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k-cordial Labeling of Triangular Book, Triangular Book with Book Mark & Jewel Graph

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Abstract

M. Hovey introduced the concept of A-cordial labeling. Let $< A, * >$ be any Abelian group. A graph $G = (V(G), E(G))$ is said to be A-*cordial* if there is a mapping $f : V(G) \to A$ which satisfies the conditions $|v_f(a) - v_f(b)| \leq 1$ and $|e_f(a) - e_f(b)| \leq 1$ for all $a, b \in A$, where $v_f(a)$ and $e_f(a)$ denote the number of vertices with label a and the number of edges with label a respectively, when the edge $e = uv$ is labeled as $f(u) * f(v)$. We note that if $A =< Z_k, +_k >$, that is additive group of modulo k then the labeling is known as *k-cordial* labeling. In this article we prove that Triangular Book and Triangular Book with Book Mark are k-cordial. In addition to this we prove that Jewel Graph is k-cordial.

AMS subject classification: 05C78.
Keywords: Abelian Group; k-Cordial Labeling; Triangular Book; Triangular Book with Book Mark; Jewel Graph.

1. Introduction

In this article we consider only finite, connected, undirected and simple graph $G = (V(G), E(G))$. We denote $|V(G)|$ = total number of vertices and $|E(G)|$ = total number of edges.

Definition 1.1. A *graph labeling* is an assignment of numbers to the vertices or edges or both subject to certain condition(s).
To understand more about graph labeling as well as bibliographic references we refer Gallian [1].

Definition 1.2. Let $< A, * >$ be any Abelian group. A graph $G = (V(G), E(G))$ is said to be A-*cordial* if there is a mapping $f : V(G) \to A$ which satisfies the following two conditions when the edge $e = uv$ is labeled as $f(u) * f(v)$

(i) $|v_f(a) - v_f(b)| \leq 1$; for all $a, b \in A$,

(ii) $|e_f(a) - e_f(b)| \leq 1$; for all $a, b \in A$.

Where,
$v_f(a) =$ the number of vertices with label a;
$v_f(b) =$ the number of vertices with label b;
$e_f(a) =$ the number of edges with label a;
$e_f(b) =$ the number of edges with label b.
We note that if $A =< Z_k, +_k >$, that is additive group of modulo k then the labeling is known as k-*cordial* labeling.

The concept of A-cordial labeling was introduced by Hovey [2] and proved the following results:

- All the connected graphs are 3-cordial.
- All the trees are 3-cordial and 4-cordial.
- Cycles are k-cordial for all odd k.

Youssef [3] proved the following results:

- The complete graph K_n is 4-cordial $\Longleftrightarrow n \leq 6$.
- The complete bipartite graph $K_{m,n}$ is 4-cordial $\Longleftrightarrow m$ or $n \not\equiv 2(mod\ 4)$.

Modha and Kanani [4] proved the following result:

- The Fan f_n is k-cordial for all k.

Modha and Kanani [5] proved the following results:

- The Bistar $B_{m,n}$ is k-cordial for all k.
- The restricted square graph $B^2_{n,n}$ of bistar is k-cordial for odd k.
- The one point union of cycle C_3 with star graph $K_{1,n}$ is k-cordial for all k.
- The comb graph $P_n \bigodot K_1$ is k-cordial for all k.

Modha and Kanani [6] proved the following results:

- The Wheel W_n is k-cordial for all odd k and for all $n = mk + j$, $m \geq 0$, $1 \leq j \leq k - 1$ except for $j = \dfrac{k-1}{2}$.
- The Total graph $T(P_n)$ of path P_n is k-cordial for all k.
- The Square graph C^2_n of cycle C_n is k-cordial for all odd k and $n \geq k$.
- The path union of n copies of cycle C_k is k-cordial graph for odd k.

Rathod and Kanani [7] proved the following results:

- The Square Graph P_n^2 of path P_n is k-cordial.
- The Pan Graph C_n^{+1} is k-cordial for all even k and $n = k + j, 0 \leq j \leq k - 1$.
- The Pan Graph C_n^{+1} is k-cordial for all even k and $n = 2tk + j$, where $t \in N \cup \{0\}$ and $0 \leq j \leq k - 1$.
- The Pan Graph C_n^{+1} is k-cordial for all even k and $n = 2tk + k + j$, where $t \in N$ and $0 \leq j \leq k - 1$.

