for $x \in (0, 1.5)$.

Finally, for all $n \geq 0$, $a_n < 0.242686\ldots$, since $\{a_n\}$ is decreasing sequence and $a_0 < 0.242686\ldots$. ■

Let us note that $a_0 = 0.242686\ldots$, is the value of the solution of equation $f(a_0)^2 g(a_0) - 1 = 0$ and (iv) follows immediately from (iii).

Next, we prove that $\frac{(6 + 2a_n - a_n^2)}{2(3 - 2a_n)} \|\Gamma_n F(x_n)\|$ is a Cauchy sequence. So, we note that

$$\begin{aligned}
&\frac{(6 + 2a_n - a_n^2)}{2(3 - 2a_n)} \|\Gamma_n F(x_n)\| \\
&\leq \frac{(6 + 2a_0 - a_0^2)}{2(3 - 2a_0)} f(a_{n-1}) g(a_{n-1}) \|\Gamma_{n-1} F(x_{n-1})\| \\
&\leq \cdots \leq \frac{(6 + 2a_0 - a_0^2)}{2(3 - 2a_0)} \|\Gamma_0 F(x_0)\| \prod_{k=0}^{n-1} f(a_k) g(a_k),
\end{aligned}$$

as $\frac{6 + 2x - x^2}{2(3 - 2x)}$ is an increasing function of x and $\{a_n\}$ is decreasing sequence for all $n \in \mathbb{N}$. Next, we analyze the factor $\prod_{k=0}^{n-1} f(a_k) g(a_k)$ by means of the following lemma.

Lemma 3.3. Let us suppose that the hypotheses of Lemma 2 are satisfied and define $\gamma = \frac{a_1}{a_0}$. Then:

(i) $\gamma = f(a_0)^2 g(a_0) \in (0, 1)$,

(ii) $a_n \leq \gamma^{2^{n-1}} a_{n-1} \leq \gamma^{2^n - 1} a_0$,

(iii) $f(a_n)g(a_n) \leq \gamma^{2^n}/f(a_0)$.

Proof. Notice that (i) is trivial. Next, we prove (ii) following an inductive procedure. So, for $n = 1$

$$a_1 \leq \gamma a_0$$

If we suppose that (ii) for n arbitrary is true, then for $n + 1$,

$$\begin{aligned} a_{n+1} &= a_n f(a_n)^2 g(a_n) \\ &\leq \gamma^{2^{n-1}} a_{n-1} f(\gamma^{2^{n-1}} a_{n-1})^2 g(\gamma^{2^{n-1}} a_{n-1}) \\ &\leq \gamma^{2^n} a_{n-1} f(a_{n-1})^2 g(a_{n-1}) = \gamma^{2^n} a_n. \end{aligned}$$

Moreover,

$$a_n \leq \gamma^{2^{n-1}} a_{n-1} \leq \gamma^{2^{n-1}} \gamma^{2^{n-2}} a_{n-2} \leq \cdots \leq \gamma^{2^n - 1} a_0.$$

So, (ii) holds. Finally, we observe that

$$\begin{aligned} f(a_n)g(a_n) &\leq f(\gamma^{2^n - 1} a_0) g(\gamma^{2^n - 1} a_0) \\ &\leq \gamma^{2^n - 1} f(a_0) g(a_0) = \frac{\gamma^{2^n}}{f(a_0)}. \end{aligned}$$

and proof is complete. ■

As a consequence of this, if we denote $\Delta = \dfrac{1}{f(a_0)}$, it follows that

$$\prod_{k=0}^{n-1} f(a_k)g(a_k) \leq \prod_{k=0}^{n-1} \gamma^{2^k} \Delta = \gamma^{2^n - 1} \Delta^n.$$

So, from $\Delta < 1$, we deduce that $\prod_{k=0}^{n-1} f(a_k)g(a_k)$ converges to zero by letting $n \to \infty$. Now, we are ready to state the following result on convergence for (2.1).

