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Functors in the Category of Graphs

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Abstract

Functors play an important role in the Theory of Graphs. They have wide range of applications in various fields like algebraic topology, geometry, logic and so on. In this article we study some properties of the two standard functors namely the morphism functors and the forgetful functors from the category of graphs into the category of sets [1].

1. INTRODUCTION

A graph G consists of a pair G = (V(G), E(G)) (also written as G = (V , E) whenever the context is clear) where V(G) is a finite set whose elements are called vertices and E(G) is a set of unordered pairs of distinct elements in V(G) whose members are called edges. The graphs as we have defined above are called simple graphs. Throughout our discussions all graphs are considered to be simple graphs [2].

Let G and G_1 be graphs. A homomorphisms $f: G \rightarrow G_1$ is a pair $f = (f^*, \tilde{f})$ where

$f^*: V(G) \rightarrow V(G_1)$ and $\tilde{f} : E(G) \rightarrow E(G_1)$ are functions such that

$\tilde{f}$ ((u, v)) = (f^*(u) , f^*(v)) for all edges (u , v) $\in$ E(G). For convenience if

(u, v) $\in$ E (G) then $\tilde{f}$ ((u, v)) is simply denoted as $\tilde{f}$ ((u, v)) [3].

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Then we have the category of graphs say $\mathscr{G}$, where objects are graphs and morphisms are as defined above, where equality, compositions and the identity morphisms are defined in the natural way. It is also proved that two homomorphisms $f = (f^*, \tilde{f})$ and $g = (g^*, \tilde{g})$ of graphs are equal if and only if

$f^* = g^*$ (Lemma 1.6) [3].

2. FUNCTORS

Definition 2.1 Morphism functors:

Let $\mathscr{G}$ and $\mathscr{S}$ denote the category of graphs and the category of sets respectively.

If X and Y are graphs then as before [X, Y] denotes the set of all homomorphisms from X to Y.

Fix a graph X in $\mathscr{G}$. For any graph Y in $\mathscr{G}$ let $H^X(Y) = [X, Y]$. Moreover for each homomorphism $f : Y \to Z$, let $H^X(f)$: $[X, Y] \to [X, Z]$ be the function defined as

$(H^X f)\, g = fg$ for each $g \in [X,Y]$.

$$H^X(f): [X, Y] \to [X, Z]$$
$$g \mapsto fg$$

Then as in any category, $H^X : \mathscr{G} \to \mathscr{S}$ is a covariant morphism functor.

Similarly we have the contravariant morphism functor $H_X : \mathscr{G} \to \mathscr{S}$ defined as follows.

$H_X(Y) = [Y, X]$ for all $Y \in \mathscr{G}$;

Also for f: $Y \to Z$ and $g \in [Z, X]$, $H_X(f) : [Z, X] \to [Y, X]$ is the function which maps g into gf.

$$H_X(f): [Z, X] \to [Y, X]$$
$$g \mapsto gf$$

Proposition 2.2 Let $f : Y \to Z$ be a homomorphism of graphs. Then f is a monomorphism in $\mathscr{G}$ if and only if $H^X(f)$ is a monomorphism in $\mathscr{S}$ for all

$X \in \mathscr{G}$.

Proof Let $f : Y \to Z$ be a monomorphism in $\mathscr{G}$. Suppose $g_1, g_2 \in [X, Y]$ such that $(H^X f)g_1 = (H^X f)g_2$. Then by definition $fg_1 = fg_2$. This implies that $g_1 = g_2$, since f is a monomorphism. Conversely suppose $H^X(f)$ is a monomorphism in $\mathscr{S}$ for each $X \in \mathscr{G}$.

Let $g_1, g_2 \in [X, Y]$ such that $fg_1 = fg_2$. Then $(H^X f)g_1 = (H^X f)g_2$ so that $g_1 = g_2$. This shows that f is a monomorphism in $\mathscr{G}$.

Corollary 2.3 $H^{X}: \mathscr{G} \to \mathscr{S}$ is a monofunctor for each $X \in \mathscr{G}$.

Corollary 2.4 Let $f : Y \to Z$ be a coretraction in $\mathscr{G}$. Then for all $X \in \mathscr{G}$, $H^X f : [X,Y] \to [X,Z]$ is a coretraction in $\mathscr{S}$.