The brief summary of definitions are as follows:

- The *Triangular Book* with n-pages is defined as n copies of cycle C_3 sharing a common edge. The common edge is called the *spine or base* of the book. This graph is denoted by $B(3, n)$. In other words it is the complete tripartite graph $K_{1,1,n}$.
- The *Triangular Book with Book Mark* is a triangular book $B(3, n)$ with a pendant edge attached at any one of the end vertices of the spine. This graph is denoted by $TB_n(u, v)(v, w)$.
- The *Jewel Graph* J_n is a graph with vertex set $V(J_n) = \{u, x, v, y, v_i : 1 \leq i \leq n\}$ and the edge set $E(J_n) = \{ux, vx, uy, vy, xy, uv_i, vv_i : 1 \leq i \leq n\}$.

For any undefined term in graph theory we rely upon Gross and Yellen [8].

2. Main Results

Theorem 2.1. The Triangular Book $B(3, n)$ with n-pages is k-cordial.

Proof. Let $G = B(3, n)$ be the triangular book graph. Let $n = mk + j$, where $m \geq 0$ and $1 \leq j \leq k - 1$. We divide n vertices into two blocks of mk and j vertices, which are denoted by $v_1, v_2, ..., v_{mk}$ and $v_1', v_2', \ldots, v_j'$. Let u and v be the spine vertices. We note that $|V(G)| = n+2$ and $|E(G)| = 2n + 1$.
To define k-cordial labeling $f : V(G) \to Z_k$ we consider the following cases:

Case 1: $m > 0$ and $j = 0$.
The labeling pattern of mk vertices $v_1, v_2, \ldots, v_{mk}$ is defined as follows:

Subcase 1: $k \equiv 0(mod\ 2)$.
$f(u) = 0$ and $f(v) = k - 1$,
$f(v_i) = 2p_i$; $\quad i \equiv p_i (\text{mod } k); \quad 1 \leq i \leq \frac{k}{2} - 1,$
$f(v_i) = 2p_i - (k - 1)$; $\quad i \equiv p_i (\text{mod } k); \quad \frac{k}{2} \leq i \leq k - 1,$
$f(v_{mk}) = 0, \forall m > 0.$

Subcase 2: $k \equiv 1(mod\ 2)$.
$f(u) = 0$ and $f(v) = k - 1$,
$f(v_i) = 2p_i - 1$; $\quad i \equiv p_i (\text{mod } k); \quad 1 \leq i \leq \frac{k-1}{2},$

$f(v_i) = 2p_i - (k-1)$; $\quad i \equiv p_i (\text{mod } k);\quad \frac{k-1}{2} + 1 \le i \le k-1,$
$f(v_{mk}) = 0, \forall m > 0.$

Case 2: $m \ge 0$ and $1 \le j \le k-1$.

Subcase 1: $k \equiv 0(mod\ 2)$.
The labeling pattern of first block of mk vertices $v_1, v_2, \ldots, v_{mk}$ is defined as follows:
$f(u) = 0$ and $f(v) = k-1$,
$f(v_i) = 2p_i$; $\quad i \equiv p_i (\text{mod } k);\quad 1 \le i \le \frac{k}{2} - 1,$
$f(v_i) = 2p_i - (k-1)$; $\quad i \equiv p_i (\text{mod } k);\quad \frac{k}{2} \le i \le k-1,$
$f(v_{mk}) = 0, \forall m > 0.$

The labeling pattern of second block of j vertices $v'_1, v'_2, \ldots, v'_j$ is defined as follows:
$f(v'_j) = 2j$; $\quad 1 \le j \le \frac{k}{2} - 1,$
$f(v'_j) = 2j - (k-1)$; $\quad \frac{k}{2} \le j \le k-1.$

Subcase 2: $k \equiv 1(mod\ 2)$.
The labeling pattern of first block of mk vertices $v_1, v_2, \ldots, v_{mk}$ is defined as follows:
$f(u) = 0$ and $f(v) = k-1$,
$f(v_i) = 2p_i - 1$; $\quad i \equiv p_i (\text{mod } k);\quad 1 \le i \le \frac{k-1}{2},$
$f(v_i) = 2p_i - (k-1)$; $\quad i \equiv p_i (\text{mod } k);\quad \frac{k-1}{2} + 1 \le i \le k-1,$
$f(v_{mk}) = 0, \forall m > 0.$

