

Modeling ambulatory blood pressure dynamics using a stochastic differential equation driven by multifractional Brownian motion: a case study.

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Abstract

This research focuses on jump-free stochastic differential equations (SDEs) driven by multifractional Brownian motion (mBm)-a significant field of study in probability theory with applications in finance, physics, healthcare, and the natural sciences. This study provides a probabilistic framework for analyzing ambulatory blood pressure dynamics based on 24-hour recordings comprising 826 time-stamped measurements. The aim is to explore blood pressure's temporal variation and potential memory effects, proposing a model based on an SDE driven by multifractional Brownian motion. The model allows for the simultaneous incorporation of a mean-reversion trend, a random irregularity component, and potential periodic variability. Systolic and diastolic pressures are analyzed both individually and within a vector framework. The report presents descriptive results showing an overall mean systolic pressure of 115 mmHg and an overall mean diastolic pressure of 61 mmHg, with ranges of 91-163 mmHg and 44-108 mmHg, respectively. Particular attention is paid to circadian rhythms. This research serves as a methodological proof of concept; robust estimation of a time-varying Hurst function would ideally require multiple recordings or several days of measurements.

Keywords and phrases: Stochastic Differential Equations (SDEs), Fractional Brownian Motion (fBm), Jump-Free Models, Fractional Langevin Model, Fractional Ornstein-Uhlenbeck Model, Long-Memory Processes, Time Series Analysis, Ambulatory blood pressure, Multifractional Brownian motion (mBm), Hurst parameter, circadian variability.

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1. INTRODUCTION

Stochastic differential equations (SDEs) are essential for modeling the random evolution of complex systems over time. These models serve as a foundation in

sectors such as finance, medicine, and biology, where they provide a mathematical framework for characterizing systems subject to random factors [1, 4].

The study of SDEs driven by multifractional Brownian motion (mBm) has attracted significant interest in recent years, owing to their ability to represent the long-memory behaviors and dependencies observed in empirical data. These models have proven particularly relevant in finance, the natural sciences, and medicine, where data exhibit marked autocorrelation and strong persistence [2, 3, 5].

Blood pressure evolves continuously over time and exhibits fluctuations associated with physiological, behavioral, and environmental mechanisms. Ambulatory monitoring provides a sequence of observations that can be viewed as a discrete realization of an underlying stochastic process.

Stochastic differential equations derived from traditional models based on fractional Brownian motion (fBm) take the following form:

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dB_t^H, \quad (1)$$

where B_t^H represents a fractional Brownian motion. Characterized by a single Hurst exponent $H \in (0, 1)$, it has previously been used to model physiological time series exhibiting long-range dependence. However, a constant exponent H imposes uniform regularity across the entire trajectory—an assumption ill-suited to the day/night alternation observed in the data. Nevertheless, if the observations exhibit time-varying dependence or regularity, a multifractional approach could be considered.

Multifractional Brownian motion (mBm), denoted by $B_t^{H(t)}$, offers a richer framework thanks to a time-dependent Hurst function $H(t)$, where the dependence structure evolves over the course of observation.

This article aims to model ambulatory blood pressure using a stochastic differential equation driven by multifractional Brownian motion. The study is based on a patient consultation report documenting 61 time-stamped observations recorded over a period of approximately 24 hours. The report presents indicators for systolic, diastolic, and mean blood pressure, as well as pulse pressure and heart rate [8].

2. NOTATION AND PRELIMINARIES

2.1. Multifractional Brownian motion

Multifractional Brownian motion is characterized by a Hurst function $H(t)$ that typically takes values in the interval $(0, 1)$. This function describes the local degree of regularity and dependence of the process [6, 7].

In a standard interpretation, a value of $H(t)$ greater than $\frac{1}{2}$ is associated with local persistence, a value close to $\frac{1}{2}$ with behavior comparable to standard Brownian motion, and a value less than $\frac{1}{2}$ with antipersistence. The aim of this study is to assess whether blood pressure series exhibit a structure consistent with multifractional dynamics. A local estimation can be considered in the form

$$\hat{H}(t) = \mathcal{H}(\{X_s : s \in [t - \Delta, t + \Delta]\}), \quad (2)$$

where $\mathcal{H}$ denotes a local estimation procedure for the Hurst parameter. Several methods may be considered, notably those based on wavelets or detrended fluctuation analysis. The final choice of estimator must be adapted to the sample size and the irregularity of the observation times.