Theorem 3.4. Theorem 1Let X, Y be Banach spaces and let $F : \Omega \subseteq X \to Y$ be a non-linear twice Fréchet differentiable operator in an open convex domain $\Omega_0 \subseteq \Omega$. Let us assume that

$$\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y, X)$$

exists at some $x_0 \in \Omega_0$ and that conditions (i)-(iii) in section 2 are satisfied. Let us denote $a_0 = M\beta\eta$. Suppose that $0 < a_0 < r = 0.242686\cdots$ (r is the smallest positive root of the polynomial $q(x) = 2x^6 - 9x^5 - 28x^4 + 104x^3 + 188x^2 - 348x + 72$). Then if

$$\overline{B(x_0, R\eta)} = \{x \in X; \|x - x_0\| \leq R\eta\} \subseteq \Omega_0,$$

where

$$R = \frac{6 + 2a_0 - a_0^2}{2(3 - 2a_0)} \frac{1}{1 - \Delta}, \quad \Delta = \frac{1}{f(a_0)},$$

the sequence $\{x_n\}$ defined in (1) and starting at x_0 converges to a solution x^* of $F(x) = 0$. In that case, the solution x^* and iterates x_n belong to $\overline{B(x_0, R\eta)}$, and x^* is the only solution of $F(x) = 0$ in $B\left(x_0, \frac{2}{M\beta} - R\eta\right) \cap \Omega_0$.

Proof. When $a_0 < 0.242686\ldots$, the convergence of the sequence $\{x_n\}$ follows immediately from the previous lemmas. On other hand, we consider $p \geq 1$;

$$\begin{aligned}
\|x_{n+p} - x_n\| &\leq \|x_{n+p} - x_{n+p-1}\| + \|x_{n+p-1} - x_{n+p-2}\| + \cdots + \|x_{n+1} - x_n\| \\
&\leq \frac{6 + 2a_n - a_n^2}{2(3 - 2a_n)}[\gamma^{2^{n+p-1}-1}\Delta^{n+p-1} + \gamma^{2^{n+p-2}-1}\Delta^{n+p-2} \\
&\quad + \cdots + \gamma^{2^n-1}\Delta^n]\|\Gamma_0 F(x_0)\| \\
&\leq \frac{6 + 2a_0\gamma^{2^n-1} - (\gamma^{2^n-1}a_0)^2}{2(3 - 2a_0)}\gamma^{2^n-1}[\Delta^{n+p-1} + \Delta^{n+p-2} \\
&\quad + \cdots + \Delta^n]\|\Gamma_0 F(x_0)\| \\
&\leq \frac{6 + 2a_0\gamma^{2^n-1} - (\gamma^{2^n-1}a_0)^2}{2(3 - 2a_0)}\gamma^{2^n-1}\Delta^n \frac{1 - \Delta^p}{1 - \Delta}\eta. \qquad (3.1)
\end{aligned}$$

So, we obtain

$$\|x_p - x_0\| \leq \frac{6 + 2a_0 - a_0^2}{2(3 - 2a_0)} \frac{1 - \Delta^p}{1 - \Delta}\eta < R\eta,$$

for $n = 0$.
To prove that $F(x^*) = 0$, notice that $\|\Gamma_n F(x_n)\| \to 0$ by letting $n \to \infty$.
As

$$\|F(x_n)\| \leq \|F'(x_n)\|\|\Gamma_n F(x_n)\|$$

and $\{\|F'(x_n)\|\}$ is a bounded sequence, we deduce that $\|F(x_n)\| \to 0$, and then $F(x^*) = 0$ by continuity of F. Now, to show unicity, suppose that

$$y^* \in B\left(x_0, \frac{2}{M\beta} - R\eta\right) \cap \Omega_0$$

is another solution of $F(x) = 0$. Then,

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))dt(y^* - x^*).$$

Using estimate

$$\begin{aligned}
&\|\Gamma_0\| \int_0^1 \|F'(x^* + t(y^* - x^*)) - F'(x_0)\| dt \\
&\leq M\beta \int_0^1 \|x^* + t(y^* - x^*) - x_0\| dt \\
&\leq M\beta \int_0^1 ((1-t)\|x^* - x_0\| + t\|y^* - x_0\|) dt \\
&< \frac{M\beta}{2}\left(R\eta + \frac{2}{M\beta} - R\eta\right) = 1.
\end{aligned}$$

We have that operator

$$\int_0^1 (F'(x^* + t(y^* - x^*)) dt$$

has an inverse, and consequently, $y^* = x^*$ and the proof of the theorem is concluded.