The labeling pattern of second block of j vertices $v'_1, v'_2, ..., v'_j$ is defined as follows:
$f(v'_j) = 2j - 1$; $\quad 1 \le j \le \frac{k-1}{2},$
$f(v'_j) = 2j - (k-1)$; $\quad \frac{k-1}{2} + 1 \le j \le k-1.$
The labeling pattern defined above satisfies the vertex conditions and edge conditions for k-cordial labeling. Therefore, the triangular book $B(3, n)$ with n-pages is k-cordial. ■

Illustration 2.2.
(a) The 5-cordial labeling of triangular book $B(3, 10)$ is shown in *Figure 1*.

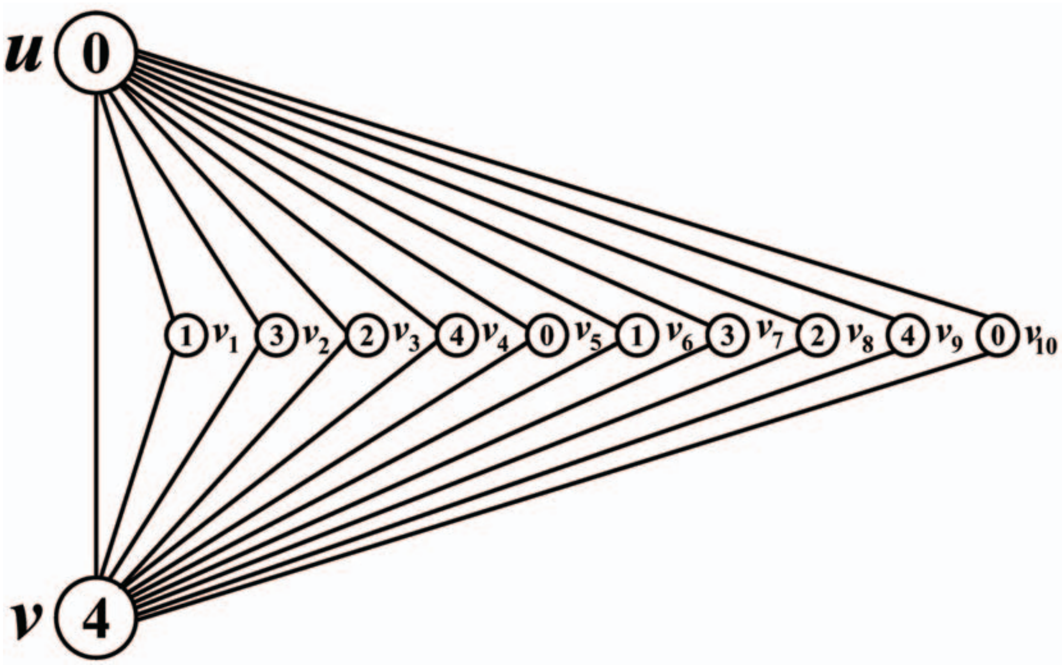

Figure 1: 5-cordial labeling of Triangular Book $B(3, 10)$.

(b) The 9-cordial labeling of triangular book $B(3, 12)$ is shown in *Figure 2*.