Proof Let $f : Y \to Z$ be a coretraction in $\mathscr{G}$. Since every coretraction is a monomorphism, f is a monomorphism in $\mathscr{S}$.

Therefore $H^X f$ is a monomorphismin $\mathscr{S}$ [3]. Hence $H^X f$ is a coretraction in $\mathscr{S}$ (since in the category of sets a morphism is a coretraction if and only if it is a monomorphism).

Corollary 2.5 The functor $H^X : \mathscr{G} \to \mathscr{S}$ preserves coretractions.

Remark 2.6 The converse of corollary 2.5 is not true i.e., if $f : Y \to Z$ is a homomorphism of graphs such that $H^X f : [X,Y] \to [X,Z]$ is a coretraction for all $X \in \mathscr{G}$, then f need not be a coretraction as the following example shows.

Consider the homomorphism $f : Y \to Z$ of graphs given below (Figure 1).

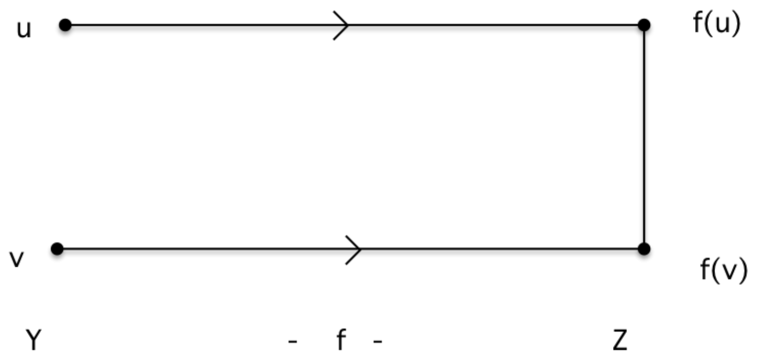

Figure 1

Here f is injective in $\mathscr{G}$ implies that f is a monomorphism in $\mathscr{G}$ [3]. Hence by proposition (2.2) $H^X f$ is a monomorphism in $\mathscr{S}$ so that $H^X f$ is a coretraction in $\mathscr{S}$. However f is not a coretraction in $\mathscr{G}$ since $(f(u), f(v))$ is an edge in Z but (u, v) is not an edge in Y. Thus by proposition 1.9 [3] f is not a coretraction in $\mathscr{G}$.

Corollary: 2.7 The functor H^X reflects monomorphisms.

Proposition 2.8 Let $f : Y \to Z$ be a homomorphism of graphs. Then f is an epimorphism in $\mathscr{G}$ if and only if $H_X f : [Z, X] \to [Y, X]$ given by $g \mapsto gf$ is a monomorphism in $\mathscr{S}$.

Proof Let $f : Y \to Z$ be an epimorphism. Suppose $g_1, g_2 \in [Z, X]$ such that $(H_X f)g_1 = (H_X f)g_2$. Then by definition $g_1 f = g_2 f$ which implies that $g_1 = g_2$ (since f is an epimorphism). Thus $H_X f$ is a monomorphism in $\mathscr{S}$.

Conversely suppose $H_X f : [Z, X] \to [Y, X]$ is a monomorphism in $\mathscr{S}$ for all $X \in \mathscr{G}$.

Let $g_1, g_2 \in [Z, X]$ such that $g_1 f = g_2 f$. Then $(H_X f)g_1 = (H_X f)g_2$ so that $g_1 = g_2$

(since $H_X f$ is a monomorphism). Therefore f is an epimorphism in $\mathscr{G}$.

Remark 2.9 By proposition [2.2] $f : Y \to Z$ is a monomorphism in $\mathscr{G}$ if and only if $H^X f : [X,Y] \to [X,Z]$ is a monomorphism in $\mathscr{S}$ for all $X \in \mathscr{G}$. However if f is an epimorphism in $\mathscr{G}$ then $H^X f$ need not be an epimorphism in $\mathscr{S}$ as the following example shows.

