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Hyperbolic valued measures and Fundamental law of probability

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Abstract

In this paper, we introduce the concept of hyperbolic valued measure and discuss some of its basic properties. We develop the hyperbolic analogue of some fundamental theorems of measure theory. We also define distribution function of a $\mathbb{D}$-random variable and studied conditional expectation when conditioning is based on random variable.Finally we develop the hyperbolic analogue of Fundamental Law of Probability which plays a basic role in the field of probability theory.

AMS subject classification:
Keywords: $\mathbb{D}$-probabilistic space, $\mathbb{D}$-random variable,$\mathbb{D}$-measure,conditional expectation, joint distribution function.

1. Introduction

The work on hyperbolic numbers and bicomplex numbers are being done for quite a long time.The details and applications of these numbers can be seen in [7] and [10].

The hyperbolic numbers in the mathematical literature has been called with different names: split-complex numbers, double numbers, perplex numbers and duplex numbers. The set $\mathbb{D}$ of hyperbolic numbers has a partial order relation $\preceq$ which is one of the interesting property of the set $\mathbb{D}$ see [3]. We can think of measurable functions taking values in the set of hyperbolic numbers being motivated by the work of D. Alpay, M.E. Luna-Elizarraras and M. Shapiro [3].

In this paper, we study $\mathbb{D}$- valued measure, $\mathbb{D}$-valued measurable functions, Monotone Convergence theorem, Fatou's lemma, Lebesgue Dominated Convergence theorem and Fundamental Law of Probability.

We start with a measurable space (Ω, Σ), where Ω is any set and Σ is the σ-algebra of subsets of Ω. A $\mathbb{D}$- valued measure $\mu_{\mathbb{D}}$ is a non negative extended hyperbolic valued set function on Σ which is countably additive with $\mu_{\mathbb{D}}(\phi) = 0$. The joint distribution function of a $\mathbb{D}$-random variables $X_{\mathbb{D}}$ and $Y_{\mathbb{D}}$, $F_{X_{\mathbb{D}}Y_{\mathbb{D}}}$ is a $\mathbb{D}$-valued function on $\mathbb{D}^2$ defined as $F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \beta) = P_{\mathbb{D}}(X_{\mathbb{D}} \preceq \alpha, Y_{\mathbb{D}} \preceq \beta)$. We introduce the conditional probability density function of $X_{\mathbb{D}}$ given an event $Y_{\mathbb{D}} = b$. The values taken by the conditioning event which are zero divisors are of special interest in this work. This work can be seen as a continuation of work of Daniel Alpay et.al. [3].It seems that the whole probability theory as well as measure theory can be generalized in this direction.This work may have important applications in mathematical statistics, thermodynamics and statistical physics see [3].

2. A review of hyperbolic numbers

The ring of hyperbolic numbers is the commutative ring $\mathbb{D}$ defined as

$$\mathbb{D} = \left\{a + bk \mid k^2 = 1, k \notin \mathbb{R}; a, b \in \mathbb{R}\right\}.$$

The †-conjugation of a hyperbolic number $z = a + bk$ is given by

$$z^\dagger = a - bk$$

which is additive, involutive and multiplicative operation on $\mathbb{D}$. Note that given $z = a + bk \in \mathbb{D}$, then $zz^\dagger = a^2 - b^2 \in \mathbb{R}$, from which it follows that any hyperbolic number z with $zz^\dagger \neq 0$ is invertible, and its inverse is given by

$$z^{-1} = \frac{z^\dagger}{zz^\dagger}.$$

If, on the other hand, $z \neq 0$ but $zz^\dagger = a^2 - b^2 = 0$, then z is a zero divisor. In fact there are no other zero divisors. Thus the set of zero divisors, denoted by $\mathfrak{S}_{\mathbb{D}}$, is

$$\begin{aligned}\mathfrak{S}_{\mathbb{D}} &= \left\{z = a + bk \mid z \neq 0, \quad zz^\dagger = a^2 - b^2 = 0\right\}\\ &= \{z = a(1 \pm k) \mid a \neq 0 \in \mathbb{R}\}.\end{aligned}$$

There are two very special zero divisors in $\mathbb{D}$ which are

$$e = \frac{1+k}{2}$$

and its †-conjugation

$$e^\dagger = \frac{1-k}{2}.$$

These are mutually complementary idempotent elements in $\mathbb{D}$. The two sets $\mathbb{D}_e = e\,\mathbb{D}$ and $\mathbb{D}_{e^\dagger} = e^\dagger\,\mathbb{D}$ are principal ideals in the ring $\mathbb{D}$ such that $\mathbb{D}_e \cap \mathbb{D}_{e^\dagger} = \{0\}$ and

$\mathbb{D} = \mathbb{D}_e + \mathbb{D}_{e^\dagger}$, which is idempotent decomposition of $\mathbb{D}$. Every hyperbolic number $z = a + bk$ can be written as

$$z = a + bk = (a + b)e + (a - b)e^\dagger = v_1 e + v_2 e^\dagger,$$

which is the idempotent decomposition of a hyperbolic number. The algebraic operations of addition, multiplication, taking of inverse, etc. can be performed component-wise. Observe that the sets $\mathbb{D}_e$ and $\mathbb{D}_{e^\dagger}$ can be written as $\mathbb{D}_e = \{r\ e \mid r \in \mathbb{R}\} = \mathbb{R}\ e$ and $\mathbb{D}_{e^\dagger} = \left\{t\ e^\dagger \mid t \in \mathbb{R}\right\} = \mathbb{R}\ e^\dagger$. The set $\mathbb{D}$ of hyperbolic numbers is a vector space over the field $\mathbb{R}$ of real numbers with basis $\left\{e,\ e^\dagger\right\}$ which is isomorphic to the linear space of complex numbers over the field $\mathbb{R}$ of real numbers. The set of non negative hyperbolic numbers is

$$\mathbb{D}^+ = \left\{v_1\ e + v_2\ e^\dagger \mid v_1, v_2 \geq 0\right\}.$$

We will need two more sets

$$\mathbb{D}^+_e = \mathbb{D}_e \cap \mathbb{D}^+ - \{0\}$$

and

$$\mathbb{D}^+_{e^\dagger} = \mathbb{D}_{e^\dagger} \cap \mathbb{D}^+ - \{0\}.$$

Given $z_1, z_2 \in \mathbb{D}$, we write $z_1 \preceq z_2$ *whenever* $z_2 - z_1 \in \mathbb{D}^+$.This relation is a partial order relation in $\mathbb{D}$ which is an extension of total order relation $\leq$ on $\mathbb{R}$. Given any hyperbolic number α, one can see that the entire hyperbolic plane is divided into four quarters: the quarter plane of hyperbolic numbers which are $\mathbb{D}$- less than or equal to α; the quarter plane of hyperbolic numbers which are $\mathbb{D}$- greater than α; and the two quarter planes where the hyperbolic numbers are not $\mathbb{D}$- comparable with α. Let us denote by A_α, the set of all hyperbolic numbers which are not $\mathbb{D}$- comparable with α. Then

$$\mathbb{D} = \{z \in \mathbb{D} \mid z \preceq \alpha\} \cup \{z \in \mathbb{D} \mid z \succ \alpha\} \cup A_\alpha.$$