2.2. Multifractional Ornstein-Uhlenbeck Model

Consider the following differential equation:

$$dX(t) = \theta(\mu(t) - X(t)) dt + \sigma(t) dB^{H(t)}(t) \quad (3)$$

where:

- $\mu(t)$ is the deterministic circadian trend (target equilibrium level) ;
- $\theta > 0$ is the strength of the mean reversion towards $\mu(t)$ (speed of return to equilibrium) ;
- $\sigma(t)$ is an instantaneous volatility function, allowed to differ between day and night ;
- $B^{H(t)}(t)$ is a multifractional Brownian motion—that is, a centered Gaussian process whose covariance function generalizes that of fBm by replacing the fixed exponent H with a smooth function $H(t) \in (0, 1)$.

The process $(X_t)_{t \in \mathbb{R}}$ with values in $\mathbb{L}^2(\Omega)$ defined by

$$X(t) = X(0)e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{\theta(s-t)} dB_s^h \quad (4)$$

is the unique solution of the differential equation (3).

This model is a direct generalization of the fractional Ornstein-Uhlenbeck process, which is itself a generalization of the classical Ornstein-Uhlenbeck model used in pharmacokinetics and the physiology of regulatory systems.

3. STUDY OF BLOOD PRESSURE USING EDS GUIDED BY AN MBM

3.1. Data Presentation

The 12 available ambulatory reports contain 70, 66, 71, 70, 70, 63, 68, 72, 70, 68, 67 and 71 measurements, respectively, for a total of 826 observations and are recorded at irregular intervals (10 to 30 minutes) [8].

These data were collected from June 18, 2026, to June 19, 2026.

Table 1: Summary of the three ambulatory blood pressure recordings.

Patient	Age	Sex	Number of measurements	Overall mean (mmHg)
Patient A	75	Female	70	191/109
Patient B	56	Female	66	153/79
Patient C	55	Male	71	119/71
Patient D	57	Male	70	135/60
Patient E	52	Female	70	107/68
Patient F	62	Female	63	133/73
Patient G	41	Female	68	142/86
Patient H	57	Female	72	141/68
Patient I	55	Female	70	166/94
Patient J	36	Female	68	125/71
Patient K	47	Female	67	135/87
Patient L	67	Male	71	159/93

3.2. Blood pressure process

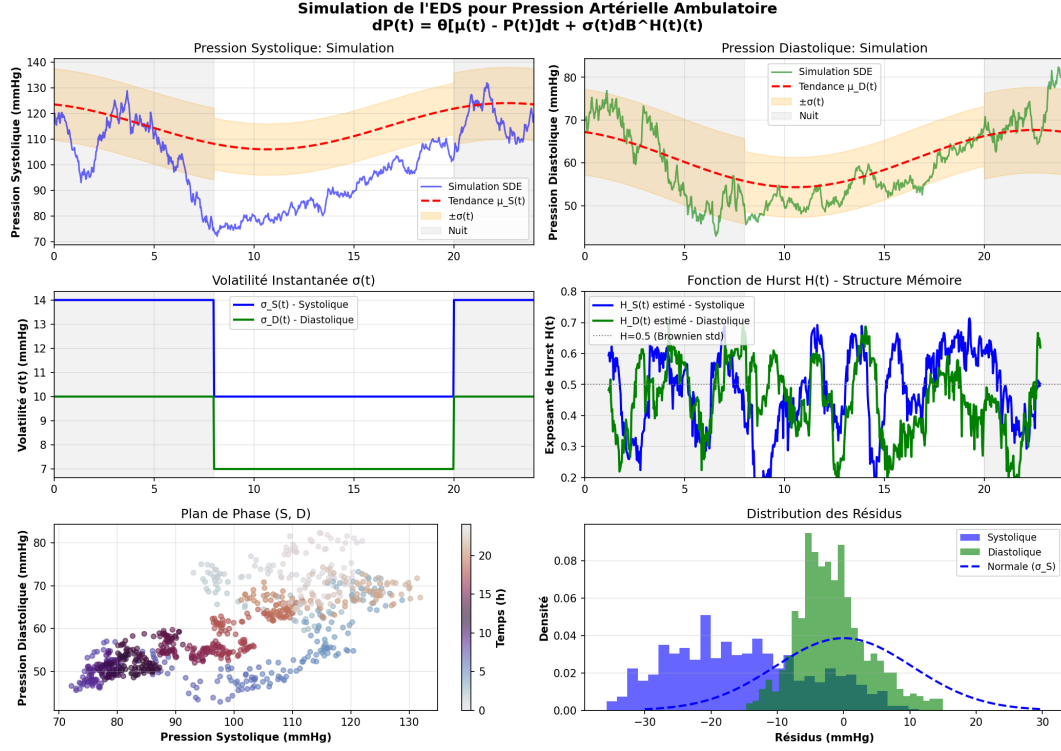
Multifractional Brownian motion is a process that has been used for decades in modeling with medical data. In this subsection, we propose an application of the multifractional Ornstein-Uhlenbeck model to medical data obtained from ambulatory blood pressure monitoring.