■

4. Global convergence

In this section, we study the dynamics of the presented method (2.1) denoted by M_1^4, Jarratt's method [20] denoted by M_2^4, fourth-order method by Argyros *et al.* [21] denoted by M_3^4 and fourth-order method by Hernández, Salanova [22] denoted by M_4^4 and fourth-order method Hueso *et al.* [23] denoted by M_4^5 by analyzing the basins of attraction [24, 25]. The above mentioned methods are given as follows:
Method M_2^4 is given as

$$\begin{aligned}
y_k &= x_k - \frac{2}{3}\Gamma_k F(x_k), \\
x_{k+1} &= x_k - \frac{1}{2}\left[(3F'(y_k) + F'(x_k))^{-1}(3F'(y_k) - F'(x_k))\right]\Gamma_k F(x_k).
\end{aligned} \tag{4.1}$$

Method M_3^4 is given as

$$\begin{aligned}
y_k &= x_k - \Gamma_k F(x_k), \\
z_k &= x_k + \frac{2}{3}(y_k - x_k), \\
x_{k+1} &= y_k - \frac{3}{4}H(x_k)\left[I - \frac{3}{2}H(x_k)\right](y_k - x_k).
\end{aligned} \tag{4.2}$$

Method M_4^4 is given as

$$\begin{aligned} y_k &= x_k - \Gamma_k F(x_k), \\ z_k &= x_k + \frac{2}{3}(y_k - x_k), \\ x_{k+1} &= y_k - \frac{3}{4}\left[I + \frac{3}{2}H(x_k)\right]^{-1} H(x_k)(y_k - x_k), \end{aligned} \tag{4.3}$$

where

$$H(x_k) = \Gamma_k[F'(z_k) - F'(x_k)].$$

Method M_4^5 is given as

$$\begin{aligned} y_k &= x_k - \Gamma_k F(x_k), \\ z_k &= x_k - \Gamma_k \left(F(y_k) + F(x_k)\right), \\ x_{k+1} &= x_k - \Gamma_k \left(F(z_k) + F(y_k) + F(x_k)\right). \end{aligned} \tag{4.4}$$

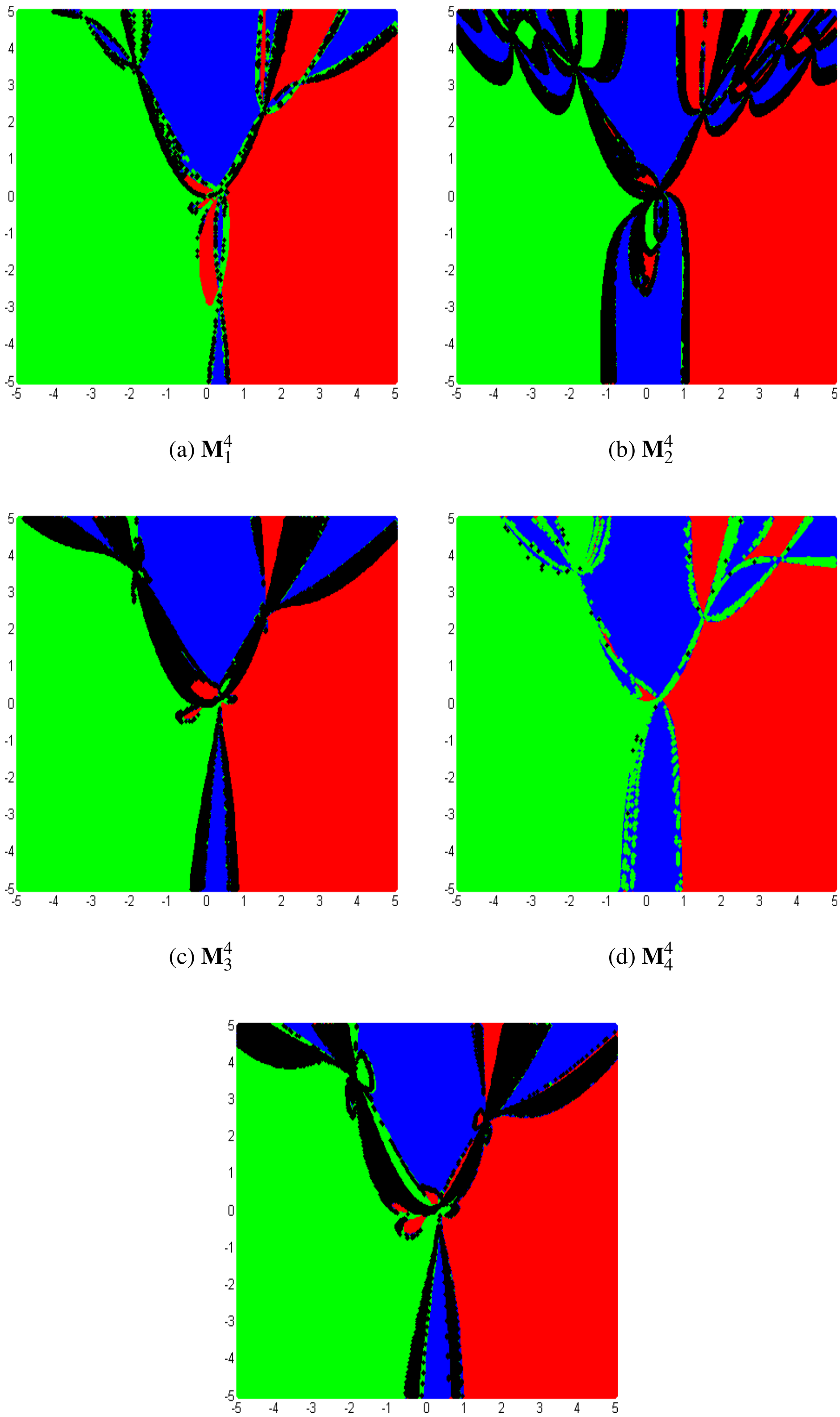
To start with, let us recall some basic dynamical concepts. Consider a Fréchet differentiable function $G : \mathbb{R}^n \to \mathbb{R}^n$. The orbit of a point $x \in \mathbb{R}^n$ is defined as the set of successive images of x denoted by $\{x, G(x), G^2(x), \cdots \ldots, G^p(x), \cdots \ldots.\}$ The dynamical behavior of the orbit of a point of $\mathbb{R}^n$ can be classified depending on its asymptotic behavior. In this way, a point x_f is a fixed point of G if $G(x_f) = x_f$. A fixed point x_f is called attracting if $\|J_G(x_f)\| < 1$, repelling if $\|J_G(x_f)\| > 1$ and neutral if $\|J_G(x_f)\| = 1$. If $\|J_G(x_f)\| = 0$, the point x_f is superattracting. Let x_{af} be an attracting fixed point of the function G, its basins of attraction $\mathcal{A}(x_{af})$ is defined as the set of pre-images of any order such that