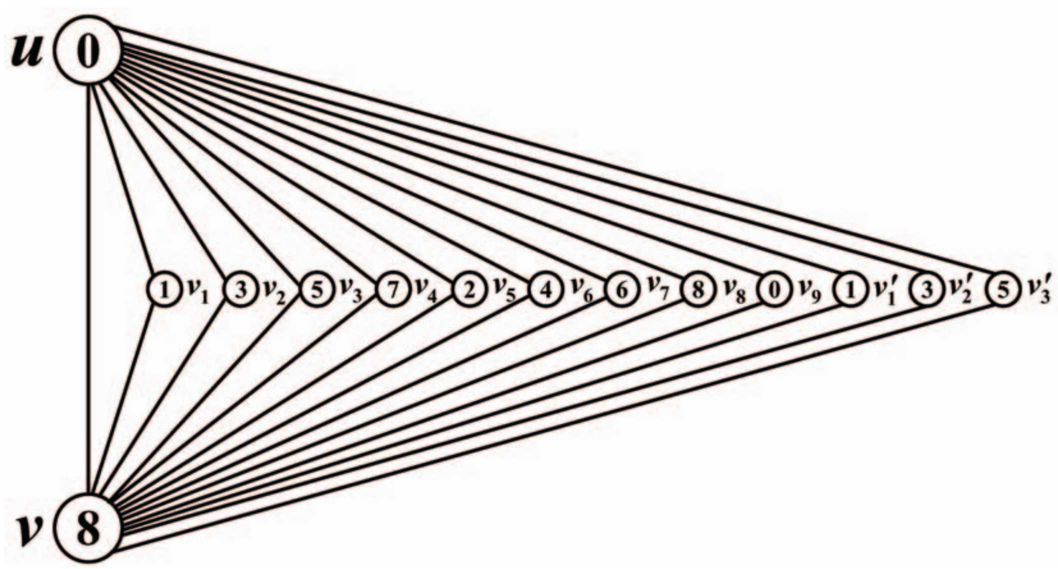

Figure 2: 9-cordial labeling of Triangular Book $B(3, 12)$.

Theorem 2.3. The Triangular Book with Book Mark $TB_n(u, v)(v, w)$ is k-cordial.

Proof. Let $G = TB_n(u, v)(v, w)$ be the triangular book with book mark obtained from triangular book $B(3, n)$ by attaching one pendant edge at any one of the spine vertices. Let u and v are vertices of spine. Let w be the pendant vertex and vw be the pendant edge. Let $n = mk + j$, where $m \geq 0$ and $1 \leq j \leq k-1$. We divide n vertices into two blocks mk and j vertices, which are denoted by $v_1, v_2, \ldots, v_{mk}$ and $v'_1, v'_2, \ldots, v'_j$. We note that $|V(G)| = n+3$ and $|E(G)| = 2(n+1)$.

To define k-cordial labeling $f : V(G) \to Z_k$ we consider the following cases:

Case 1: $m > 0$ and $j = 0$.
The labeling pattern of mk vertices $v_1, v_2, \ldots, v_{mk}$ is defined as follows:

Subcase 1: $k \equiv 0(mod\ 2)$.
$f(u) = 0,\ f(v) = k-1$ and $f(w) = 1$,
$f(v_i) = 2p_i\ ;\quad i \equiv p_i(\text{mod } k);\quad 1 \leq i \leq \frac{k}{2} - 1,$
$f(v_i) = 2(k - p_i) - 3\ ;\quad i \equiv p_i(\text{mod } k);\quad \frac{k}{2} \leq i \leq k-2,$
$f(v_{mk-1}) = k-1,$
$f(v_{mk}) = 0, \forall m > 0.$

Subcase 2: $k \equiv 1(mod\ 2)$.
$f(u) = 0,\ f(v) = k-1$ and $f(w) = 1$,
$f(v_i) = 2p_i\ ;\quad i \equiv p_i(\text{mod } k);\quad 1 \leq i \leq \frac{k-1}{2} - 1,$
$f(v_i) = 2(k - p_i) - 3\ ;\quad i \equiv p_i(\text{mod } k);\quad \frac{k-1}{2} \leq i \leq k-2,$
$f(v_{mk-1}) = k-1,$
$f(v_{mk}) = 0, \forall m > 0.$

Case 2: $m \geq 0$ and $1 \leq j \leq k-1$.

Subcase 1: $k \equiv 0(mod\ 2)$.
The labeling pattern of first block of mk vertices $v_1, v_2, \ldots, v_{mk}$ is defined as follows:
$f(u) = 0,\ f(v) = k-1$ and $f(w) = 1$,
$f(v_i) = 2p_i\ ;\quad i \equiv p_i(\text{mod } k);\quad 1 \leq i \leq \frac{k}{2} - 1,$
$f(v_i) = 2(k - p_i) - 3\ ;\quad i \equiv p_i(\text{mod } k);\quad \frac{k}{2} \leq i \leq k-2,$
$f(v_{mk-1}) = k-1,$
$f(v_{mk}) = 0, \forall m > 0.$