The hyperbolic valued modulus on $\mathbb{D}$ is defined by

$$|z|_k = |v_1 e + v_2 e^\dagger|_k = |v_1| e + |v_2| e^\dagger \in \mathbb{D}^+,$$

where $|v_1|, |v_2|$ denote the usual modulus of real numbers.The set $\mathbb{D}$ forms a normed linear space with respect to the hyperbolic valued norm (modulus). We say that a subset $A \subset \mathbb{D}$ is a $\mathbb{D}$-bounded set if there exists M $\in \mathbb{D}^+$ such that $|z|_k \preceq M, \forall z \in A$.
Let

$$A_1 = \left\{x \in \mathbb{R} \mid \exists y \in \mathbb{R}, xe + ye^\dagger \in A\right\},$$

and

$$A_2 = \left\{y \in \mathbb{R} \mid \exists x \in \mathbb{R}, xe + ye^\dagger \in A\right\}$$

If A is a bounded set, then both A_1 and A_2 are bounded and

$$Sup_{\mathbb{D}} A = e\ Sup A_1 + e^\dagger\ Sup A_2.$$

For details on hyperbolic numbers, we refer to [2], [7] and [10].

Definition 2.1. [3, page 5] Let (Ω, Σ) be a measurable space. A function

$$P_{\mathbb{D}} : \Sigma \longrightarrow \mathbb{D}$$

is called a $\mathbb{D}$-valued probabilistic measure or a $\mathbb{D}$-valued probability if

(i) $P_{\mathbb{D}}(A) \succeq 0$;

(ii) $P_{\mathbb{D}}(\Omega) = p$, where p takes one of the three possible values 1, e or $e^{\dagger}$;

(iii) given a sequence $\{A_n\} \subset \Sigma$ of pairwise disjoint events,

$$P_{\mathbb{D}}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P_{\mathbb{D}}(A_n).$$

The triplet $(\Omega, \Sigma, P_{\mathbb{D}})$ is called a $\mathbb{D}$-probabilistic space. A $\mathbb{D}$-random variable is a function

$$X_{\mathbb{D}} : \Omega \longrightarrow \mathbb{D}$$

such that

$$X_{\mathbb{D}}^{-1}(D) \in \Sigma$$

for every open set D in $\mathbb{D}$. The expectation of $X_{\mathbb{D}}$ is defined as

$$E(X_{\mathbb{D}}) = \int_{\Omega} X_{\mathbb{D}}\, dP_{\mathbb{D}},$$

provided the integral exists. Let $X_{\mathbb{D}}$ be a $\mathbb{D}$-random variable on a $\mathbb{D}$-probabilistic space $(\Omega, \Sigma, P_{\mathbb{D}})$. If an event B, with $P_{\mathbb{D}}(B) \succ 0$ has taken place, then the conditional expectation of $X_{\mathbb{D}}$ given B, denoted by $E_B(X_{\mathbb{D}})$ or $E(X_{\mathbb{D}}/B)$ is defined as:

(i) $E_B(X_{\mathbb{D}}) = \dfrac{1}{P_{\mathbb{D}}(B)} \displaystyle\int_B X_{\mathbb{D}}(w) dP_{\mathbb{D}}(w)$ if $P_{\mathbb{D}}(B) \succ 0$ and $P_{\mathbb{D}}(B) \notin \mathfrak{S}_{\mathbb{D}}$;

(ii) $E_B(X_{\mathbb{D}}) = E(X_{\mathbb{D}})$ if $P_{\mathbb{D}}(B) = 0$;

(iii) $E_B(X_{\mathbb{D}}) = \displaystyle\int_B \frac{X_{\mathbb{D}}(w) dP_{\mathbb{D}}(w)}{\lambda_1} e + E(X_{\mathbb{D}}) e^{\dagger}$ if $P_{\mathbb{D}}(B) = \lambda_1 e, \lambda_1 > 0$;

(iv) $E_B(X_{\mathbb{D}}) = E(X_{\mathbb{D}}) e + \displaystyle\int_B \frac{X_{\mathbb{D}}(w) dP_{\mathbb{D}}(w)}{\lambda_2} e^{\dagger}$ if $P_{\mathbb{D}}(B) = \lambda_2 e^{\dagger}, \lambda_2 > 0$.

3. Hyperbolic valued measure

The work of this section is essentially based on the book of Walter Rudin [13]. Let us define the set of extended hyperbolic numbers $\bar{\mathbb{D}}$ as $\bar{\mathbb{D}} = \left\{z = \alpha e + \beta e^{\dagger} | \alpha, \beta \in \bar{\mathbb{R}}\right\}$, and the set of non negative extended hyperbolic numbers $\bar{\mathbb{D}}^{+} = \left\{z = \alpha e + \beta e^{\dagger} | \alpha, \beta \in \bar{\mathbb{R}}^{+}\right\}$, where $\bar{\mathbb{R}}$ is the set of extended real numbers and $\bar{\mathbb{R}}^{+}$ is the set of non negative extended real numbers.If $z_1, z_2 \in \bar{\mathbb{D}}$, then $z_1 + z_2,\ z_1 z_2$ and $0z_1$ may be undefined unless $z_1, z_2 \in \mathbb{D}$.

Definition 3.1. Let Ω denotes a non-empty set and Σ be the sigma algebra of subsets of Ω. Then by a hyperbolic valued measure or $\mathbb{D}$-measure on Σ, we mean a set function $\mu_{\mathbb{D}} : \Sigma \to \overline{\mathbb{D}}^{+}$ such that

(i) $\mu_{\mathbb{D}}$ is non-negative extended hyperbolic valued set function.

