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Domination Stability in Fuzzy Graphs

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Abstract

In general γ_f (G) can be made to increase or decrease by the removal of nodes from G. In this paper we studied the stability of fuzzy dominating set. We investigate the stability of fuzzy paths and fuzzy cycles. Also we obtain sharp bounds and characterizations for the domination stability of fuzzy graphs.

Keywords: Fuzzy dominating set, fuzzy domination number, fuzzy domination stability.

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1. INTRODUCTION

Brigham [1] introduced vertex domination critical graphs. Harary et al [2] explained an interesting application in voting situations using the concept of domination. Nagoor Gani [4] and Vijayalakshmi [4] discussed domination critical nodes in fuzzy graph Rosenfeld [5] introduced the notion of fuzzy graph and several fuzzy analogous of graph theoretic concepts such as paths, cycles, connectedness and etc. Somasundaram and Somasundaram [6] discussed domination in fuzzy graphs. Sumner [7] discussed domination critical graphs. Nader Jafari Rad, Elahe Sharifi and Marcin krzywkowski [4] introduced domination stability in graphs.

2. PRELIMINARIES

A *fuzzy graph* G=(σ,μ) is a non-empty set V together with a pair of functions $\sigma: V \to [0,1]$ and μ:VxV→[0,1] such that μ(u,v) ≤ $\sigma(u) \wedge \sigma(v)$ for all u,v ∈ V, where $\sigma(u) \wedge \sigma(v)$is the minimum of $\sigma(u)$ *and* $\sigma(v)$.The *underlying crisp graph* of the fuzzy graph G=(σ,μ) is denoted as $G^*=(\sigma^*, \mu^*)$ where σ^*={ u∈ $V/\ \sigma(u)$ >0} and μ^*= {(u,v)∈VxV / μ(u,v)>0}.Let G =(σ,μ) be a fuzzy graph and τ be any fuzzy subset of σ,i.e $\tau(u) \leq \sigma(u)$for all u,. Then the fuzzy subgraph of G = (σ,μ) induced by τ is the maximal fuzzy sub graph of G = (σ,μ) that has fuzzy node set τ. Evidently this is just the fuzzy graph (τ,ρ), where ρ(u,v) = $\tau(u) \wedge \tau(v)$for all u, v ∈ V.

A fuzzy graph G=(σ,μ) is a *complete fuzzy graph* if μ(u,v) = $\sigma(u) \wedge \sigma(v)$for all u,v ∈ σ^* Two nodes u and v are said to be *neighbours* if μ(u,v) > 0.The *strong neighbourhood* of u is N_S(u) = { v ∈ V : (u,v) is a strong arc}. N_S[u] = N_S(u) ∪ {u} is the *closed strong neighbourhood* of u. Let G=(σ,μ) be a fuzzy graph. Two nodes u and v of G are *strong adjacent* if (u,v) is strong arc. The *strong degree* of a node v is the minimum number of nodes that are strong adjacent to v. It is denoted by d_S(v). The minimum cardinality of strong neighbourhood δ_S(G) = min{$|N_S(u)|$:u∈ $V(G)$} and the maximum cardinality of strong neighbourhood Δ_S (G) = max {$|N_S(u)|$: u∈ $V(G)$} . Let G be a fuzzy graph. Let S be a set of vertices in G. Let u∈S then the *private neighbourhood* of u is pn(u,S) = { v: N_S(v)∩S = {u}}. The *external private neighbourhood* epn(v,S) = pn(u,S) \ S. A node u is called a *fuzzy end node* of G = (σ,μ) if it has atmost one strong neighbour in G = (σ,μ).