The labeling pattern of second block of j vertices $v'_1, v'_2, \ldots, v'_j$ is defined as follows:
$f(v'_j) = 2j\ ;\quad 1 \leq j \leq \frac{k}{2} - 1,$
$f(v'_j) = 2(k - j) - 3\ ;\quad \frac{k}{2} \leq j \leq k-2,$
$f(v'_{k-1}) = k-1.$

Subcase 2: $k \equiv 1(mod\ 2)$.
The labeling pattern of first block of mk vertices $v_1, v_2, \ldots, v_{mk}$ is defined as follows:
$f(u) = 0$, $f(v) = k - 1$ and $f(w) = 1$,

$f(v_i) = 2p_i$; $\quad i \equiv p_i(\text{mod } k)$; $\quad 1 \leq i \leq \dfrac{k-1}{2} - 1$,

$f(v_i) = 2(k - p_i) - 3$; $\quad i \equiv p_i(\text{mod } k)$; $\quad \dfrac{k-1}{2} \leq i \leq k - 2$,

$f(v_{mk-1}) = k - 1$,
$f(v_{mk}) = 0, \forall m > 0$.

The labeling pattern of second block of j vertices $v'_1, v'_2, \ldots, v'_j$ is defined as follows:

$f(v'_j) = 2j$; $\quad 1 \leq j \leq \dfrac{k-1}{2}$,

$f(v'_j) = 2(k - j) - 3$; $\quad \dfrac{k-1}{2} + 1 \leq j \leq k - 2$,

$f(v'_{k-1}) = k - 1$.

The labeling pattern defined above satisfies the vertex conditions and edge conditions for k-cordial labeling. Therefore, the triangular book with book mark $TB_n(u, v)(v, w)$ is k-cordial for all n and k. ■

Illustration 2.4.
(a) The 6-cordial labeling of triangular book with book mark $TB_6(u, v)(v, w)$ is shown in *Figure 3*.

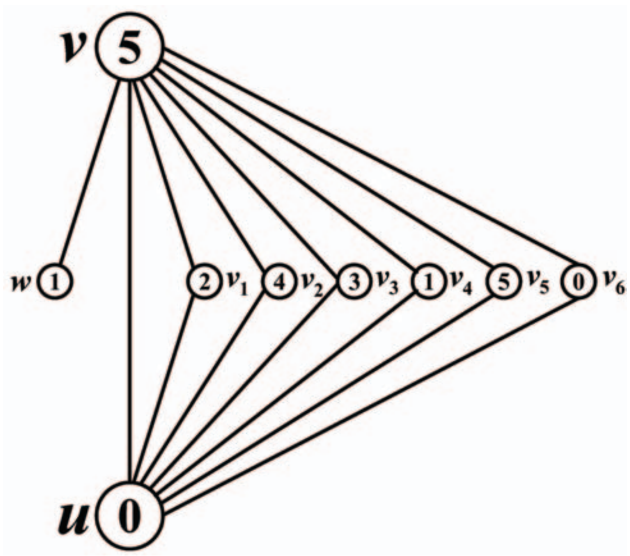

Figure 3: 6-cordial labeling of triangular book with book mark $TB_6(u, v)(v, w)$.

(b) The 7-cordial labeling of triangular book with book mark $TB_{15}(u, v)(v, w)$ is shown in *Figure 4*.