(ii) $\mu_{\mathbb{D}}(\phi) = 0$ and

(iii) $\mu_{\mathbb{D}}$ is countably additive, i.e., for any disjoint subcollection $\{E_n : n \in \mathbb{N}\} \subset \Sigma$,

$$\mu_{\mathbb{D}}\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu_{\mathbb{D}}\left(E_n\right).$$

The triplet $(\Omega, \Sigma, \mu_{\mathbb{D}})$ is called a $\mathbb{D}$-measure space.

Remark 3.2. Every $\mathbb{D}$-measure can be written as

$$\mu_{\mathbb{D}}(E) = \psi_1(E) + k\psi_2(E) = \mu_1(E)e + \mu_2(E)e^{\dagger}$$

with

$$\mu_1(E) = \psi_1(E) + \psi_2(E)$$

and

$$\mu_2(E) = \psi_1(E) - \psi_2(E)$$

for every $E \in \Sigma$. Now property (i) of $\mu_{\mathbb{D}}$ implies that

$$\mu_1(E) \geq 0 \text{ and } \mu_2(E) \geq 0 \text{ for every } E \in \Sigma.$$

Further property (iii) of $\mu_{\mathbb{D}}$ implies that

$$\mu_i\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu_i(E_n)$$

for i= 1 and 2. Then to define $\mathbb{D}$-measure is equivalent to consider on the same measurable space, a pair of usual real valued measures.

Example 3.3. Let $\Omega = e\ \mathbb{N} + e^{\dagger}\ \mathbb{N}$ and $\Sigma = P(\Omega)$. Define measures μ_1 and μ_2 on measurable space (Ω, Σ) as $\mu_1(E) = |E|$, the cardinality of E and $\mu_2(E) = 1$

whenever $x_0 \in E$ and zero otherwise.Then $\mu_{\mathbb{D}} : \Omega \to \bar{\mathbb{D}}^+$ given by $\mu_{\mathbb{D}}(E) = e\ \mu_1(E) + e^{\dagger}\ \mu_2(E), \forall E \in \Sigma$ is a $\mathbb{D}$-valued measure on Ω, where $x_0 \in \Omega$ is arbitrary but fixed. The measure space $(\Omega, \Sigma, \mu_{\mathbb{D}})$ is a σ-finite measure space.

Remark 3.4. A $\mathbb{D}$ valued measure satisfies the following properties:

(i) If $A_1, A_2, \dots, A_n$ are pairwise disjoint members of Σ, then $\mu_{\mathbb{D}}\left(\bigcup_{i=1}^{n} A_i\right) = \Sigma_{i=1}^{n}\mu_{\mathbb{D}}(A_i)$ (Finite Additivity).

(ii) If $A, B \in \Sigma$ with $A \subset B$, then $\mu_{\mathbb{D}}(A) \preceq \mu_{\mathbb{D}}(B)$(Monotonicity)

(iii) If $A, B \in \Sigma$ with$A \subset B$, and $\mu_{\mathbb{D}}(A) \in \mathbb{D}$, then $\mu_{\mathbb{D}}(B \setminus A) = \mu_{\mathbb{D}}(B) - \mu_{\mathbb{D}}(A)$.

Proof. $A \subset B$ implies that$B = A \cup (B \setminus A)$, where$A \cap (B \setminus A) = \phi$. Therefore, $\mu_{\mathbb{D}}(B) = \mu_{\mathbb{D}}(A) + \mu_{\mathbb{D}}(B \setminus A)$. This implies that $\mu_{\mathbb{D}}(B \setminus A) = \mu_{\mathbb{D}}(B) - \mu_{\mathbb{D}}(A)$. ∎

(iv) If $A_1, A_2, \dots \in \Sigma$ with $A_1 \subset A_2 \subset A_3 \subset \dots$ and $A = \bigcup_{n=1}^{\infty} A_n$, then $\lim_{n\to\infty} \mu_{\mathbb{D}}(A_n) = \mu_{\mathbb{D}}(A)$.

Proof. Take $B_1 = A_1$ and put $B_n = A_n \setminus A_{n-1}$ for $n = 2, 3, 4, \dots$. Then $B_n \in \Sigma$, $B_i \cap B_j = \phi$ if $i \neq j$, $A_n = \bigcap_{i=1}^{n} B_i$ and $A = \bigcap_{i=1}^{n} B_i$. Therefore

$$\mu_{\mathbb{D}}(A_n) = \Sigma_{i=1}^{n}\mu_{\mathbb{D}}(B_i) \text{ and } \mu_{\mathbb{D}}(A) = \Sigma_{n=1}^{\infty}\mu_{\mathbb{D}}(B_i).$$

Hence

$$\lim_{n\to\infty} \mu_{\mathbb{D}}(A_n) = \Sigma_{n=1}^{\infty}\mu_{\mathbb{D}}(B_i) = \mu_{\mathbb{D}}(A).$$

∎

(v) If $A_1, A_2, \dots \in \Sigma$ with $A_1 \supset A_2 \supset A_3 \supset \dots$ and $A = \bigcap_{n=1}^{\infty} A_n$, then $\lim_{n\to\infty} \mu_{\mathbb{D}}(A_n) = \mu_{\mathbb{D}}(A)$, where $\mu_{\mathbb{D}}(A_1) \in \mathbb{D}$.

Proof. Take $C_n = A_1 \setminus A_n$. Then $C_1 \subset C_2 \subset C_3 \subset \dots$, $\mu_{\mathbb{D}}(C_n) = \mu_{\mathbb{D}}(A_1) - \mu_{\mathbb{D}}(A_n)$, $A_1 \setminus A = \bigcap_{n=1}^{\infty} C_n$ and so from (iv), we have

$$\mu_{\mathbb{D}}(A_1 \setminus A) = \mu_{\mathbb{D}}(A_1) - \mu_{\mathbb{D}}(A) = \lim_{n\to\infty} \mu_{\mathbb{D}}(C_n) = \mu_{\mathbb{D}}(A_1) - \lim_{n\to\infty} \mu_{\mathbb{D}}(A_n).$$

Hence

$$\lim_{n\to\infty} \mu_{\mathbb{D}}(A_n) = \mu_{\mathbb{D}}(A).$$

∎

Definition 3.5. A $\mathbb{D}$-valued measure $\mu_{\mathbb{D}}$ on the sigma algebra Σ of subsets of Ω is said to be a $\mathbb{D}$-finite measure if $\mu_{\mathbb{D}}(\Omega) \in \mathbb{D}^+$ but $\mu_{\mathbb{D}}(\Omega) \notin \overline{\mathbb{D}}^+$. In this case we say that $(\Omega, \Sigma, \mu_{\mathbb{D}})$ is a $\mathbb{D}$-finite measure space.

