Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 10 (2017), pp. 7051–7057
© Research India Publications
http://www.ripublication.com/gjpam.htm

Comparative study on modification and development of layers in a differentially rotating disks

R. Kuzhali
Assistant Professor, Department of Mathematics,
Auxilium College (Autonomous), Vellore-632006. Tamilnadu, India.

B. Girija
M. Phil. Scholar, Department of Mathematics,
Auxilium College (Autonomous), Vellore-632006. Tamilnadu, India.

Abstract

An experiment is considered, when the fluid is contained in parallel disks rotating differentially and changes in the Ekman layers are studied when the Rossby number R takes the value greater than $O(E^{\frac{1}{4}})$. Two cases are considered and two different approaches are used to find the modifications of these layers. The nonlinear Ekman condition [3] is used for the case, where the rotation rate increases outwards and time-dependent general scaling approach [4] is used for the case, where the rotation rate decreases outwards. The discussion is restricted to the range $E^{\frac{1}{4}} << R << E^{\frac{1}{6}}$, since the flow is linear in other regions. A comparative study is made on thickness of layers with respect to increase and decrease in rotation rate, and the results are discussed.

AMS subject classification: 76U05.
Keywords: Rossby number, Ekman number, Ekman and Stewartson layers, differential rotation.

1. Introduction

Among many problems in fluid dynamics, the rotating fluid problem has great impact in many fields, especially in meteorology, oceanography and astrophysics. The linear theory with $R = 0$ and E sufficiently small had been studied by many researchers.

Hide [1] and Bracilon [2] studied the modification of $E^{\frac{1}{4}}$ layers for two coaxial rotating cylinders where the fluid is injected through the surface of one cylinder and removed through other and they get that, on the source cylinder, the layer has thickness $O(R)$ as R increases and for sink cylinder, the layer has thickness $O(R^{-1}E^{\frac{1}{2}})$. They both used linear Ekman condition to the modification of vertical layers, but which is inappropriate once $R >> E^{\frac{1}{4}}$.

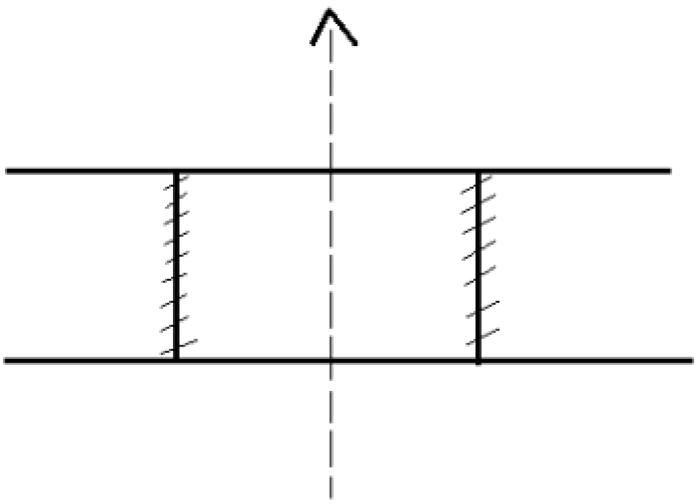
Bennets and Hockings [3] correctly developed nonlinear Ekman condition and also they discussed the experiment for differentially rotating disks. Smith [4] also discussed the same experiment within the range $E^{\frac{1}{4}} << R << E^{\frac{1}{6}}$, using time-dependent general scaling approach.

In this paper, we consider two cases, which are,

Case(i): The rotation rate of outer portion increases, and

Case(ii): The rotation rate of outer portion decreases.

Here nonlinear Ekman condition [3] is used for *Case (i)* and time-dependent general scaling approach [4] is used for *Case (ii)*.



Infinite parallel differentially rotating disks

A comparative study is made on thickness of layers with respect to increase and decrease in rotation rate.

2. Modification of layers in *Case (i)*

Let the experiment takes place within the container of height a. The velocity components (u, v, w) in terms of velocity functions ψ and χ are given by,

$$u = \frac{d}{ar}\frac{\partial \psi}{\partial y}, \quad v = \frac{d}{ar}\chi, \quad w = -\frac{d}{ar}\frac{\partial \psi}{\partial r},$$

where d denotes the distance between axis of rotation to the location of layers and ψ is a stream function and χ is proportional to the azimuthal circulation of the motion relative to the rigid rotation Ωk For this equation of continuity $\nabla u = 0$ is satisfied, and eliminating pressure from Navier-Stokes equation,

$$\frac{1}{2}\nabla p + k \times u = \frac{1}{2}E\nabla^2 u - R(u \cdot \nabla)u$$

we obtain,

$$-\frac{\partial \chi}{\partial y} = \frac{1}{2}E\Delta^4\psi + \frac{Rd}{ar}\frac{\partial(\psi, \Delta^2\psi)}{\partial(r, y)} + \frac{2Rd}{ar^2}\frac{\partial \psi}{\partial y}\Delta^2\psi + \frac{2Rd}{ar^2}\chi\frac{\partial \chi}{\partial y}$$

$$\frac{\partial \psi}{\partial y} = \frac{1}{2}E\Delta^2\chi + \frac{Rd}{ar}\frac{\partial(\psi, \chi)}{\partial(r, y)}$$

where

$$\Delta^2 = \frac{\partial^2}{\partial r^2} - \frac{\partial}{r\partial r} + \frac{\partial^2}{\partial y^2}$$