Consider the graphs X, Y and Z where $V(X) = \{u_1, u_2\}$ and $(u_1, u_2) \in E(X)$;

$V(Y) = \{v_1, v_2\}$ *and* $E(Y) = \varphi$; (Figure 2)

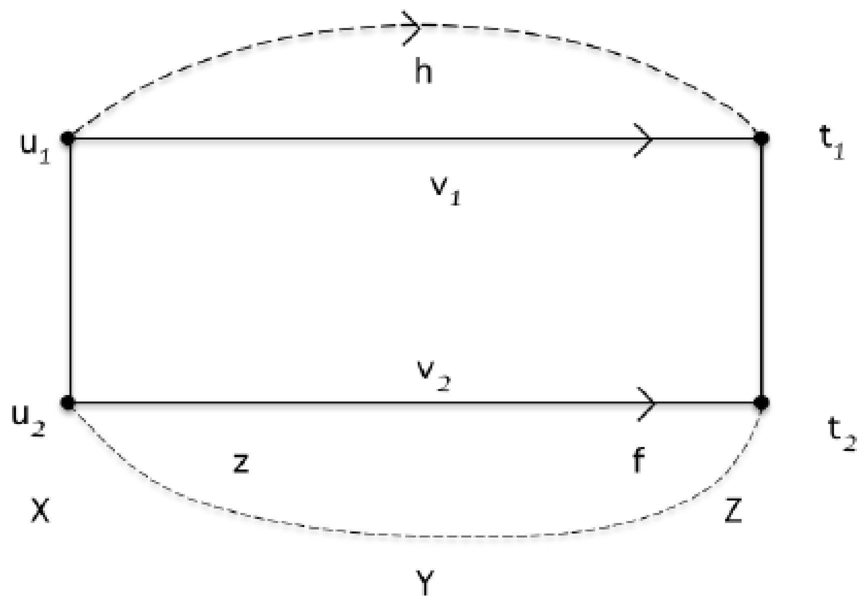

Figure 2

$V(Z) = \{t_1, t_2\}$and $(t_1, t_2) \in E(Z)$.

Let $f: Y \to Z$ be the homomorphism given by $f^*(v_1) = t_1$, $f^*(v_2) = t_2$ and

$E(Y) = \varphi$ and $h : X \to Z$ be the homomorphism given by $h^*(u_1) = t_1$, $h^*(u_2) = t_2$ and $h(u_1, u_2) = (t_1, t_2)$. Then there is no homomorphism from $X \to Y$ and so there does not exist a homomorphism $g: X \to Y$ such that $f\,g = h$ i.e. such that $(H^*f) = h$. Thus since f *is* surjective f is an epimorphism in $\mathscr{G}$ (Proposition 1.20 in [3]) but

H^*f is not an epimorphism in $\mathscr{S}$ (Proposition 2.8).

Remark: 2.10 Let Y and Z be graphs and let $f : Y \to Z$ be an epimorphism. Then

$H_X f : [Z,X] \to [Y,X]$ need not be an epimorphism as the following example shows.

Consider the graphs X, Y and Z (Figure 3)

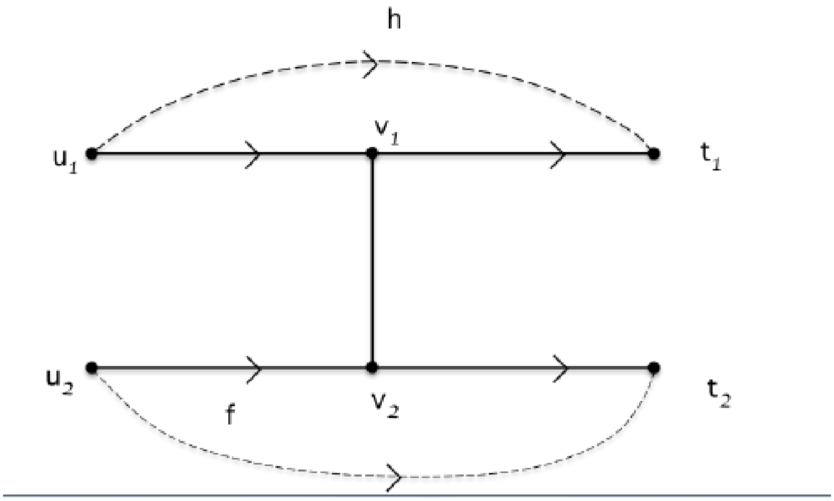

Figure 3

Let $V(Y) = \{u_1, u_2\}$ $E(Y) = \varphi$; $V(Z) =\{v_1,v_2\}$,

$E(Z) =\{(v_1,v_2)\}$; $V(X) =\{t_1,t_2\}$, $E(X) = \varphi$.