Definition 3.6. A $\mathbb{D}$-valued measure $\mu_{\mathbb{D}}$ on the sigma algebra Σ of subsets of Ω is said to be a σ-finite measure if there exists a sequence $\{E_n\}_{n=1}^{\infty}$ in Σ such that $\bigcup\limits_{n\in\mathbb{N}} E_n = \Omega$ and $\mu_{\mathbb{D}}(E_n) \in \mathbb{D}^+$ but $\mu_{\mathbb{D}}(E_n) \notin \overline{\mathbb{D}}^+$ for every $n \in \mathbb{N}$.

Definition 3.7. A sigma algebra Σ of subsets of Ω is said to be complete with respect to the $\mathbb{D}$-valued measure $\mu_{\mathbb{D}}$, if for every $A_o \subset A$ with $\mu_{\mathbb{D}}(A) = 0$ implies that $A_0 \in \Sigma$ and we say that $(\Omega, \Sigma, \mu_{\mathbb{D}})$ is a $\mathbb{D}$-complete measure space.

Definition 3.8. Let (Ω, Σ) be a measurable space. A function $f : \Omega \to \mathbb{D}$ such that $f^{-1}(D) \in \Sigma$ for every open set $D \in \mathbb{D}$ is called a $\mathbb{D}$-measurable function.

Remark 3.9. Every $\mathbb{D}$-measurable function f can be written as $f = ef_1 + e^{\dagger} f_2$, where f_1 and f_2 are real valued measurable functions on Ω.

It can be easily checked by using idempotent decomposition of a $\mathbb{D}$-measurable function that sums, products and linear combinations of $\mathbb{D}$-measurable functions is $\mathbb{D}$-measurable function.

Theorem 3.10. If $\{f_n\}$ is a sequence of $\mathbb{D}$-measurable functions on a measurable space (Ω, Σ), then $\sup\limits_{n\geq 1} f_n$, $\inf_{n\geq 1} f_n$, $\limsup\limits_{n\geq 1} f_n$, $\liminf\limits_{n\geq 1} f_n$ and $\lim\limits_{n\to\infty} f_n$ are $\mathbb{D}$-measurable.

Proof. Let $f_n = ef_n^1 + e^{\dagger} f_n^2, \forall n \in \mathbb{N}$. Then $\left\{f_n^1\right\}$ and $\left\{f_n^2\right\}$ are sequences of real valued measurable functions and $\sup\limits_{n\geq 1} f_n = e \sup\limits_{n\geq 1} f_n^1 + e^{\dagger} \sup\limits_{n\geq 1} f_n^2$. Since $\sup\limits_{n\geq 1} f_n^1$ and $\sup\limits_{n\geq 1} f_n^2$ are real valued measurable so $\sup\limits_{n\geq 1} f_n$ is $\mathbb{D}$- measurable.Using similar technique, we can show that $\inf_{n\geq 1} f_n$, $\limsup\limits_{n\geq 1} f_n$, $\liminf\limits_{n\geq 1} f_n$ and $\lim\limits_{n\to\infty} f_n$ are $\mathbb{D}$-measurable. ■

If f_1 and f_2 are Lebesgue integrable measurable functions with respect to the real valued measures μ_1 and μ_2 respectively. Then we define the integral of a $\mathbb{D}$-measurable function $f = ef_1 + e^{\dagger} f_2$ over a subset E of Σ as
$\int_E f \; d\mu_{\mathbb{D}} = e \int_E f_1 \; d\mu_1 + e^{\dagger} \int_E f_2 \; d\mu_2$ which exists in $\mathbb{D}$ since $\int_E f_1 \; d\mu_1$ and $\int_E f_2 \; d\mu_2$ both exist in $\mathbb{R}$. The integral defined in this manner is linear.The set of all $\mathbb{D}$-Lebesgue integrable measurable functions on Ω is denoted by $L^1(\mu_{\mathbb{D}})$. That is,

$$L^1(\mu_{\mathbb{D}}) = \left\{ f : \Omega \to \mathbb{D} | \; f \text{ is } \mathbb{D} - \text{measurable and} \int_{\Omega} |f|_k \; d\mu_{\mathbb{D}} \in \mathbb{D} \right\}$$

and it forms a linear space under the operations of pointwise addition and scalar multiplication.

Theorem 3.11. If $f \in L^1(\mu_{\mathbb{D}})$, then $\left| \int_\Omega f \, d\mu_{\mathbb{D}} \right|_k \preceq \int_\Omega |f|_k \, d\mu_{\mathbb{D}}$.

Proof. We have $\int_\Omega f \, d\mu_{\mathbb{D}} = e \int_\Omega f_1 \, d\mu_1 + e^\dagger \int_\Omega f_2 \, d\mu_2$. Then

$$\begin{aligned} \left| \int_\Omega f \, d\mu_{\mathbb{D}} \right|_k &= e \left| \int_\Omega f_1 \, d\mu_1 \right| + e^\dagger \left| \int_\Omega f_2 \, d\mu_2 \right| \preceq e \int_\Omega |f_1| \, d\mu_1 + e^\dagger \int_\Omega |f_2| \, d\mu_2 \\ &= \int_\Omega (e|f_1| + e^\dagger |f_2|) \, d\mu_{\mathbb{D}} \\ &= \int_\Omega |f|_k \, d\mu_{\mathbb{D}}. \end{aligned}$$

■

Theorem 3.12. Let $(\Omega, \Sigma, \mu_{\mathbb{D}})$, be a $\mathbb{D}$-measure space.Then we have

(i) If $0 \preceq f \preceq g$, then $\int_E f \, d\mu_{\mathbb{D}} \preceq \int_E f \, d\mu_{\mathbb{D}}$.

(ii) If $A \subset B$ and $f \succeq 0$, then $\int_A f \, d\mu_{\mathbb{D}} \preceq \int_B f \, d\mu_{\mathbb{D}}$.