A *path* ρ in a fuzzy graph is a sequence of distinct nodes $u_0,u_1,u_2,...u_n$ such that $\mu(u_{i-1},u_i)>0$; $1\leq i \leq n$ here n≥ 0 is called the *length* of the path ρ .The consecutive pairs (u_{i-1},u_i) are called the *arcs* of the path. A path ρ is called a *cycle* if $u_o=u_n$ and n≥3. An arc (u,v) is said to be a *strong arc* if μ(u,v) ≥ μ^∞ (u,v) and the node v is said to be a *strong neighbour* of u. If μ(u,v) = 0 for every v∈V then u is called *isolated node*. Two nodes that are joined by a path are said to be *connected*. The relation connected is a reflexive , symmetric and transitive. The equivalence classes of nodes under this relation are the *connected components* of the given fuzzy graph. A fuzzy graph G=(σ,μ) is *fuzzy bipartite* if it has a spanning fuzzy sub graph H= (τ,π) which is bipartite where for all edges (u,v) not in H, weight of (u,v) in G is strictly less than the strength of pair (u ,v) in H. (i.e) μ(u,v) $<\ \pi^\infty$(u,v). A fuzzy bipartite graph G with fuzzy bipartition (V_1,V_2) is said to be a *complete fuzzy bipartite* if for each node of V_1, every node of V_2 is a strong neighbour. Let G= (σ,μ) be a fuzzy graph and u be a node in G then there exist a node v such that (u,v) is a strong arc then we say that *u dominates v*. Let G=(σ,μ) be a fuzzy graph. A set D of V is said to be *fuzzy dominating set* of G if every v ∈ V-D there exist u ∈ D such that u dominates v. A fuzzy dominating set D of G is called a *minimal fuzzy dominating set* of G if no proper subset of D is a fuzzy dominating set. The *fuzzy domination number* γ_f(G) of the fuzzy graph G is the smallest number of nodes in any fuzzy dominating set of G. A fuzzy dominating set D of a fuzzy graph G such that $|D|$ = γ_f(G) is called minimum fuzzy dominating set. If G is a disconnected fuzzy graph with components $G_1, G_2, \ldots G_k$ then stγ_f(G) = min { stγ_f(G_1), stγ_f(G_2),..... stγ_f(G_k)}.

3. FUZZY DOMINATING CRITICAL NODES

Definition 3:

Let G=(σ, μ) be a fuzzy graph. A node v of G is said to be *fuzzy dominating critical node* if its removal either increases (or) decreases the fuzzy domination number.

We partition the nodes of G into three disjoint sets according to how their removal affects γ_f (G). Let V= $V_f^0 \cup V_f^+ \cup V_f^-$ for

$$V_f^0 = \{ v \in V : \gamma_f (G-v) = \gamma_f (G)\}$$

$$V_f^+ = \{ v \in V : \gamma_f (G-v) > \gamma_f (G)\}$$

$$V_f^- = \{ v \in V : \gamma_f (G-v) < \gamma_f (G)\}$$

Definition 3.2:

The *domination stability (or)* $\gamma_f -$ *stability* of a fuzzy graph is the minimum number of nodes whose removal changes the fuzzy domination number.

γ_f^+- *Stability* of a fuzzy graph G denoted by ${\gamma_f}^+$ (G) is defined as the minimum number of nodes whose removal increases γ_f (G).

γ_f^- - *Stability* of a fuzzy graph G denoted by γ_f^- (G) is defined as the minimum number of nodes whose removal decreases γ_f (G).

We denote the $\gamma_f -$ stability of G by stγ_f(G). Thus the domination stability of a fuzzy graph G is stγ_f(G) = min {γ_f^- (G) , ${\gamma_f}^+$ (G)}.

Proposition 3.3: [4]

A fuzzy graph G has a domination critical node if and only if ρ(G) = $\emptyset$. Here ρ(G) = min{$|\mathrm{epn}(v, S)|$: v∈S, S is a γ_f (G) – set}.

Proposition 3.4:

For every fuzzy graph G we have stγ_f(G) ≤ δ_S(G) +1.

Observation 3.5:

If G is a fuzzy star then stγ_f(G) = 1.

Observation 3.6:

For complete fuzzy bipartite graphs $K_{m,n}$ with 2≤ $m \leq n$ we have stγ_f($K_{m,n}$) = m-1.

Observation 3.7:

γ_f (P_n) = γ_f (C_n) = $\lfloor (n+2)/3 \rfloor$.

4. γ_f - STABILITY OF FUZZY PATHS :

Proposition 4.1:

For fuzzy paths P_n we have stγ_f(P_n) = 2 if n≡2 (mod3) and stγ_f(P_n) =1 otherwise.

Proof:

First assume that n ≡ 0(mod3). Let us observe that γ_f(P_n-v) = γ_f (P_n) +1, where v is a node adjacent to fuzzy end node. Consequently, stγ_f(P_n) = 1.Next assume that n≡1 (mod3). If v is a fuzzy end node then γ_f(P_n-v) = γ_f (P_n) -1, and consequently stγ_f(P_n) = 1. Now assume that n=3k+2 for some integer k. Using observation 3.7 we get γ_f(P_n) = k+1.Let v be an arbitrary node of P_n. We show that the removal of v does not change the fuzzy domination number. If v is a fuzzy end node then γ_f(P_n-v) = γ_f (P_{n-1}) = k+1 = γ_f (P_n). Now assume that the strong degree of node v is 2. Let Pn_1 and Pn_2 be the components of P_n-v.