Let $f : Y \to Z$ be the homomorphism where $f^*(u_1) = v_1$, $f^*(u_2) = v_2$ & $\tilde{f} = \varphi$.

Consider the homomorphism $h : Y \to X$ where $h^*(u_1) = t_1$, $h^*(u_2) = t_2$ and $\tilde{h} = \varphi$. f is surjective and hence an epimorphism (Proposition 1.20 in [3]). But there exists no homomorphism $g : Z \to X$ such that $gf = h$. Therefore $H_x f$ is not surjective in $\mathscr{S}$ and so is not an epimorphism.

3.1 The Forgetful Functors

If the objects of a category $\mathscr{A}$ are sets with certain structures like groups,

R-modules, topological spaces etc., and the morphisms are structure preserving maps (like homomorphism between groups, continuous maps etc.), then we have a functor

$T : \mathscr{A} \to \mathscr{S}$ which assigns to each object its underlying set and to each morphism the corresponding set map. In particular if $\mathscr{G}$ is the category of graphs then we have the Forgetful functor $\mathbb{F} : \mathscr{G} \to \mathscr{S}$ defined as follows:

If $G \in \mathscr{G}$ then $\mathbb{F}(G) = V(G)$ and if $f : G_1 \to G_2$ is a homomorphism of graphs, then $\mathbb{F}(f) = f^* : V(G_1) \to V(G_2)$. In this section we investigate some of the properties of this functor $\mathbb{F}$.

Proposition 3.2 The functor $\mathbb{F}$ has the following properties:

i. $\mathbb{F}$ preserves and reflects monomorphisms.
ii. $\mathbb{F}$ preserves and reflects epimorphisms.
iii. $\mathbb{F}$ preserves coretractions but does not reflect.
iv. $\mathbb{F}$ preserves retractions but does not reflect.
v. $\mathbb{F}$ preserves isomorphisms but not reflect.
vi. $\mathbb{F}$ is not representative.
vii. $\mathbb{F}$ is faithful.
viii. $\mathbb{F}$ is not an embedding.
ix. $\mathbb{F}$ is not full.
x. $\mathbb{F}$ is not an equivalence.

Proof (i) Let $f : Y \rightarrow Z$ be a homomorphism of graphs, then f is a monomorphism in $\mathscr{G}$ if and only if $f^*: V(Y) \rightarrow V(Z)$ is injective in $\mathscr{S}$ (Proposition 1.19 in [3]) i.e. if and only if $\mathbb{F}(f)$ is injective in $\mathscr{S}$ i.e. if and only $\mathbb{F}(f)$ is a monomorphism in $\mathscr{S}$. Thus $\mathbb{F}$ preserves and reflects monomorphisms.

ii) f is an epimorphism in $\mathscr{G}$ if and only if $f^*: V(Y) \rightarrow V(Z)$ is surjective in $\mathscr{S}$

(Proposition 1.20 in [3]) i.e. if and only if $\mathbb{F}(f)$ is surjective in $\mathscr{S}$ i.e. if and only if $\mathbb{F}(f)$ is an epimorphism in $\mathscr{S}$. Thus $\mathbb{F}$ preserves and reflects epimorphisms.

iii) Let f be a coretraction in $\mathscr{G}$. Then f is a monomorphism in $\mathscr{G}$ and hence

$f^* = \mathbb{F}(f)$ is a monomorphism in $\mathscr{S}$. Hence by (Proposition 5.1 in [1]) $\mathbb{F}(f)$ is a coretraction in $\mathscr{S}$. Thus $\mathbb{F}$ preserves coretraction.

However $\mathbb{F}(f) = f^*$ is a coretraction in $\mathscr{S}$ does not imply that f is a coretraction in $\mathscr{G}$ as the following example shows. Consider the homomorphism $f : Y \rightarrow Z$ in $\mathscr{G}$ defined by the following diagram (Figure 4).