$$\mathcal{A}(x_{af}) = \{x \in \mathbb{R}^n : G^p(x) \to x_{af} \text{ for } p \to \infty\}$$

To study dynamical behavior, we consider a system of quadratic equations, representing the intersection of two conics in $\mathbb{R}^2$ given as

$$\left.\begin{aligned} x^2 + 2y &= 3 \\ 2xy &= 1 \end{aligned}\right\}$$

presents three simple real roots that are superattractive fixed points for the methods in previous section. For generating basins of attraction associated with roots of nonlinear system of equations, we take a square $[-5, 5] \times [-5, 5]$ of 1024×1024 points, which contains all roots of concerned nonlinear system of equations and we apply the iterative method starting in every point in the square. We assign a color to each point according to the root to which the corresponding orbit of the iterative method, starting from the point, converges. If the corresponding orbit does not reach any root of the polynomial, with tolerance 10^{-3} in a maximum of 25 iterations, we mark those points with black color.



(a) $\mathbf{M}_1^4$ (b) $\mathbf{M}_2^4$

(c) $\mathbf{M}_3^4$ (d) $\mathbf{M}_4^4$

(e) $\mathbf{M}_5^4$

Figure 1: Basins of attraction for system of equations $x_1^2 + 2x_2 = 3, 2x_1x_2 = 1$ for various methods.

For the given test problem, it can be observed in Fig.1 that all the roots of the polynomial system have their respective basins of attraction with different colors. Also the Julia set can be seen as black lines of unstable behavior. Based on Fig. 1, we can see that M_4^4 is the best, followed by M_1^4, while the rest of the methods contain more divergent points than M_1^4 in the considered region.

5. Conclusion

In this paper, the recurrence relations are developed for establishing the convergence of a fourth-order method for solving $F(x) = 0$ in Banach spaces. Based on recurrence relations, we prove a semilocal convergence, which shows the existence-uniqueness theorem for this method. Numerical examples are worked out to demonstrate our approach and show our method can be of practical interest.

References

[1] L. V. Kantorovich and G. P. Akilov, *Functional Analysis* , Pergamon Press, Oxford, 1982.