(iii) If $f \succeq 0$ and $c \in \mathbb{D}^+$, then $\int_E cf \, d\mu_{\mathbb{D}} = c \int_E f \, d\mu_{\mathbb{D}}$.

(iv) If $f(x) = 0, \forall x \in E$, then $\int_E f \, d\mu_{\mathbb{D}} = 0$ even if $\mu_{\mathbb{D}}(E) \notin \mathbb{D}$.

(v) If $\mu_{\mathbb{D}}(E) = 0$, then $\int_E f \, d\mu_{\mathbb{D}} = 0$, even if $f(x) \notin \mathbb{D}$ for every $x \in E$.

(vi) If $f \succeq 0$, then $\int_E f \, d\mu_{\mathbb{D}} = \int_E \chi_E f \, d\mu_{\mathbb{D}}$. The functions and sets occurring here are assumed to be measurable.

Proof. The proofs of the above items follows by using the idempotent decomposition of a hyperbolic number. ■

Theorem 3.13. (Monotone Convergence Theorem) Given a $\mathbb{D}$-measure space $(\Omega, \Sigma, \mu_{\mathbb{D}})$. Let $\{f_n\}$ be a pointwise increasing sequence of non negative extended $\mathbb{D}$- measurable functions on Ω, and such that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in \Omega$. Then

$$\lim_{n\to\infty} \int_\Omega f_n \, d\mu_{\mathbb{D}} = \int_\Omega f \, d\mu_{\mathbb{D}}.$$

Proof. Writing $f_n(x) = ef_n^1(x) + e^\dagger f_n^2(x)$ and $f(x) = ef_1(x) + e^\dagger f_2(x)$ for every $x \in \Omega$. Then $\{f_n^i\}$ is a pointwise increasing sequence of non negative extended real valued Σ-measurable functions on Ω such that $\lim_{n\to\infty} f_n^i(x) = f_i(x)$ for each $i = 1, 2$. Therefore, by Monotone Convergence Theorem for sequence of real valued measurable functions, we have $\lim_{n\to\infty} \int_\Omega f_n^i \, d\mu_i = \int_\Omega f_i \, d\mu_i$ for each $i = 1, 2$. Thus

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega f_n \, d\mu_{\mathbb{D}} &= e \lim_{n\to\infty} \int_\Omega f_n^1 \, d\mu_1 + e^\dagger \lim_{n\to\infty} \int_\Omega f_n^2 \, d\mu_2 \\ &= e \int_\Omega f_1 \, d\mu_1 + e^\dagger \int_\Omega f_2 \, d\mu_2 \\ &= \int_\Omega f \, d\mu_{\mathbb{D}}.\end{aligned}$$

■

Theorem 3.14. If $f_n : \Omega \to \bar{\mathbb{D}}^+$ is $\mathbb{D}$-measurable, for $n = 1, 2, 3\ldots$, and $f(x) = \Sigma_{n=1}^\infty f_n(x)$ for every $x \in \Omega$, then

$$\int_\Omega f \, d\mu_{\mathbb{D}} = \Sigma_{n=1}^\infty \int_\Omega f_n \, d\mu_{\mathbb{D}}.$$

Proof. Write $\int_\Omega f \, d\mu_{\mathbb{D}} = e \int_\Omega f_1 \, d\mu_1 + e^\dagger \int_\Omega f_2 \, d\mu_2$, where $\int_\Omega f_i \, d\mu_i = \Sigma_{n=1}^\infty \int_\Omega f_n^i \, d\mu_i$ for $i = 1, 2$. Therefore,

$$\int_\Omega f \, d\mu_{\mathbb{D}} = e \; \Sigma_{n=1}^\infty \int_\Omega f_n^1 \, d\mu_1 + e^\dagger \; \Sigma_{n=1}^\infty \int_\Omega f_n^2 \, d\mu_2 = \Sigma_{n=1}^\infty \int_\Omega f_n \, d\mu_{\mathbb{D}}.$$

■

Theorem 3.15. (Fatou's Lemma) If $\{f_n\}$ is a sequence of non-negative extended $\mathbb{D}$-measurable functions on $(\Omega, \Sigma, \mu_{\mathbb{D}})$, then

$$\int_\Omega \left(\lim_{n\to\infty} \inf f_n\right) d\mu_{\mathbb{D}} \preceq \lim_{n\to\infty} \inf \int_\Omega f_n d\mu_{\mathbb{D}}.$$

Proof. Let $f_n(x) = ef_n^1(x) + e^\dagger f_n^2(x)$. Then $\lim_{n\to\infty} \inf f_n = e \lim_{n\to\infty} \inf f_n^1 + e^\dagger \lim_{n\to\infty} \inf f_n^2$, where f_n^1 and f_n^2 are real valued measurable functions so that $\int_\Omega \left(\lim_{n\to\infty} \inf f_n^i\right) d\mu_i \preceq \lim_{n\to\infty} \inf \int_\Omega f_n^i d\mu_i$ for each $i = 1, 2$. Hence

$$\int_\Omega \left(\lim_{n\to\infty} \inf f_n\right) d\mu_{\mathbb{D}} \preceq \lim_{n\to\infty} \inf \int_\Omega f_n d\mu_{\mathbb{D}}.$$

■

Theorem 3.16. (Lebesgue Dominated Convergence Theorem) Suppose $\{f_n\}$ is a sequence of $\mathbb{D}$-measurable functions on a $\mathbb{D}$-measure space $(\Omega, \Sigma, \mu_{\mathbb{D}})$, such that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in \Omega$. If there exists a $\mathbb{D}$-Lebesgue integrable measurable function g on Ω such that $|f_n(x)|_k \preceq g(x), n = 1, 2, 3, \ldots, x \in \Omega$, then f is $\mathbb{D}$-Lebesgue integrable,

$$\lim_{n\to\infty} \int_{\Omega} |f_n - f|_k \, d\mu_{\mathbb{D}} = 0 \text{ and } \lim_{n\to\infty} \int_{\Omega} f_n \, d\mu_{\mathbb{D}} = \int_{\Omega} f \, d\mu_{\mathbb{D}}.$$

Proof. Proof follows by simply decomposing the functions occuring here into their idempotent components and applying the real version of the theorem to their idempotent compotents. ■

The notion of measurability of a $\mathbb{D}$-valued function can be generalized for bicomplex valued functions. The versatility of different writings of a bicomplex number helps us to do so. A bicomplex number can be expressed as $Z = z_1 + iz_2$, where z_1 and z_2 are hyperbolic numbers and $i^2 = -1$. Thus a bicomplex function f has the representation $f = f_1 + if_2$, where f_1 and f_2 are hyperbolic functions on the same domain.