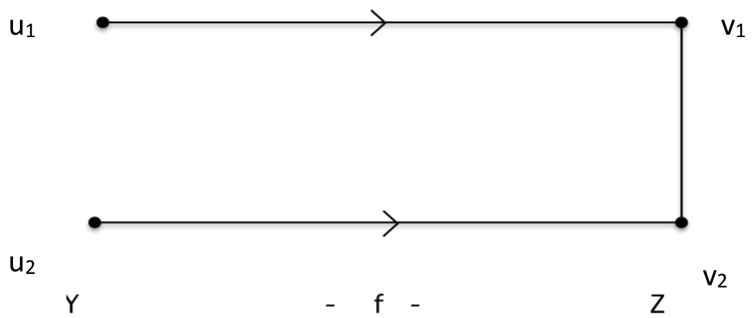

Figure 4

Here $f^* = \mathbb{F}(f)$ is injective in $\mathscr{S}$ implies that $\mathbb{F}(f)$ is a coretraction in $\mathscr{S}$.

However f is not a coretraction in $\mathscr{S}$ so that F does not reflect coretractions.

iv) Let $f : Y \to Z$ be a retraction in $\mathscr{G}$. Then by (Proposition 1.16 in [3]) f is an epimorphism and hence $f^*: V(Y) \to V(Z)$ is surjective in $\mathscr{S}$(Proposition1.16 in [3]). Hence $\mathbb{F}(f) = f^*$ is a retraction in $\mathscr{S}$. Therefore $\mathbb{F}$ preserves retractions.

However $\mathbb{F}(f)$ is a retraction in $\mathscr{S}$ does not imply that f is a retraction in $\mathscr{G}$. Consider the example given in (iii). $\mathbb{F}(f) = f^*$ is surjective and hence a retraction in $\mathscr{S}$. But f is not a retraction in $\mathscr{G}$ since there is no homomorphism from Z to Y. Thus $\mathbb{F}$ does not reflect retractions.

v) Let $f : Y \to Z$ be an isomorphism in $\mathscr{G}$.. Then f is both a coretraction and retraction in $\mathscr{G}$ and hence f is both a monomorphism and an epimorphism in $\mathscr{G}$. This shows that $f^*: V(Y) \to V(Z)$ is both injective and surjective. i.e. f^* is both a monomorphism and an epimorphism in $\mathscr{S}$ and so is an isomorphism in $\mathscr{S}$(since $\mathscr{S}$ is balanced). Thus $\mathbb{F}(f)$ is an isomorphism and hence $\mathbb{F}$ preserves isomorphisms.

However $\mathbb{F}$ does not reflect isomorphisms as an example in (iii) shows. Here $\mathbb{F}(f) = f^*: V(Y) \to V(Z)$ is a bijection and hence an isomorphism in $\mathscr{S}$. But f is not an isomorphism in $\mathscr{G}$ as seen earlier. Thus $\mathbb{F}$ does not reflect isomorphisms.

vi) $\mathbb{F}$ is not representative. For consider any infinite set A. Since we are considering only graphs with finite number of vertices, there is no graph G such that

$\mathbb{F}(G) = V(G)$ is isomorphic to A.

However every finite set A has a graph G such that $\mathbb{F}(G) \cong \mathscr{A}$. For if A has n elements, then consider any graph G with n-vertices (for example the complete graph on n vertices).Then $\mathbb{F}(G) = V(G)$ has n-elements and hence is isomorphic to A.

vii) $\mathbb{F}$ is faithful.

Let X, Y be graphs. Let $\overline{\mathbb{F}}: [X,Y] \to [\mathbb{F}(X), \mathbb{F}(Y)] = [V(X),V(Y)]$ be the function induced by $\overline{\mathbb{F}}$. i.e.

$$\overline{\mathbb{F}} : [X,Y] \to [V(X), V(Y)]$$
$$f \mapsto \mathbb{F}(f) = f^*$$

Then $f_1, f_2 \in [X, Y]$ and $f_1 \neq f_2$ implies that $f_1^* \neq f_2^*$ (Lemma 1.6 in [3]).

i.e. $\mathbb{F}(f_1) \neq \mathbb{F}(f_2)$. Thus by definition $\mathbb{F}$ is faithful.

viii) We know that $\mathbb{F}$ is faithful. Consider the two non isomorphic graphs on six vertices given below [Figure 5]. $G1 \ncong G2$.

