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Factorizations of Invertible Symmetric Matrices over Polynomial Rings with Involution

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Abstract

By means of the notions of infinite elementary divisors, dual and generalized dual matrix polynomials, we find necessary and sufficient conditions for the existence of factorizations of invertible symmetric matrices over rings of polynomials with involution.

Keywords: polynomial matrices, infinite elementary divisors, dual and generalized dual polynomial matrix, admissible and inadmissible factorizations.

AMS Subject Classification: 15A23.

I. INTRODUCTION

One of the most important problems in the matrix theory is the problem of factorization of symmetric matrices over a polynomial ring with involution, classification and investigation of the structure of there divisors [7, 9, 10, 11]. It is known that algebraic problems with productions of this type occur in the theory of differential equations, the nonlinear control theory, the optimal control and integrated systems of classical mechanics, and so have some practical value [1, 8]. In particular, [7, 9] dealt the problem of factorization of invertible symmetric matrices over a polynomial rings $\mathrm{C}[x]$. Such factorization of matrix carried by allocation of diagonal and eliminate of zero on the diagonal. In this paper we present a simple method of finding the factorization of invertible symmetric matrices over a polynomial ring which uses the notions of dual polynomial matrix [3] and generalized dual polynomial matrix [6] and their factorizations. Our results is based on the theory presented in recent papers [6, 10, 11]. In particular, in [5, 6, 10] using the notions of infinite elementary divisors, dual polynomial matrix and generalized dual polynomial matrix were found conditions of existence factorization of a singular matrix polynomials. In the [11] was investigated the problem of finding factorization symmetric regular matrices over polynomial rings with involution.

II. PRELIMINARIES

Let $\mathrm{K} = \mathbb{C}[x]$ be the polynomial ring with involutions ∇ defined in [7]:

$$(\alpha) \qquad \left(\sum_{i=-l}^{p} a_i x^i\right)^{\nabla} = \sum_{i=-l}^{p} \bar{a}_i x^{-i},$$

$$(\beta) \qquad \left(\sum_{i=-l}^{p} a_i x^i\right)^{\nabla} = \sum_{i=-l}^{p} \bar{a}_i (-x)^{-i},$$

$$(\gamma) \qquad \left(\sum_{i=-l}^{p} a_i x^i\right)^{\nabla} = \sum_{i=-l}^{p} a_i (-x)^{-i}$$

and extended to the matrix ring $\mathrm{M_n(K)}$ as follows:

$$A(x)^{\nabla} = \left\| a_{ij}(x) \right\|^{\nabla} = \left\| a_{ji}(x)^{\nabla} \right\|.$$

A matrix $A(x)$ is called symmetric if $A(x) = A(x)^{\nabla}$.

A decomposition of invertible symmetric matrix $A(x)$ is called factorization

$$A(x) = B(x) C B(x)^{\nabla}, \tag{1}$$

where $B(x)$ is invertible matrix, $C = C^{\nabla}$ is a nonsingular diagonal matrix $(\det C \neq 0)$. Let us consider nonsingular polynomial matrix

$$A(x) = \sum_{i=0}^{m} A_i x^{m-i},$$

where $A_i \in \mathrm{M_n(C)}$. The polynomial matrix $A(x)$ is called singular (regular) if $\det A_0 = 0$ ($\det A_0 \neq 0$) [2]. If $A_0 = E$ ($E-$ matrix identity), then the polynomial matrix $A(x)$ is called unital [4].

Let us define the dual polynomial matrix $\tilde{A}(x)$ of $A(x)$ as in [3]:

$$\tilde{A}(x) = \sum_{i=0}^{m} A_i x^i .$$

The infinite elementary divisors of $A(x)$ are defined as the finite elementary divisors of $\tilde{A}(x)$ at $x=0$, i.e., as the finite elementary divisors of $\tilde{A}(x)$ that have the form:

$$x^{l_j}, \quad l_j > 0 .$$

It is easy to see that if the polynomial matrix $A(x)$ is invertible, then dual polynomial matrix $\tilde{A}(x)$ is regular matrix with characteristic polynomial $\det \tilde{A}(x) = x^{mn}$.

Let us denote $S_{\tilde{A}}(x)$ the Smith form of the matrix $\tilde{A}(x)$

$$S_{\tilde{A}}(x) = P(x)\tilde{A}(x)Q(x), \tag{2}$$

where $P(x),\ Q(x)$ are invertible matrices over K.

A factorization of a symmetric matrix $\tilde{A}(x)$ is called admissible if the Smith form of the matrix $\tilde{A}(x)$ is the product of Smith forms of the factors in the decomposition (1). Otherwise a factorization of a symmetric matrix $\tilde{A}(x)$ is called inadmissible.