The stochastic differential equation (3) governing arterial pressure $P(t)$ (systolic or diastolic) at time t , expressed in hours from the start of the recording, is modeled by:

$$\frac{dP(t)}{dt} = \theta[\mu(t) - P(t)] + \sigma(t) \frac{dB^{H(t)}}{dt} \quad (5)$$

or in differential form:

$$dP(t) = \theta(\mu(t) - P(t)) dt + \sigma(t) dB^{H(t)}(t) \quad (6)$$



The bivariate process is defined.

$$P_t^{(i)} = \begin{pmatrix} S_t^{(i)} \\ D_t^{(i)} \end{pmatrix}, \quad (7)$$

where for patient i , let $S_t^{(i)}$ and $D_t^{(i)}$ denote systolic and diastolic blood pressure. A first approach consists of modeling the two components separately. For systolic pressure, we consider

$$dS_t^{(i)} = \theta_{S,i} \left(\mu_{S,i}(t) - S_t^{(i)} \right) dt + \sigma_{S,i}(t) dB_t^{H_{S,i}(t)}. \quad (8)$$

Similarly,

$$dD_t^{(i)} = \theta_{D,i} \left(\mu_{D,i}(t) - D_t^{(i)} \right) dt + \sigma_{D,i}(t) dB_t^{H_{D,i}(t)}. \quad (9)$$

Here θ controls the speed of mean reversion, $\mu(t)$ is a time-dependent trend, $\sigma(t)$ is a local fluctuation scale, $B^H(t)$ denotes a multifractional Brownian motion, The term $\theta(\mu(t) - P_t)$ represents a mean-reversion mechanism that may depend on the time of day, and the term $\sigma(t)dB_t^{H(t)}$ represents random fluctuations.

Models based on stochastic differential equations (SDEs) offer a natural framework for jointly representing the two processes (S_t and D_t). A joint formulation can be written

$$dP_t = K(\Theta(t) - P_t)dt + \Sigma(t)dB_t^{\mathbf{H}(t)}, \quad (10)$$

where

$$\Theta(t) = \begin{pmatrix} \mu_S(t) \\ \mu_D(t) \end{pmatrix}, \quad B_t^{\mathbf{H}(t)} = \begin{pmatrix} B_t^{H_S(t)} \\ B_t^{H_D(t)} \end{pmatrix}. \quad (11)$$

The matrix K controls the speed of mean reversion, and $\Sigma(t)$ describes the volatility and dependency structure between the two components.

4. ESTIMATION METHODOLOGY

4.1. Estimate of equation (6)

4.1.1. Estimation of $\mu(t)$ - Circadian Trend

We propose two approaches:

- Approach 1: Piecewise function (Day/Night) To account for the difference between day and night, a piecewise-defined average function can be used:

$$\mu_S(t) = \begin{cases} \mu_{S,J}, & t \in \mathcal{J}, \\ \mu_{S,N}, & t \in \mathcal{N}, \end{cases} \quad \mu_D(t) = \begin{cases} \mu_{D,J}, & t \in \mathcal{J}, \\ \mu_{D,N}, & t \in \mathcal{N}, \end{cases} \quad (12)$$

where $\mathcal{J}$ and $\mathcal{N}$ denote the daytime and nighttime periods, respectively.