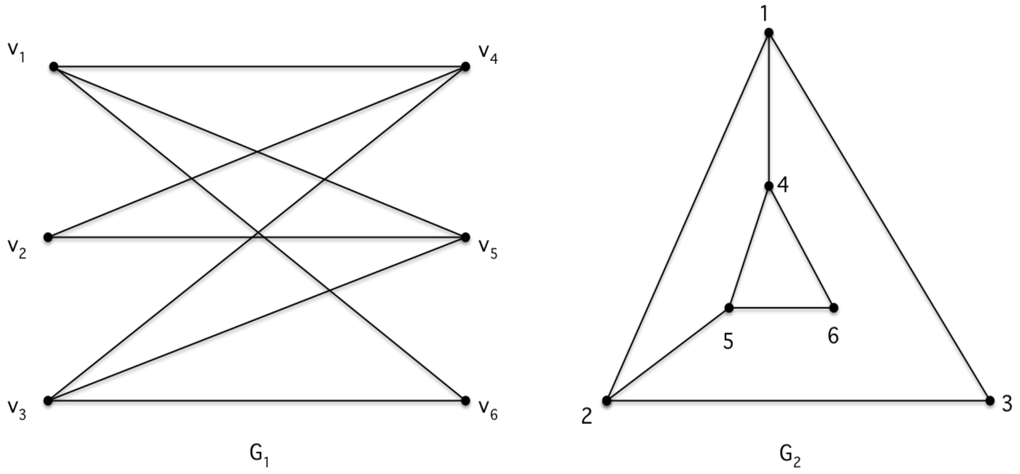

Figure 5

However $V(G_1)$ and $V(G_2)$ each have six elements and so are isomorphic in $\mathscr{S}$ .i.e. $\mathbb{F}(G_1) \cong \mathbb{F}(G_2)$. Thus $G_1 \ncong G_2$ but $\mathbb{F}(G_1) \cong \mathbb{F}(G_2)$ and so $\mathbb{F}$ is not an imbedding.

ix) Consider the graphs X and Y given below (Figure 6).

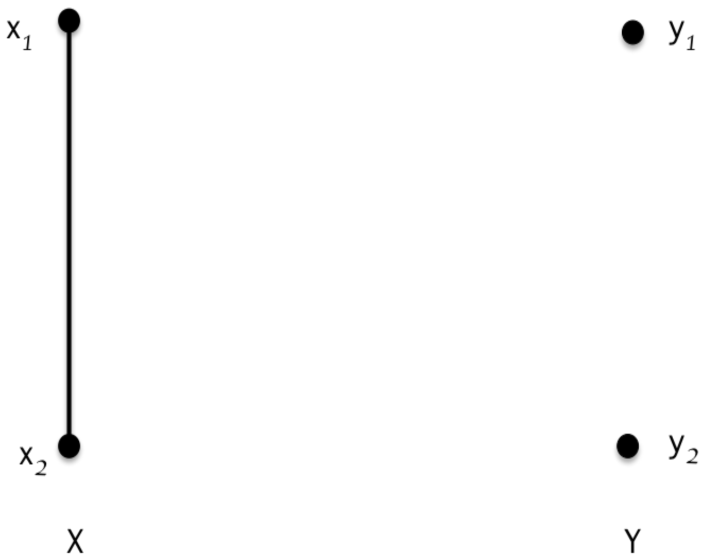

Figure 6

Then $\mathbb{F}(X) = V(X) = \{x_1, x_2\}$ and

$\mathbb{F}(Y) = V(Y) = \{y_1, y_2\}$.

We have a function $g : \mathbb{F}(X) \rightarrow \mathbb{F}(Y)$ defined as $g(x_i) = y_i$, $i = 1, 2$. However there is no $f \in [X, Y]$ such that $\mathbb{F}(f) = f^* = g$ (since the edge (x_1, x_2) has no image).

x) This follows simply from (ix) and the definition of equivalence.

4. CONCLUSIONS

We have shown some properties of two standard functors mainly the morphism functors and the forgetful functors from the category of graphs into the category of sets.

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