Without loss of generality we may assume that either n_1 ≡ 0(mod3) and n_2 ≡ (mod3), or n_1 ≡ 2(mod3) and n_2 ≡ 2(mod3). In the first case we get γ_f(P_n-v) = γ_f (Pn_1) + γ_f (Pn_2) = $\lfloor (n_1+2)/3 \rfloor$ + $\lfloor (n_2+2)/3 \rfloor$ = n_1/3 + (n_2+2)/3 = k+1 = γ_f (P_n). In the second case we similarly obtain γ_f(P_n-v) = γ_f (P_n). We conclude that stγ_f(P_n) ≥ 2. Now proposition 3.4 implies that stγ_f(P_n) = 2.

5. γ_f – STABILITY OF FUZZY CYCLES :

Proposition 5.1:

We have stγ_f(C_n) = i if n ≡ i (mod3) for i = 1,2, while stγ_f(C_n) = 3 if n ≡ 0(mod3).

Proof:

First assume that n = 3k+1 for some integer k. Then for any node v we have γ_f(C_n-v) = γ_f (P_{n-1}) = k = γ_f(C_n) -1, and thus stγ_f(C_n) = 1. Now assume that n = 3k+2. For any node v we have γ_f(C_n-v) = γ_f (P_{n-1}) = k+1 = γ_f(C_n). Thus stγ_f(C_n) ≥ 2.

Now γ_f(C_n-u-v) = γ_f (P_{n-2}) = k = γ_f(C_n) -1, where u and v are two adjacent nodes. This implies that stγ_f(C_n)=2. Finally assume that n=3k. It is easy to observe that the removal of any node does not change the fuzzy domination number. Since C_n – v = P_{n-1} and by proposition 4.1 we have stγ_f(P_{n-1}) = 2, we conclude that stγ_f(C_n) = 3.

6. BOUNDS AND CHARACTERIZATIONS FOR THE DOMINATION STABILITY OF A FUZZY GRAPH:

Proposition 6.1:

If G is a fuzzy graph of order n, then stγ_f(G) ≤ n with equality if and only if G = K_n.

Proof:

The bound for an arbitrary fuzzy graph is obvious. Clearly, stγ_f(K_n) = n. Now if G ≠ K_n, then δ_S(G) <n-1 and using proposition 2 we get stγ_f(G) < 2.

Theorem 6.2:

For any fuzzy graph G with γ_f (G) ≥ 2 we have stγ_f(G) ≤ $\lfloor n/ \gamma_f$ (G)$\rfloor$ and this bound is sharp.

Proof:

Let D be a γ_f (G) – set, and let x be a node of D with minimum number of fuzzy private neighbours in V(G)\ D . Then the removal of x and its private neighbours in V(G)\ D decreases the fuzzy domination number, which implies that stγ_f(G) ≤ $\lfloor n/ \gamma_f$ (G)$\rfloor$. To see the sharpness, consider a cycle C_n, where n≡2 (mod3).

Proposition 6.3:

For every connected fuzzy graph $G \neq K_1$ we have γ_f (G) $\leq$ n/2 with equality if and only if G is the cycle C_4.

Proposition 6.4:

If $\gamma \geq 2$ and $1 \leq k \leq 2\gamma-2$, then there is no connected fuzzy graph G of order n with γ_f (G) = γ and stγ_f(G) = n-k.

Proof:

Suppose that G is a connected fuzzy graph with γ_f (G) ≥ 2 and stγ_f(G) = n-k. Using proposition 6.3 we get n > k n /2(γ_f (G)-1) ≥ k γ_f (G)/ (γ_f (G)-1). This implies that n-k = stγ_f(G) > n / γ_f (G),contracting theorem 6.2.

Proposition 6.5:

There is no fuzzy graph G of order n with stγ_f(G) = n-1.

Proof:

Suppose that G is a fuzzy graph of order n with stγ_f(G) = n-1. Proposition 6.4 implies that γ_f (G) = 1. Let x be a universal node of G. From stγ_f(G) = n-1 we obtain that there is no pair of non-adjacent nodes in N(x). Consequently, $G=K_n$. But then stγ_f(G) = n, a contradiction.