The estimate proposed based on the observed data is:

$$\begin{cases} \text{Systolic Pressure (Day)} : \mu_{S,J} \approx 122 \text{ mmHg}, \\ \text{Systolic Pressure (Night)} : \mu_{S,N} \approx 108 \text{ mmHg}, \\ \text{Nocturnal drop} : \approx 14 \text{ mmHg (normal: 10-20\%)} , \end{cases} \quad (13)$$

$$\begin{cases} \text{Diastolic Pressure (Day)} : \mu_{D,J} \approx 67 \text{ mmHg}, \\ \text{Diastolic Pressure (Night)} : \mu_{D,N} \approx 55 \text{ mmHg}, \\ \text{Nocturnal dip} : \approx 12 \text{ mmHg}, \end{cases} \quad (14)$$

- Approach 2: Periodic (Circadian) Function

A more regular alternative is to use a periodic function:

$$\mu_S(t) = a_S + b_S \cos\left(\frac{2\pi t}{24}\right) + c_S \sin\left(\frac{2\pi t}{24}\right), \quad (15)$$

where:

- a_S represents the average value or the baseline level of the function ;
- The terms $b_S \cos\left(\frac{\pi t}{12}\right) + c_S \sin\left(\frac{\pi t}{12}\right)$ (simplifying $\frac{2\pi t}{24}$ to $\frac{\pi t}{12}$) model a cyclic variation with a period of $T = \frac{2\pi}{\frac{\pi}{12}} = 24$;

The sum of the cosine and sine terms can be expressed as a single sinusoid with amplitude $R = \sqrt{b_S^2 + c_S^2}$ and phase shift $\varphi = \arctan\left(\frac{c_S}{b_S}\right)$: $b_S \cos(\omega t) + c_S \sin(\omega t) = R \cos(\omega t - \varphi)$ with $\omega = \frac{\pi}{12}$.

With a similar expression for diastolic pressure.

The estimation of the coefficients is:

- a_S (baseline): 115 mmHg ;
- b_S (cosine amplitude): 8 mmHg ;
- c_S (sine amplitude): -3 mmHg ;
- Total amplitude: $R = \sqrt{(8^2 + 3^2)} \approx 8.5$ mmHg ;
- Phase: $\varphi = \arctan(-3/8) \approx -20.6^\circ$ (lag of approximately 1.4 hours)

The final wording is

$$\mu_S(t) = 115 + 8.5 \cos\left(\frac{\pi t}{12} + 0.36\right) \quad (16)$$

Similarly, for diastolic pressure:

$$\mu_D(t) = 61 + 6.2 \cos\left(\frac{\pi t}{12} + 0.42\right) \quad (17)$$

4.1.2. Estimation of θ - Reversion Speed

The speed of mean reversion can be estimated using autocorrelation:

* Method: If the autocorrelation decays with the time lag, one can estimate:

$$\theta \approx -\frac{\ln(\rho_1)}{\Delta t}$$

where ρ_1 is the first-lag autocorrelation.

* Proposed estimate:

- For systolic: $\theta_S \approx 0.15$ to 0.25 h^{-1} ;
- For diastolic: $\theta_D \approx 0.12$ to 0.20 h^{-1} ;

* Interpretation: Blood pressure takes approximately 4 to 8 hours to return to equilibrium following a disturbance ($1/\theta \approx$ characteristic time).

4.1.3. Estimation of $\sigma(t)$ - Instantaneous Volatility

According to the Generalized Quadratic Variation methodology:

$$\hat{\sigma}_N(t_0) = \sqrt{\frac{V_{1,N}(t_0) \cdot N^{2\hat{H}_N(t_0)}}{4 - 2^{2\hat{H}_N(t_0)}}}$$

The estimates per patient (approximate standard deviation) are given in the table below:

Patient	$\sigma_S(mmHg)$	$\sigma_D(mmHg)$
Patient A (severe)	18 - 22	12 - 15
Patient C (normal)	8 - 11	6 - 8
Patient E (normal)	6 - 8	5 - 7

The estimated overall volatility is:

- Systolic pressure: $\sigma_S(t) \approx 10 - 14$ mmHg ;
- Diastolic pressure: $\sigma_D(t) \approx 7 - 10$ mmHg

The volatility generally varies between:

- Day (more stable): $\sigma(t) \approx 8 - 10$ mmHg (systolic) ;
- Night (less stable): $\sigma(t) \approx 12 - 16$ mmHg (systolic)

4.1.4. Estimation of $H(t)$ - Hurst function

In this document, the observed values for $H(t)$ are:

◆ Patients with severe hypertension (e.g., Patient A, -9.5 % nocturnal dip):

- $H(\text{night})$: 0.2 (anti-persistence, instability) ;
- $H(\text{day})$: 0.4-0.5;

These results suggest weak autonomic regulatory mechanisms, reflecting the body's reduced capacity to effectively adjust blood pressure in response to physiological variations.