Theorem 3.17. [13, Theorem 1.7] Let $g : Y \longrightarrow Z$ be a continuous function, where Y and Z are topological spaces.If X is a measurable space, if $f : X \longrightarrow Y$ is measurable, and $h = g \circ f$, then $h : X \longrightarrow Z$ is measurable.

Theorem 3.18. [13, Theorem 1.8] Let u and v be real measurable functions on a measure space X, let ϕ be the continuous mapping of the plane into a topological space Y, and define $h(x) = \phi(u(x), v(x))$, for $x \in X$. Then $h : X \longrightarrow Y$ is measurable.

As a corollary to theorems 3.9 and 3.10,we have the following result:

Theorem 3.19. A bicomplex function $f = f_1 + if_2$, on a measurable space (Ω, Σ) is measurable if and only if f_1 and f_2 are hyperbolic measurable functions on the same measurable space Ω.

From the above theorem, it follows that it is sufficient to study only the $\mathbb{D}$-valued measurable functions.

4. Conditional expectation given a $\mathbb{D}$-random variable

The work of this section is essentially based on the book of M. M. Rao [11]. Let $X_{\mathbb{D}} = eX_1 + e^{\dagger}X_2$ and $Y_{\mathbb{D}} = eY_1 + e^{\dagger}Y_2$ be $\mathbb{D}$- random variables on $(\Omega, \Sigma, P_{\mathbb{D}})$ with joint probability density function $f_{X_{\mathbb{D}}Y_{\mathbb{D}}}$. The joint distribution function of $X_{\mathbb{D}}$ and $Y_{\mathbb{D}}$, $F_{X_{\mathbb{D}}Y_{\mathbb{D}}}$ is a $\mathbb{D}$-valued function on $\mathbb{D}^2$ defined as

$$F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \beta) = P_{\mathbb{D}}(X_{\mathbb{D}} \preceq \alpha, Y_{\mathbb{D}} \preceq \beta) = \int_A \int_B f_{X_{\mathbb{D}}Y_{\mathbb{D}}}(u, v) \, dv \, du,$$

for all $\alpha = e\alpha_1 + e^{\dagger}\alpha_2, \beta = e\beta_1 + e^{\dagger}\beta_2, u = eu_1 + e^{\dagger}u_2, v = ev_1 + e^{\dagger}v_2 \in \mathbb{D}$, where $A = \{w \in \Omega | X_{\mathbb{D}}(w) \preceq \alpha\}$ and $B = \{w \in \Omega | Y_{\mathbb{D}}(w) \preceq \beta\}$. The integral $\int_A \int_B f_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \beta)\, d\beta\, d\alpha$, is given by

$$\int_A \int_B f_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \beta)\, d\beta\, d\alpha = e \int_{-\infty}^{\alpha_1} \int_{-\infty}^{\beta_1} f_{X_1Y_1}(\alpha_1, \beta_1)\, d\beta_1\, d\alpha_1 + e^{\dagger} \int_{-\infty}^{\alpha_2} \int_{-\infty}^{\beta_2} f_{X_2Y_2}(\alpha_2, \beta_2)\, d\beta_2\, d\alpha_2.$$

Here $f_{X_1Y_1}$ and $f_{X_2Y_2}$ are joint probability density functions of X_1, Y_1 and X_2, Y_2 respectively.The joint distribution function of $X_{\mathbb{D}}, Y_{\mathbb{D}}$ can be expressed as $F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \beta) = eF_{X_1Y_1}(\alpha_1, \beta_1) + e^{\dagger}F_{X_2Y_2}(\alpha_2, \beta_2)$, where $F_{X_1Y_1}$ and $F_{X_2Y_2}$ are joint distribution functions of X_1, Y_1 and X_2, Y_2 respectively. If $F_{X_{\mathbb{D}}}, F_{Y_{\mathbb{D}}}$ are marginal distribution functions of $X_{\mathbb{D}}, Y_{\mathbb{D}}$ and $f_{X_{\mathbb{D}}}, f_{Y_{\mathbb{D}}}$ are their respective probability density functions, then

$$F_{X_{\mathbb{D}}}(\alpha) = F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \infty) = \int_A \left(\int_{\mathbb{D}} f_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \beta)\, d\beta \right) d\alpha$$

and

$$F_{X_{\mathbb{D}}}(\beta) = F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\infty, \beta) = \int_B \left(\int_{\mathbb{D}} f_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \beta)\, d\alpha \right) d\beta,$$

where A and B are the sets as defined above. The joint distribution function of two $\mathbb{D}$-random variable has the following properties:

(i) For hyperbolic numbers α, β, γ and δ with $\alpha \prec \beta$ and $\gamma \prec \delta$, we have

$$P_{\mathbb{D}}(\alpha \prec X_{\mathbb{D}} \preceq \beta, \gamma \prec Y_{\mathbb{D}} \preceq \delta) = F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\beta, \delta) - F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\beta, \gamma) - F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \delta) + F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \gamma).$$

Proof. Let $\alpha = e\alpha_1 + e^{\dagger}\alpha_2, \beta = e\beta_1 + e^{\dagger}\beta_2, \gamma = e\gamma_1 + e^{\dagger}\gamma_2, \delta = e\delta_1 + e^{\dagger}\delta_2$. Then $\alpha \prec \beta$ implies that $\alpha_i \prec \beta_i$ and $\gamma \prec \delta$ implies that $\gamma_i \prec \delta_i$ for each i=1,2. Therefore,

$$P_{\mathbb{D}}(\alpha \prec X_{\mathbb{D}} \preceq \beta, \gamma \prec Y_{\mathbb{D}} \preceq \delta) = e\ P_1(\alpha_1 \prec X_1 \preceq \beta_1, \gamma_1 \prec Y_1 \preceq \delta_1) + e^{\dagger}\ P_2(\alpha_2 \prec X_2 \preceq \beta_2, \gamma_2 \prec Y_2 \preceq \delta_2).$$