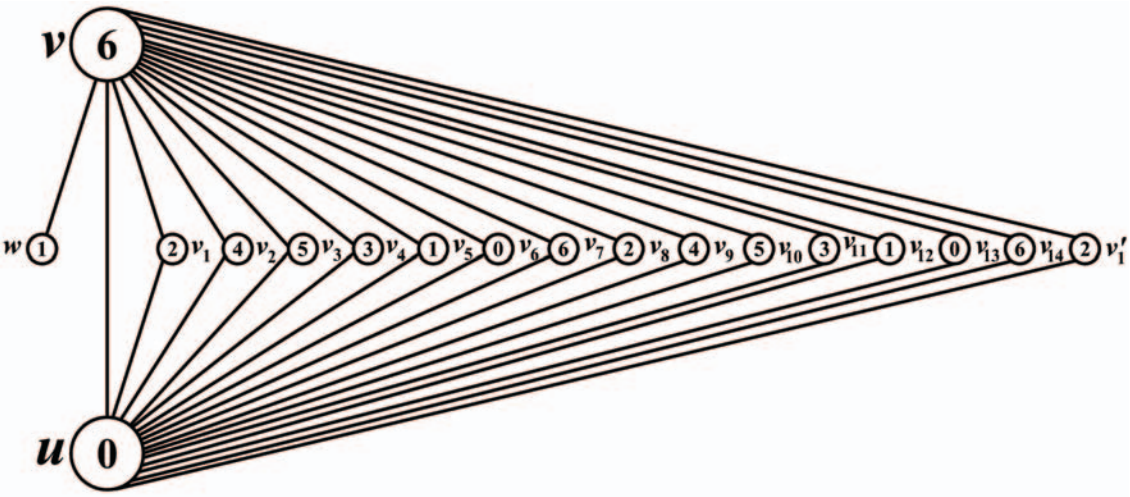

Figure 4: 7-cordial labeling of triangular book with book mark $TB_{15}(u, v)(v, w)$.

Theorem 2.5. The Jewel Graph J_n is k-cordial.

Proof. Let $G = J_n$ be the jewel graph. Let vertex set $V(J_n) = \{u, x, v, y, v_i : 1 \leq i \leq n\}$ and the edge set $E(J_n) = \{ux, vx, uy, vy, xy, uv_i, vv_i : 1 \leq i \leq n\}$. Let $n = mk + j$, where $m \geq 0$ and $1 \leq j \leq k-1$. We divide n vertices into two blocks mk and j vertices, which are denoted by $v_1, v_2, \ldots, v_{mk}$ and $v'_1, v'_2, \ldots, v'_j$. We note that $|V(G)| = n + 4$ and $|E(G)| = 2n + 5$.
To define k-cordial labeling $f : V(G) \to Z_k$ we consider the following cases:

Case 1: $m > 0$ and $j = 0$.
The labeling pattern of mk vertices $v_1, v_2, \ldots, v_{mk}$ is defined as follows:

Subcase 1: $k \equiv 0(mod\ 2)$.
$f(u) = 0,\ f(v) = k-1,\ f(x) = 1,\ f(y) = k-2,$
$f(v_i) = 2p_i + 1\ ; \qquad i \equiv p_i (\text{mod}\ k); \quad 1 \leq i \leq \frac{k}{2} - 2,$
$f(v_i) = 2(k - p_i) - 6\ ; \qquad i \equiv p_i (\text{mod}\ k); \quad \frac{k}{2} - 1 \leq i \leq k - 4,$
$f(v_{mk-3}) = k - 1,$
$f(v_{mk-2}) = 1,$
$f(v_{mk-1}) = k - 2,$
$f(v_{mk}) = 0, \forall m > 0.$

Subcase 2: $k \equiv 1(mod\ 2)$.
$f(u) = 0,\ f(v) = k-1,\ f(x) = 1,\ f(y) = k-2,$
$f(v_i) = 2p_i + 1\ ; \qquad i \equiv p_i (\text{mod}\ k); \quad 1 \leq i \leq \frac{k-1}{2} - 2,$
$f(v_i) = 2(k - p_i) - 6\ ; \qquad i \equiv p_i (\text{mod}\ k); \quad \frac{k-1}{2} - 1 \leq i \leq k - 4,$
$f(v_{mk-3}) = k - 1,$

$f(v_{mk-2}) = 1,$
$f(v_{mk-1}) = k-2,$
$f(v_{mk}) = 0, \forall m > 0.$

Case 2: $m \geq 0$ and $1 \leq j \leq k-1$.