Proposition 6.6:

For every integers n and k such that $1 \leq k \leq n$ and $k \neq n-1$ there exists a fuzzy graph G of order n with stγ_f(G) = k.

Proof:

We construct a fuzzy graph $G_{k,n}$ from a complete fuzzy graph K_k with nodes $v_1,v_2,...v_k$ by adding n-k new nodes $u_1,u_2,....u_{n-k}$ together with new arcs v_iu_j, $1 \leq i \leq k$, $1 \leq j \leq n-k$. It can be easily verified that the removal of any subset of k-1 nodes of $G_{k,n}$ does not change the fuzzy domination number. On the other hand, γ_f ($G_{k,n} - v_1-v_2-...-v_k$) = n-k $\neq$ 1.Thus stγ_f($G_{k,n}$) = k.

Proposition 6.7:

For a fuzzy graph G we have stγ_f(G) = n-2 if and only if G =$G_{n-2,n}$.

Proof:

It was already seen that stγ_f($G_{n-2,n}$) = n-2. Now let G be a fuzzy graph of order n with stγ_f(G) = n-2. Theorem 7 implies that γ_f (G) = 1. Clearly, G ≠ K_n. Let y and z be two non-adjacent nodes of G. If there is a node x∈ V(G)\ {y,z} such that x∉ N(y)∩ N(z), then the removal of x, y and z yields that stγ_f(G) < n-2, a contradiction. Thus every node of V(G)\ {y,z} is adjacent to both y and z. If there are two non-adjacent nodes x_1 and x_2 in V(G)\ {y,z}, then similarly we obtain stγ_f(G) < n-2, which is a contradiction. Thus the graph induced by V(G)\ {y,z} is complete. Consequently, G =$G_{n-2,n}$.

7. UPPER BOUND ON THE DOMINATION STABILITY OF A FUZZY GRAPH WITH DOMINATION NUMBER ATLEAST TWO:

Proposition 7.1:

For any fuzzy graph G with γ_f (G) ≥ 2 we have stγ_f(G) ≤ min {δ_S(G)+1,n- δ_S(G)-1}.

Proof:

Let D be a γ_f (G) – set. If there is a node x ∈ D such that epn (x, D) = ∅, then x is a domination critical node and stγ_f(G) = 1. Thus assume that for every x ∈ D we have epn(x,D) ≠ ∅.

Let v∈D, and let A be the set of private neighbours of v in V(G)\D. Then γ_f (G[A∪{v}) = 1 < γ_f (G).Using proposition 2 we get stγ_f(G) ≤ min {δ_S (G)+1,n- δ_S(G)-1}.

Observation 7.2:

If G is a fuzzy graph with γ_f (G) = 1 or γ_f ($\bar{G}$) = 1,then stγ_f(G) + stγ_f($\bar{G}$) ≤ n+1, and this bound is sharp.

Theorem 7.3:

If G is a fuzzy graph with γ_f (G) ≥ 2 and γ_f ($\bar{G}$) ≥ 2, then stγ_f(G) + stγ_f($\bar{G}$) ≤ n-1, and this bound is sharp.

Proof:

If n is odd, then using theorem 6.2 we get stγ_f(G) + st$\gamma_f(\bar{G})$ ≤ (n-1)/2 + (n-1)/2 = n-1. Thus assume that n is even. Using proposition 7.1 we get stγ_f(G) + st$\gamma_f(\bar{G})$ ≤ min {δ_S(G)+1,n-δ_S (G)-1}+ min {δ_S ($\bar{G}$)+1,n- $\delta_S(\bar{G})$-1} ≤ n/2 + n/2 = n. Suppose now that stγ_f(G) + st$\gamma_f(\bar{G})$ = n. This implies that δ_S(G) = $\delta_S(\bar{G})$ = n/2-1, and consequently, Δ_S (G) = $\Delta_S(\bar{G})$ = n/2. If x is a node of maximum strong degree in G, then $|V(G)\setminus N[x]|$ n/2-1. Clearly, γ_f (G[N[x]]) = 1 < γ_f (G). This implies that stγ_f(G) ≤ $|V(G)\setminus N[x]|$ = n/2-1. Similarly we get st$\gamma_f(\bar{G})$ ≤ n/2-1. Consequently, stγ_f(G) + st$\gamma_f(\bar{G})$ ≤ n-2. To see the sharpness, consider a cycle C_5.

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