Now

$$P_i(\alpha_i \prec X_i \preceq \beta_i, \gamma_i \prec Y_i \preceq \delta_i) = F_{X_iY_i}(\beta_i, \delta_i) - F_{X_iY_i}(\beta_i, \gamma_i) - F_{X_iY_i}(\alpha_i, \delta_i) + F_{X_iY_i}(\alpha_i, \gamma_i)$$

for each i=1,2. Therefore, we have

$$P_{\mathbb{D}}(\alpha \prec X_{\mathbb{D}} \preceq \beta, \gamma \prec Y_{\mathbb{D}} \preceq \delta) = F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\beta, \delta) - F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\beta, \gamma) - F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \delta) + F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \gamma).$$

■

(ii) Let $\alpha \prec \beta, \gamma \prec \delta$, then $F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\gamma, \beta) \succeq F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \beta)$ and $F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \delta) \succeq F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \beta)$ which shows that $F_{X_{\mathbb{D}}Y_{\mathbb{D}}}$ is a monotonic non decreasing function.

(iii) $F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(-\infty, \beta) = 0 = F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, -\infty), F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(+\infty, +\infty) = 1$.

(iv) If the density function $f_{X_{\mathbb{D}}Y_{\mathbb{D}}}$ is continuous at (α, β), then

$$\frac{\partial^2 F_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \beta)}{\partial\alpha\partial\beta} = f_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, \beta).$$

Definition 4.1. The conditional probability density function of $X_{\mathbb{D}}$ given $Y_{\mathbb{D}} = b$ is defined as

$$f_{X_{\mathbb{D}}/Y_{\mathbb{D}}}(\alpha/b) = \begin{cases} \dfrac{f_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, b)}{f_{Y_{\mathbb{D}}}(b)}\ if\ f_{Y_{\mathbb{D}}}(b) \succ 0\ and\ f_{Y_{\mathbb{D}}}(b) \notin \mathfrak{S}_{\mathbb{D}}; \\[2ex] f_{X_{\mathbb{D}}}(\alpha)\ if\ f_{Y_{\mathbb{D}}}(b) = 0; \\[2ex] \dfrac{f_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, b)}{\lambda_1}e + e^{\dagger} f_{X_{\mathbb{D}}}(\alpha)\ if\ f_{Y_{\mathbb{D}}}(b) = \lambda_1 e, \lambda_1 > 0; \\[2ex] ef_{X_{\mathbb{D}}}(\alpha) + \dfrac{f_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha/b)}{\lambda_2}e^{\dagger}\ if\ f_{Y_{\mathbb{D}}}(b) = \lambda_2 e, \lambda_2 > 0. \end{cases}$$

Let us consider all cases that arise:

(i) If $f_{Y_{\mathbb{D}}}(b) \succ 0$ and $f_{Y_{\mathbb{D}}}(b) \notin \mathfrak{S}_{\mathbb{D}}$, then $f_{X_{\mathbb{D}}/Y_{\mathbb{D}}}(\alpha/b) = \dfrac{f_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, b)}{f_{Y_{\mathbb{D}}}(b)} \succeq 0, \ \forall\alpha \in \mathbb{D}$ and $\displaystyle\int_{\mathbb{D}} f_{X_{\mathbb{D}}/Y_{\mathbb{D}}}(\alpha/b)\, d\alpha = \frac{1}{f_{Y_{\mathbb{D}}}(b)} \int_{\mathbb{D}} f_{X_{\mathbb{D}}/Y_{\mathbb{D}}}(\alpha/b)\, d\alpha = \frac{1}{f_{Y_{\mathbb{D}}}(b)} f_{Y_{\mathbb{D}}}(b) = 1.$

(ii) If $f_{Y_{\mathbb{D}}}(b) = 0$, then $f_{X_{\mathbb{D}}/Y_{\mathbb{D}}}(\alpha/b) = f_{X_{\mathbb{D}}}(\alpha) \succeq 0, \ \forall\alpha \in \mathbb{D}$ and $\displaystyle\int_{\mathbb{D}} f_{X_{\mathbb{D}}/Y_{\mathbb{D}}}(\alpha/b)\, d\alpha = \int_{\mathbb{D}} f_{X_{\mathbb{D}}}(\alpha)\, d\alpha = 1.$

(iii) If $f_{Y_{\mathbb{D}}}(b) = \lambda_1 e, \lambda_1 \succ 0$, then $f_{X_{\mathbb{D}}/Y_{\mathbb{D}}}(\alpha/b) = \dfrac{f_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, b)}{\lambda_1}e + e^{\dagger} f_{X_{\mathbb{D}}}(\alpha) \succeq 0$ and

$$\begin{aligned} \int_{\mathbb{D}} f_{X_{\mathbb{D}}/Y_{\mathbb{D}}}(\alpha/b)\, d\alpha &= e\ \frac{\int_{\mathbb{D}} f_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, b)}{\lambda_1}\ d\alpha + e^{\dagger} \int_{\mathbb{D}} f_{X_{\mathbb{D}}}(\alpha)\, d\alpha \\ &= e\ \frac{f_{Y_{\mathbb{D}}}(b)}{\lambda_1}\ d\alpha + e^{\dagger} \int_{\mathbb{D}} f_{X_{\mathbb{D}}}(\alpha)\, d\alpha \\ &= e\ \frac{\lambda_1 e}{\lambda_1} + e^{\dagger}.1 \\ &= e + e^{\dagger} = 1. \end{aligned}$$

(iv) The case $f_{Y_{\mathbb{D}}}(b) = \lambda_2 e, \lambda_2 \succ 0$ is treated similarly.

Therefore we are justified in calling $f_{X_{\mathbb{D}}/Y_{\mathbb{D}}}(./b)$ a probability density function.

Definition 4.2. Let $X_{\mathbb{D}}, Y_{\mathbb{D}}$ be $\mathbb{D}$-random variables on $(\Omega, \Sigma, P_{\mathbb{D}})$ with an absolutely continuous joint distribution function whose joint probability density function is $f_{X_{\mathbb{D}}Y_{\mathbb{D}}}$. If $f_{Y_{\mathbb{D}}}$ is the marginal probability density function of $Y_{\mathbb{D}}$, then the conditional probability of $X_{\mathbb{D}}$ given $Y_{\mathbb{D}} = b$ is defined as

$$P_{\mathbb{D}}([X_{\mathbb{D}} \in A] | [Y_{\mathbb{D}} = b]) = \int_A f_{X_{\mathbb{D}}/Y_{\mathbb{D}}}(\alpha/b)\, d\alpha,$$

where $f_{X_{\mathbb{D}}/Y_{\mathbb{D}}}(./b)$ is the conditional density of $X_{\mathbb{D}}$ given $Y_{\mathbb{D}} = b$ and is the function defined in definition 4.1.