As in [3], in the Ekman layers $\partial/\partial y >> \partial/\partial x$ and the governing equations are,

$$-\frac{\partial \chi}{\partial y} = \frac{1}{2}E\frac{\partial^4\psi}{\partial y^4} + R\left(\frac{\partial \psi}{\partial x}\frac{\partial^3\psi}{\partial y^3} - \frac{\partial \psi}{\partial y}\frac{\partial^3\psi}{\partial x\partial y^2}\right)$$

$$\frac{\partial \psi}{\partial y} = \frac{1}{2}E\frac{\partial^2\chi}{\partial y^2} + R\left(\frac{\partial \psi}{\partial x}\frac{\partial \chi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \chi}{\partial x}\right)$$

The appropriate boundary conditions at the wall are,

$$\chi = f_0, \quad \psi = g_0, \quad \frac{\partial \psi}{\partial y} = 0,$$

where f_0 and g_0 are constants, and for large values χ and ψ must approach limiting values,

$$\chi \to F^*(x), \quad \psi \to G^*(x).$$

Then we get the governing equations for layers as,

$$\frac{\partial \chi}{\partial y} = 0$$

$$\frac{\partial\psi}{\partial y}\left(1+R\frac{\partial\chi}{\partial x}\right)=\frac{1}{2}E\frac{\partial^2\chi}{\partial x^2}.$$

Now let the rotation rate decreases for the central portion of the rotating disks. Let the angular velocities of inner and outer potions be,

$$\omega=\Omega\left(1-\frac{1}{2}\epsilon\right)\quad \text{for } ar<d,$$

$$\omega=\Omega\left(1+\frac{1}{2}\epsilon\right)\quad \text{for } ar>d.$$

The velocity scale is taken as $U=\frac{1}{2}|\epsilon|\,\Omega d$, and the Rossby number is $R=|\epsilon|\,\frac{d}{4a}$. For $\epsilon>0$, by using the nonlinear Ekman condition, as in [3],

$$G=Fh(F',FG'/G) \tag{2.1}$$

and

$$\chi=sgn\epsilon+F(\xi),\quad \psi=-2E^{\frac{1}{2}}yG(\xi),$$

we get,

$$G(1+F')=-\mu F''$$

where μ is given by

$$\mu=\frac{1}{4}R^{-2}E^{\frac{1}{2}}$$

together with the boundary conditions

$$F(0)=-sgn\epsilon,\quad F(\infty)=G(\infty)=0$$

The asymptotic value for h in (2.1) is given by

$$G\approx-0.56F(F')^{-\frac{3}{4}} \tag{2.2}$$

If $sgn\epsilon=1$, the total vorticity component in the layer is proportional to $1+F'$, and since $F'>0$ it is increased by the inertial terms. When $F'>1$, (2.1) and (2.2) combined and we get,

$$\mu F''=0.56F(F')^{\frac{1}{4}}$$

By solving the above equation, we get,

$$F=-(1+0.096\mu^{-\frac{4}{7}}\xi)^{-7}$$

$$G=0.75\mu^{\frac{3}{7}}(1+0.096\mu^{-\frac{4}{7}}\xi)^{-1}.$$

The change in value of F takes place in layer of thickness $\mu^{\frac{4}{7}}$. Therefore, when the rotation rate increases outwards, the solution consists of layer of thickness $E^{\frac{2}{7}}R^{-\frac{1}{7}}$, when $R>E^{\frac{1}{4}}$. This layer is called as modified Stewartson layer. From this we can say that, when the Rossby number R increases, for this case, we get that the thickness of layer decreases.

3. Modification of layers in *Case (ii)*

The velocities in the radial, azimuthal and axial directions are Ωau,Ωav, and Ωaw respectively, and the pressure is given by $\rho\Omega^2 a^2 p$. The governing equations are,

$$u_r + \frac{1}{r}u + w_z = 0,$$

$$u_t + uu_r + wu_z - \frac{1}{r}v^2 = -p_r + E\left(u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u + u_{zz}\right)$$

$$v_t + uv_r + wv_z + \frac{1}{r}uv = E\left(v_{rr} + \frac{1}{r}v_r - \frac{1}{r^2}v + v_{zz}\right)$$