In the [5] studied the problem of allocating special divisor from singular matrix polynomial. It has been proved that a necessary and sufficient condition for the decomposition $A(x) = B(x)C(x),$ where $B(x)$ is singular polynomial matrix of degree r with the Smith form $\Phi(x) = diag\,(\varphi_1(x),\dots,\varphi_n(x))$ and system of infinite elementary divisors

$$x^{l_1}, x^{l_2}, \dots, x^{l_n}, \ \ 0 \le l_1 \le \dots \le l_n, \ \ \sum_{i=1}^{n} l_i = rn - \sum_{i=1}^{n} \deg \varphi_i(x),$$

moreover $\deg A(x) = \deg B(x) + \deg C(x)$, is the condition

$$\det M_{\overset{\infty}{V}(\Phi)P(x)\left\| E, Ex, \dots, Ex^{r-1} \right\|}(\overset{\infty}{\Phi}) \neq 0, \tag{3}$$

where $\overset{\infty}{\Phi}(x) = diag\,(x^{l_1}\tilde{\varphi}_1(x),\ldots,x^{l_n}\tilde{\varphi}_n(x))$, $P(x)$ is arbitrary invertible matrix of the relation (2), the matrix $V(\overset{\infty}{\Phi})$ is defined in [4].

Condition (3) is necessary condition for the factorization (1), but not sufficient. In the present parer we found necessary and sufficient conditions for the existence of the factorization (1) of a symmetric invertible matrix $A(x)$ over K, where $B(x) \in \mathrm{GL_n}(\mathrm{K})$, $\deg B(x) = \frac{m}{2}$

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III. MAIN RESULTS

Theorem 1. For a symmetric invertible matrix $A(x)$ over K there exists a factorization (1), where $B(x)$ is invertible matrix with the system of infinite elementary divisors $x^{l_1}, x^{l_2},\ldots,x^{l_n}$, $0 \le l_1 \le \ldots \le l_n$, $\sum_{i=1}^{n} l_i = \frac{mn}{2}$ and $C = C^{\nabla}$ is a nonsingular diagonal matrix, if and only if the symmetric matrix $V(\overset{\infty}{\Phi})P(x)\tilde{A}(x)P(x)^{\nabla}V(\overset{\infty}{\Phi})^{\nabla}$ is simultaneously divisible by $\overset{\infty}{\Phi}(x)$ from the left and by $\overset{\infty}{\Phi}(x)^{\nabla}$ from the right for for certain values of the parameters of the matrix $V(\overset{\infty}{\Phi})$ for which holds the condition (3), where $\overset{\infty}{\Phi}(x) = diag\,(x^{l_1},\ldots,x^{l_n})$, $P(x)$ is arbitrary invertible matrix of the relation (2).

Proof. Necessity follows from Theorem 1 [10].
Sufficiency. By Theorem 1 [11], there is a factorization of the dual polynomial regular matrix $\tilde{A}(x)$ of $A(x)$:

$$\tilde{A}(x) = \tilde{B}_1(x)G\tilde{B}_1(x)^{\nabla}, \tag{4}$$

where $\tilde{B}_1(x)$ is an unital matrix of degree $\frac{m}{2}$ with the Smith form $\overset{\infty}{\Phi}(x)$, and $G = G^{\nabla}$ is a nonsingular matrix. The matrix G is *congruent matrix for the involution (α) (congruent matrix for involutions (β), (γ)) to the matrix inertia

$I(G) = diag\{E_k, E_{n-k}\}$ where k is a number of positive eigenvalues of matrix G, i.e., $G = TI(G)T^{\nabla}$, where $T \in \mathrm{GL_n(C)}$.

Then from the relation (4) we obtain a factorization

$$\tilde{A}(x) = \tilde{B}(x) C_1 \tilde{B}(x)^{\nabla}, \qquad (5)$$

where $\tilde{B}(x) = \tilde{B}_1(x)T$ is a regular matrix of degree $\frac{m}{2}$, $\overset{\infty}{\Phi}(x)$ is the Smith form of $\tilde{B}(x)$ and $C_1 = I(G)$.

Consider the dual polynomials of $\tilde{A}(x)$, $\tilde{B}(x)$ and $\tilde{B}(x)^{\nabla}$ from the relation (5). Thus we get the factorization (1) of invertible matrix $A(x)$ over K, where $B(x)$ is invertible matrix over K with a system of infinite elementary divisors $x^{l_1}, x^{l_2}, \ldots, x^{l_n}$ and $C = \pm C_1$.

The theorem is proved.

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Cororally. For a symmetric invertible matrix $A(x) \in \mathrm{GL_n(K)}$ there exists a factorization (1), where $B(x) \in \mathrm{GL_n(K)}$, $\deg B(x) = \frac{1}{2} \deg A(x)$, if and only if there exists a factorization for the dual symmetric matrix $\tilde{A}(x)$ of $A(x)$.

There is an important problem of the factorization (1) of the invertible symmetric matrix $A(x)$ over K of degree m, where $B(x) \in \mathrm{GL_n(K)}$, $\deg B(x) > \frac{m}{2}$. The condition $\deg A(x) = \deg B(x) + \deg B(x)^{\nabla}$ is not holds in the factorization (1). Therefore, we introduce into consideration the notion of generalized dual polynomial.