[2] Q. Wu and Y. Zhao, Third-order convergence theorem by using majorizing function for a modified Newton method in Banach space *Appl. Math. Comput.*, **175** (2006), 1515–1524.

[3] L. B. Rall, *Computational Solution of Nonlinear Operator Equations*, Robert E. Krieger, New York, 1979.

[4] V. Candela and A. Marquina, Recurrence Relations for Rational Cubic Methods II: the Chebyshev Method, *Computing*, **45** (1990), 355–367.

[5] V. Candela and A. Marquina, Recurrence Relations for Rational Cubic Methods I: the Halley Method, *Computing*, **44** (1990), 169–184.

[6] M. A. Hernández, Reduced Recurrence Relations for the Chebyshev Method, *J. optim. Theory. Appl.*, **98** (1998), 385–397.

[7] M. A. Hernández, Second-Derivative-Free Variant of Chebyshev Method for Nonlinear Equations, *J. optim. Theory. Appl.*, **104** (2000), 501–515.

[8] M. A. Hernández, Chebyshev's approximation algorithms and applications, *Comput. Math. Appl.*, **41** (2001), 433–445.

[9] J. A. Ezquerro and M. A. Hernández, Relaxing Convergence Conditions for an inverse-free Jarratt-type Approximations, *J. Comput. Appl. Math.*, **83** (1997), 131–135.

[10] J. A. Ezquerro and M. A. Hernández, A modification of the super-Halley method under mild differentiability conditions, *J. Comput. Appl. Math.*, **114** (2000), 405–409.

[11] J. A. Ezquerro and M. A. Hernández, Recurrence relations for Chebyshev-type methods, *Appl. Math. Optim.*, **41** (2000), 227–236.

[12] J. M. Gutiérrez and M. A. Hernández, Third-order iterative methods for operators with bounded second derivative, *J. Comput. Appl. Math.*, **82** (1997), 171–183.

[13] J. M. Gutiérrez and M. A. Hernández, Recurrence Relations for the Super-Halley Method, *Comput. Math. Appl.*, **36** (1998), 1–8.

[14] P. K. Parida and D. K. Gupta, Recurrence relations for a Newton-like method in Banach spaces, *J. Comput. Appl. Math.*, **206** (2007), 873–887.

[15] X. Wang, C. Gu and J. Kou, Semilocal convergence of a multipoint fourth-order super-Halley method in Banach spaces, *Numer. Algorithms*, **56** (2011), 497–516.

[16] C. Chun, P. Stanica and B. Neta, Third-order Family of Methods in Banach Spaces, *Comput. Math. Appl.*, **61** (2011), 1665–1675.

[17] L. Zheng and C. Gu, Recurrence relations for semilocal convergence of a fifth-order method in Banach spaces, *Numer. Algorithms*, **59** (2012), 623–638.

[18] A. Cordero, M. A. Hernández, N. Romero and J. R. Torregrosa, Semilocal Convergence by using recurrence relations for a fifth-order method in Banach spaces, *J. Comput. Appl. Math.*, **273** (2015), 205–213.

[19] J. R. Sharma, R. K. Guha and R. Sharma, An efficient fourth order weighted-Newton method for systems of nonlinear equations, *Numer. Algorithms*, **62** (2013), 307–323.

[20] P. Jarratt, Some fourth order multipoint iterative methods for solving equations, *Math. Comput.*, **20** (1966), 434–437.

[21] I. K. Argyros, D. Chen and Q. Qian, An inverse-free Jarratt-type approximation in a Banach space, *J. Appr. Th. Appl.*, **12(1)** (1996), 19–30.

[22] M. A. Hernández and M. A. Salanova, Sufficient conditions for semilocal convergence of a fourth-order multipoint iterative method for solving equations in Banach spaces, *Southwest. J. Pure Appl. Math.*, **1** (1999), 29–40.

[23] J. Hueso, L. Martínez and J. R. Torregrosa, Third and fourth order iterative methods free from second derivative for nonlinear systems, *Appl. Math. Comput.*, **209** (2011), 190–197.

[24] A. Cordero, F. Soleymani and J. R. Torregrosa, Dynamical analysis of iterative methods for nonlinear systems or how to deal with dimension ?, *Appl. Math. Comput.*, **244** (2014), 398–412.

[25] Á. A. Magreñán, A new tool to study real dynamics:the convergence plane, *Appl. Math. Comput.*, **248** (2014), 215–224.