$$w_t + uw_r + ww_z = -p_z + E\left(w_{rr} + \frac{1}{r} + w_r + w_{zz}\right)$$

At time $t = 0$, let the angular velocities of inner and outer portions be,

$$\omega = \Omega\left(1 + \frac{1}{2}\epsilon\right)$$

$$= \Omega\left(1 - \frac{1}{2}\epsilon\right)$$

To develope general approach to the problem, introduce the formal scaling given by as in [4],

$$r - 1 = E^a R^\alpha \eta, \quad t = E^{-d} R^{-\delta}\tau, \quad u = E^b R^\beta F,$$

$$v = r + E^c R^\gamma G, \quad w = E^{b-a} R^{\beta-\alpha} H, \quad p = \frac{1}{2}r^2 + E^{c+a} R^{\gamma+\alpha} P$$

where $a, b, c, d, \alpha, \beta, \gamma, \delta$ are constants and F, G, H, P are functions of η, z, τ. In steady state, the inner layer have two parts of thickness $O(E^{\frac{1}{3}})$ and $O(E^{\frac{1}{2}}R^{-1})$. For $t = E^{-\frac{1}{3}}$ the scaling is given by,

$$r - 1 = E^{\frac{1}{3}}\eta, \quad t = E^{-\frac{1}{3}}\tau, \quad u = RE^{\frac{1}{2}}F,$$

$$v = r + RE^{\frac{1}{6}}G, \quad w = RE^{\frac{1}{6}}H, \quad p = \frac{1}{2}r^2 + RE^{\frac{1}{2}}P$$

with the equations,

$$G_\tau + 2F = G_{\tau\tau}, \quad H_\tau + P_z = H_{\eta\eta}$$

For $t > E^{-\frac{1}{3}}$ the vertiacl layer continue to grow and it is of thickness $O(E^{\frac{1}{2}}R^{-1})$. This layer developed in the time scale $O(R^{-2})$. With $R >> E^{\frac{1}{4}}$, the non-linear terms in the angular momentum equation must be included, and the scaling is given by,

$$r - 1 = E^{\frac{1}{2}}R^{-1}\eta, \quad t = R^{-2}\eta, \quad u = RE^{\frac{1}{2}}F,$$

$$v = r + E^{\frac{1}{2}} R^{-1} G, \quad w = R^2 H, \quad p = \frac{1}{2} r^2 + E R^{-2} P,$$

with the equations,

$$G_\tau + 2F + F G_\eta = G_{\tau\tau}, \quad P_z = 0$$

For $t >> R^{-2}$, the layer $E^{\frac{1}{2}} R^{-1}$ layer is quasi-steady. Now the next stage of development commences, and we take,

$$r - 1 = R\eta, \quad t = E^{-\frac{1}{2}} \tau, \quad u = R E^{\frac{1}{2}} F, \quad v = r + RG, \quad w = E^{\frac{1}{2}} H,$$

$$p = \frac{1}{2} r^2 + R^2 P,$$

for the equations,

$$G_\tau + 2F + F G_\eta = 0, \quad P_z = 0$$

The solution consists of layer of thickness R. When $R = O(E^{\frac{1}{6}})$, the $E^{\frac{1}{3}}$ and $E^{\frac{1}{2}} R^{-1}$ layers coalesce in both transient and steady states. The nonlinearity is present still the layer of thickness R for $t >> R^{-2}$.

4. Conclusion

The increment of Rossby number causes the change in $E^{\frac{1}{4}}$ layer and nonlinearities were present in the region $E^{\frac{1}{4}} << R << E^{\frac{1}{6}}$. When the rotation rate increases outwards, the non-linear layer has thickness $E^{\frac{2}{7}} R^{-\frac{1}{7}}$ and for the decrement of rotation rate outwards the solution has non-linear layer of thickness R. Therefore, when the Rossby number R increased,

- for ***Case (i)***, there is a decrease in thickness of layer, and
- for ***Case (ii)***, there is an increase in thickness of layer.

References

[1] Hide R., *On source-sink flows in a rotating fluid,* J. Fluid Mech., 32 (1968), pp. 737–764.

[2] Barcilon V., *Some inertial modifications of the linear viscous theory of steady rotating fluid flows,* Phys. Fluids, 13 (1970), pp. 537–544.

[3] Bennetts D.A., and Hocking L.M., *On nonlinear Ekman and Stewartson layers in a rotating fluid*, Proc.Roy.Soc.A, 333(1973), pp 469–489.

[4] Smith S.H., *The development of vertical layers in a rotating fluid*, SIAM Journal on Applied Mathematics, Vol.40, No.1 (Feb. 1981), pp 78–89.

[5] Greenspan H.P., and Howard L.N., *On a time dependent motion of a rotating fluid*, J.Fluid Mech., 17(1963), pp. 385–404.

[6] Stewartson K., *On almost rigid rotations*, J.Fluid Mech., 3(1957), pp. 17–26.

[7] Smith S.H., *The non-linear split disc problem*, Quart. J. Mech. Appl. Math., 29(1976), pp. 399–414.