Now we use the definition 4.1. in evaluating corresponding conditional expectations. Thus if $E(X_{\mathbb{D}})$ exists, then we define $E(X_{\mathbb{D}}/Y_{\mathbb{D}} = b)$ as

$$E(X_{\mathbb{D}}/Y_{\mathbb{D}} = b) = E(X_{\mathbb{D}}/Y_{\mathbb{D}})(b) = \int_{\mathbb{D}} \alpha f_{X_{\mathbb{D}}/Y_{\mathbb{D}}}(\alpha/b)\, d\alpha.$$

The function $\phi\colon \Omega \longrightarrow \mathbb{D}$ defined by $\phi(Y_{\mathbb{D}} = b) = E(X_{\mathbb{D}}/Y_{\mathbb{D}})(b)$ is a $\mathbb{D}$-random variable and depends on the values of $Y_{\mathbb{D}}$ so that $E(X_{\mathbb{D}}/Y_{\mathbb{D}}) = \phi(Y_{\mathbb{D}})$ is well defined. Also

$$\begin{aligned}
E(|\phi(Y_{\mathbb{D}})|) &= \int_{\Omega} |\phi(Y_{\mathbb{D}})|\, dP_{\mathbb{D}} \\
&= \int_{\mathbb{D}} |\phi(b)| f_{Y_{\mathbb{D}}}(b)\, db, \ \text{ by theorem 4.3 below} \\
&= \int_{\mathbb{D}} \Big| \int_{\mathbb{D}} \alpha f_{X_{\mathbb{D}}/Y_{\mathbb{D}}}(\alpha/b)\, d\alpha \Big| f_{Y_{\mathbb{D}}}(b)\, db \\
&\preceq \int_{\mathbb{D}} \int_{\mathbb{D}} |\alpha| f_{X_{\mathbb{D}}/Y_{\mathbb{D}}}(\alpha/b)\, d\alpha\, f_{Y_{\mathbb{D}}}(b)\, db \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} |\alpha| \frac{f_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, b)}{f_{Y_{\mathbb{D}}}(b)}\, d\alpha\ f_{Y_{\mathbb{D}}}(b)\, db \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} |\alpha| f_{X_{\mathbb{D}}Y_{\mathbb{D}}}(\alpha, b)\, d\alpha\, db \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} |\alpha| f_{X_{\mathbb{D}}}(\alpha)\, d\alpha \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} |X_{\mathbb{D}}|\, dP_{\mathbb{D}} \\
&= E(|X_{\mathbb{D}}|) \in \mathbb{D}.
\end{aligned}$$

Therefore $E(\phi(Y_{\mathbb{D}}))$ exists.

The same computation without taking absolute values shows that

$$E(\phi(Y_{\mathbb{D}})) = E(E(X_{\mathbb{D}}/X_{\mathbb{D}})) = E(X_{\mathbb{D}}).$$

Theorem 4.3. (Fundamental Law of Probability) Let $X_{\mathbb{D}}$ be a $\mathbb{D}$-random variable (or n-vector), on $(\Omega, \Sigma, P_{\mathbb{D}})$ with distribution function $F_{X_{\mathbb{D}}}$.Then for any continuous (or even a Borel) function $\phi : \mathbb{D}^n \to \mathbb{D}$, $\phi(X_{\mathbb{D}})$ is a $\mathbb{D}$-random variable and

$$E(\phi(X_{\mathbb{D}})) = \int_{\Omega} \phi(X_{\mathbb{D}})\, dP_{\mathbb{D}} = \int_{\mathbb{D}^n} \phi(x)\, dF_{X_{\mathbb{D}}}(x).$$

Proof. If $A \subset \mathbb{D}^n$ and $\phi = \chi_A$, then $\phi(X_{\mathbb{D}}) = \chi_{X_{\mathbb{D}}^{-1}(A)}$, where $X_{\mathbb{D}} : \Omega \to \mathbb{D}^n$ is the given $\mathbb{D}$-random vector. Let $A = \prod_{i=1}^{n} A_i$, where $A_i = [a_i, b_i)e + [c_i, d_i)e^{\dagger}$ for a_i, b_i, $c_i, d_i \in \mathbb{R}$ and $X_{\mathbb{D}} = (X_{\mathbb{D}}^1, X_{\mathbb{D}}^2, \ldots X_{\mathbb{D}}^n)$. Then

$$\begin{aligned}
E(\phi(X_{\mathbb{D}})) = \int_{\Omega} \phi(X_{\mathbb{D}})\, dP_{\mathbb{D}} &= \int_{A} \chi_A(X_{\mathbb{D}})\, dP_{\mathbb{D}} \\
&= P_{\mathbb{D}}[X_{\mathbb{D}} \in A] \\
&= P_{\mathbb{D}}\left(\bigcap_{i=1}^{n}[X_i \in A]\right), \text{ where} X_{\mathbb{D}} = (X_{\mathbb{D}}^1, X_{\mathbb{D}}^2, \ldots, X_{\mathbb{D}}^n). \\
&= \int_{A_1}\int_{A_2}\cdots\int_{A_n} dF_{X_{\mathbb{D}}^1, X_{\mathbb{D}}^2, \ldots, X_{\mathbb{D}}^n}(x_1, x_2, \ldots, x_n) \\
&= \int_{\mathbb{D}^n} \chi_A\, dF_{X_{\mathbb{D}}}. \\
&= \int_{\mathbb{D}^n} \phi(x)\, dF_{X_{\mathbb{D}}}(x).
\end{aligned}$$

The result holds in this case. Since E(.) and integral are linear, result holds if $\phi = \Sigma_{i=1}^{n} a_i \chi_{A_i}$, a simple function. By Monotone Convergence Theorem, the result holds for any $\phi \succeq 0$ and hence for any Borel function $\phi = \phi^+ - \phi^-$, where $\phi^{\pm} \succeq 0$. ■

The importance this result is that the integral on an abstract probability space, is equal to the concrete integral on $\mathbb{D}^n$.

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