Subcase 1: $k \equiv 0(mod\ 2)$.
The labeling pattern of first block of mk vertices $v_1, v_2, \ldots, v_{mk}$ is defined as follows:
$f(u) = 0,\ f(v) = k-1,\ f(x) = 1,\ f(y) = k-2,$
$f(v_i) = 2p_i + 1; \quad i \equiv p_i (\text{mod } k); \quad 1 \leq i \leq \frac{k}{2} - 2,$
$f(v_i) = 2(k - p_i) - 6; \quad i \equiv p_i (\text{mod } k); \quad \frac{k}{2} - 1 \leq i \leq k-4,$
$f(v_{mk-3}) = k-1,$
$f(v_{mk-2}) = 1,$
$f(v_{mk-1}) = k-2,$
$f(v_{mk}) = 0, \forall m > 0.$

The labeling pattern of second block of j vertices $v'_1, v'_2, \ldots, v'_j$ is defined as follows:
$f(v'_j) = 2j + 1; \quad 1 \leq j \leq \frac{k}{2} - 2,$
$f(v'_j) = 2(k - j) - 6; \quad \frac{k}{2} - 1 \leq j \leq k-4,$
$f(v'_{k-3}) = k-1.$
$f(v'_{k-2}) = 1.$
$f(v'_{k-1}) = k-2.$

Subcase 2: $k \equiv 1(mod\ 2)$.
The labeling pattern of first block of mk vertices $v_1, v_2, \ldots, v_{mk}$ is defined as follows:
$f(u) = 0,\ f(v) = k-1,\ f(x) = 1,\ f(y) = k-2,$
$f(v_i) = 2p_i + 1; \quad i \equiv p_i (\text{mod } k); \quad 1 \leq i \leq \frac{k-1}{2} - 2,$
$f(v_i) = 2(k - p_i) - 6; \quad i \equiv p_i (\text{mod } k); \quad \frac{k-1}{2} - 1 \leq i \leq k-4,$
$f(v_{mk-3}) = k-1,$
$f(v_{mk-2}) = 1,$
$f(v_{mk-1}) = k-2,$
$f(v_{mk}) = 0, \forall m > 0.$

The labeling pattern of second block of j vertices $v'_1, v'_2, \ldots, v'_j$ is defined as follows:
$f(v'_j) = 2j + 1\ ; \quad 1 \leq j \leq \frac{k-1}{2} - 2,$
$f(v'_j) = 2(k - j) - 6\ ; \quad \frac{k-1}{2} - 1 \leq j \leq k-4,$
$f(v'_{k-3}) = k-1.$
$f(v'_{k-2}) = 1.$

$f(v'_{k-1}) = k-2.$

The labeling pattern defined above satisfies the vertex conditions and edge conditions for k-cordial labeling. Therefore, the jewel graph J_n is k-cordial for all n and k. ■

Illustration 2.6. The 8-cordial labeling of Jewel graph J_8 is shown in *Figure 5*.

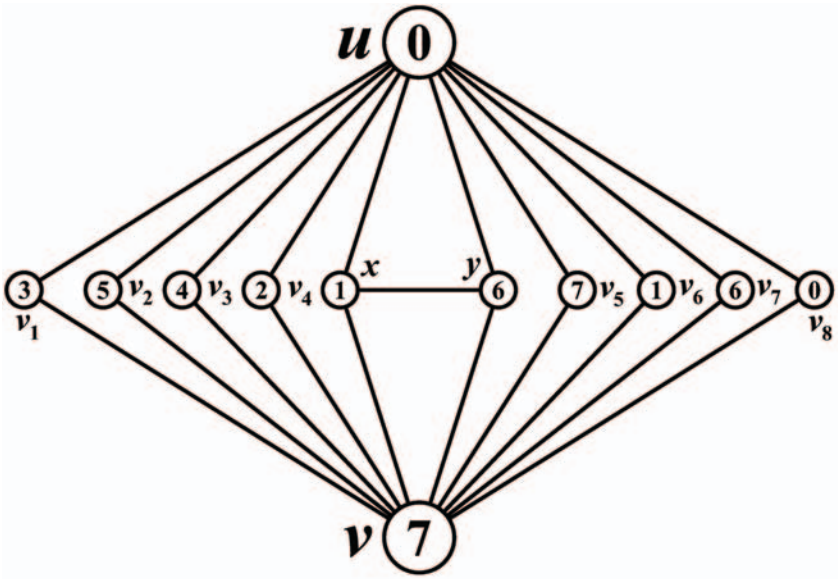

Figure 5: 8-cordial labeling of Jewel graph J_8.

3. Concluding Remarks

Here we have contributed three new results to the theory of k-cordial labeling. To derive similar results for other graph families is an open problem.

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