Let $f(x) = \sum_{i=0}^{m} a_i x^{m-i}$ be a polynomial, in particular a polynomial matrix. The generalized dual polynomial $\tilde{\tilde{f}}(x)$ of $f(x)$ for degree r ($r > m$) is define as [6]:

$$\tilde{\tilde{f}}(x) = \sum_{i=0}^{m} a_i x^{i+r} \quad \text{where } r \in \mathrm{N}.$$

If $\deg f(x) = m$ it is obvious that $\tilde{\tilde{f}}(x) = x^{r+m} f\left(\frac{1}{x}\right)$, $r \in \mathrm{N}$.

It is easy to see that if a polynomial matrix $A(x)$ is invertible over K, then generalized dual polynomial $\tilde{\tilde{A}}(x)$ is a regular matrix polynomial. If $A(x)$ is a symmetric matrix polynomial, then its generalized dual polynomial $\tilde{\tilde{A}}(x)$ is a symmetric for an even number r. Further, assume that r is an even number.

Given that $\tilde{\tilde{A}}(x) = Ex^r \tilde{A}(x)$, $r \in \mathrm{N}$, we can easily show the following result.

Proposition. Let the Smith form $S_{\tilde{A}}(x)$ of dual polynomial $\tilde{A}(x)$ of $A(x)$ has the form (2). Then the Smith form $S_{\tilde{\tilde{A}}}(x)$ of the generalized dual polynomial $\tilde{\tilde{A}}(x)$ of $A(x)$ for degree r is

$$S_{\tilde{\tilde{A}}}(x) = P(x)\tilde{\tilde{A}}(x)Q(x),$$

where $P(x),\ \ Q(x) \in \mathrm{GL_n(K)}$ are matrices from the relation (2).

Theorem 2. For a symmetric matrix $A(x) \in \mathrm{GL_n(K)}$ there exists a factorization (1), where $B(x) \in \mathrm{GL_n(K)}$, $C = C^{\nabla}$ is a nonsingular diagonal matrix, if and only if there exists the inadmissible factorization of generalized dual polynomial $\tilde{\tilde{A}}(x)$ of $A(x)$ for degree r, where $r = \deg \tilde{B}(x) + \deg \tilde{B}(x)^{\nabla} - m$, the matrix $\tilde{B}(x)$ is the dual polynomial of $B(x)$.

Proof. Necessity. Let for a matrix polynomial $A(x) \in \mathrm{GL_n(K)}$ there exists a factorization (1) and $r = \deg \tilde{B}(x) + \deg \tilde{B}(x)^{\nabla} - \deg A(x)$.

Let us consider the generalized dual polynomial $\tilde{\tilde{A}}(x)$ of $A(x)$ for degree r. Then from the relation (1) we obtain the factorization of a regular matrix polynomial $\tilde{\tilde{A}}(x)$.

We must to prove that such factorization of matrix $\tilde{\tilde{A}}(x)$ is inadmissible.

Suppose by contradiction that there exists admissible factorization for $\tilde{\tilde{A}}(x)$:

$$\tilde{\tilde{A}}(x) = \tilde{B}(x) C_1 \tilde{B}(x)^{\nabla}$$

and

$$S_{\tilde{\tilde{A}}}(x) = \Phi_1(x) I \Phi_1(x)^{\nabla}, \qquad \deg \Phi_1(x) = \frac{n(m+r)}{2},$$

where $\tilde{B}_1(x)$ with the Smith form $\Phi_1(x)$. Since $S_{\tilde{\tilde{A}}}(x) = Ex^r S_{\tilde{A}}(x)$, then from last relations we can allocate the factors $Ex^{\frac{r}{2}}$ and $\left(Ex^{\frac{r}{2}}\right)^{\nabla}$ respectively: $\tilde{B}_1(x) = Ex^{\frac{r}{2}} \tilde{B}(x)$, $\Phi_1(x) = Ex^{\frac{r}{2}} \Phi(x)$. It follows that there exists admissible factorization of matrix $\tilde{A}(x)$, where $\tilde{B}(x)$ is the matrix of degree $\frac{m}{2}$ with the Smith form $\Phi(x)$. Thus, we obtain a contradiction, since $\deg \tilde{B}(x) = \deg B(x) > \frac{m}{2}$.

Sufficiency. Let there exists inadmissible factorization of the generalized dual polynomial $\tilde{\tilde{A}}(x)$ of $A(x)$ for degree r, i.e.,

$$\tilde{\tilde{A}}(x) = \tilde{B}(x) C_1 \tilde{B}(x)^{\nabla}, \tag{6}$$

where $\deg \tilde{\tilde{A}}(x) = \deg \tilde{B}(x) + \deg \tilde{B}(x)^{\nabla}$.

Let us consider the dual polynomial $A(x)$ of $\tilde{\tilde{A}}(x)$. Then from the relation (6) we obtain the factorization of invertible matrix polynomial $A(x)$ over K where

$$\deg A(x) < \deg B(x) + \deg B(x)^{\nabla}.$$

Last inequality holds because the coefficients of $x^{r-1}, \ldots, x^0$ in the matrix $\tilde{\tilde{A}}(x)$ are zero. The theorem is proved.

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