◆ Normotensive patients (e.g., Patient C/J, normal profile):

- $H(\text{day})$: 0.75 (persistence, healthy physiological adaptation) ;
- $H(\text{night})$: 0.65;

These results indicate stable and predictable regulation, characterized by regular and controlled blood pressure variations.

◆ Average typical profile:

- H(day): 0.60-0.70 ;
- H(night): 0.55-0.65 ;

4.2. Estimated Differential Equation

4.2.1. Univariate Model – Systolic (8)

$$dS(t) = \theta_S [\mu_S(t) - S(t)] dt + \sigma_S(t) dB^{H_S(t)}(t)$$

With estimates:

$$dS(t) = 0.20 \left[115 + 8.5 \cos \left(\frac{\pi t}{12} + 0.36 \right) - S(t) \right] dt + \sigma_S(t) dB^{H_S(t)}(t)$$

where:

- $\sigma_S(t) = 10$ mmHg (day) to 14 mmHg (night) ;
- $H_S(t) = 0.70$ (day) to 0.60 (night) ;

4.2.2. Bivariate Model – Systolic Diastolic (10)

$$d\mathbf{P}(t) = K[\Theta(t) - \mathbf{P}(t)]dt + \Sigma(t)d\mathbf{B}^{H(t)}(t)$$

The parameter matrix is:

$$K = \begin{pmatrix} 0.20 & 0 \\ 0 & 0.15 \end{pmatrix}$$

$$\Theta(t) = \begin{pmatrix} 115 + 8.5 \cos(\omega t) \\ 61 + 6.2 \cos(\omega t) \end{pmatrix}$$

$$\Sigma(t) = \begin{pmatrix} 12 & 2 \\ 2 & 8 \end{pmatrix} \text{ mmHg}$$

The cross-covariance ≈ 2 indicates a moderate correlation between systolic and diastolic values.

5. RESULTS AND DISCUSSION

Patients with severe hypertension exhibit an $H(t)$ function close to 0.2 during the night, reflecting strong anti-persistence and instability in autonomic regulatory mechanisms. Conversely, normotensive profiles show an $H(t)$ value of 0.75, indicating persistence and healthy physiological adaptation.

The nocturnal drop in blood pressure is a key prognostic indicator. In non-dipper or reverse-dipper patients, with a -9,5 % nocturnal drop, the local Hölder exponent drops drastically between 01:00 and 05:00.

Using an SDE driven by multifractional Brownian motion makes it possible to simultaneously consider three potential data characteristics: mean reversion, random variability, and a memory or regularity structure that may vary over time.

However, the present study is based on a single 24-hour recording and 826 observations. This size is sufficient for an exploratory case study. In particular, the local estimation of $H(t)$ generally requires more observations to ensure sufficiently long time windows.

It is therefore preferable to interpret the results as a methodological proof of concept. A more comprehensive study should include several days of measurements for a single subject and, ideally, multiple subjects. Such a database would make it possible to study inter-individual variability and evaluate the model's robustness. The multifractal SDE model provides a comprehensive mathematical representation of blood pressure dynamics, simultaneously incorporating: reversion to a circadian mean, time-dependent stochastic volatility, complex memory structure via $H(t)$

6. CONCLUSION

This article proposes a framework for modeling ambulatory blood pressure using a stochastic differential equation driven by multifractional Brownian motion. Multifractional Brownian motion offers a robust mathematical framework for capturing the complexity of blood pressure dynamics. The approach allows for the integration of temporal variability, reversion to a mean level, and potential non-classical memory. By transforming ambulatory blood pressure monitoring time series into $H(t)$ trajectories, clinicians can better identify patients at high cardiovascular risk even before the onset of target-organ damage.

The case study under consideration comprises 826 time-stamped observations collected over approximately 24 hours. Available statistics reveal substantial variation in blood pressure and a circadian component, prompting the introduction of time-dependent dynamics.

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