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Some Estimators the Parameter of Maxwell-Boltzmann Distribution

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Abstract

In this paper, the problem of point estimation for the one parameter θ of Maxwell-Boltzmann distribution has been investigated using simulation technique, to estimate the parameter of Maxwell-Boltzmann by many methods. These methods are divided in two sections; the first section includes Non-Bayesian statistical methods, such as minimum variance unbiased estimator method, while the second section includes Bayesian statistical methods, such as (extension Jeffrey Bayesian estimator, standard Bayesian informative prior Method, and Shrinkage Method).

Comparing between these four mentioned methods by employing mean square error measure. At last simulation technique used to generate many number of samples sizes to compare between these methods.

Keywords: Maxwell-Boltzmann distribution, Minimum Variance Unbiased Method, Extension Jeffrey Bayesian Method, Informative Bayesian prior Method, Shrinkage Method, Mean square error, simulation technique.

1. INTRODUCTION:

The statistical mechanics deal with Maxwell-Boltzmann distribution which description the energy and velocity in gas, when the molecules motion freely between the levels of energy, and do not interaction with each other, as employment of the temperature of gas system. In the statistical mechanics, the Maxwell-Boltzmann distribution grant the velocity and energy in gas.

In the statistical mechanics, the Maxwell-Boltzmann distribution grant the velocity and energy molecules in thermal equilibrium. In (1989) Tyagi and Bhattacherya introduction the Maxwell distribution in lifetime model [1]. In (1998) Chaturvedi and Rani used classical and Bayesian method of the generalized Maxwell distribution in lifetime to found the reliability estimation function [2]. In (2005) Bekker and Roux made the characteristic of reliability function in Maxwell distribution and estimate the Bayesian as study [3]. In (2009) krishna and Malik estimated the reliability function in Maxwell distribution by using type-two censored sample [4]. In (2011) Kasmi, and others utilizing the maximum likelihood estimator in type one censored sample of mixture Maxwell distribution [5]. In (2012) Kasmi and others utilizing the Bayesian estimation for two component of Maxwell distribution by using type-one censored sample [6]. In (2013) Al-Baldawi compare between some Bayesian estimator with maximum likelihood estimator for Maxwell distribution using non-informative priors [7].

In (2016) Rasheed and Khalifa estimated the parameter of Maxwell distribution by using Bayes estimator under quadratic loss function using Non-informative prior [8].

The aim of this paper is to study four estimation methods, the first is minimum variance unbiased estimator method, the second is extension Jeffrey Bayesian estimator method, the third is informative Bayesian prior estimator method, the fourth is Shrinkage estimator method, then compare between them by using Mean Square Error (MSE) utilizing Monte Carlo simulation technique with various sample sizes.

2. MAXWELL-BOLTZMANN DISTRIBUTION.

The random variable (x) has Maxwell-Boltzmann distribution which contain one parameter, it has the following cumulative distribution function (cdf):

$$F(x;a) = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{x^2}{2a^2}\right)$$

Where $\gamma\left(\frac{3}{2},\frac{x^2}{2a^2}\right)$ is the lower incomplete Gamma function defined by

$\gamma(a,t) = \int_0^t y^{a-1}e^{-y}dy \qquad (a,t > 0)$,

And a is scale parameter .

The probability density function (pdf) for the Maxwell-Boltzmann distribution defined as follows:

$$f(x;a) = \begin{Bmatrix} \sqrt{\frac{2}{\pi}}\frac{x^2 e^{-\frac{x^2}{2a^2}}}{a^3} & x \geq 0, a > 0 \\ 0 & o.\omega \end{Bmatrix}$$

let $a^2 = \theta$ then

$$f(x;\theta) = \begin{Bmatrix} \sqrt{\frac{2}{\pi}}\frac{x^2 e^{-\frac{x^2}{2\theta}}}{\theta^{3/2}} & x \geq 0, \theta > 0 \\ 0 & o.\omega \end{Bmatrix}$$

The mean, variance for this distribution define as follows:

$$\mu_x = 2a\sqrt{\frac{2}{\pi}} \qquad , \qquad \sigma_x^2 = \frac{a^2(3\pi - 8)}{\pi}$$

the moment generating function (mgf) of this distribution is as follows:

$$M_x(t) = \frac{e^{a^2t^2/2}}{\sqrt{\pi}}\left[a^2t^2\Gamma\left(\frac{1}{2},\frac{a^2t^2}{2}\right) + 2\,\Gamma\left(\frac{3}{2},\frac{a^2t^2}{2}\right)\right] + 2\sqrt{\frac{2}{\pi}}at$$

3. ESTIMATION METHODS:

In this study dealing with four estimation methods, which are as follows:

3.1. Minimum Variance Unbiased Estimator Method(MVUEM).

To prove that the estimator of Maxwell-Boltzmann distribution is (MVUE) property, we must prove that is unbiased estimator, sufficient and complete statistic, and has minimum variance.

1-Unbiased estimator

let $\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^2}{3n}$ is MLE for θ

$$E(\hat{\theta}) = E\left(\frac{\sum_{i=1}^{n} x_i^2}{3n}\right)$$

$$E(\hat{\theta}) = \frac{1}{3n}\sum_{i=1}^{n} E(x_i^2)$$

but $E(x_i^2) = 3\theta$

$$E(\hat{\theta}) = \frac{n}{3n}3\theta$$

$$E(\hat{\theta}) = \theta$$

2- Sufficient and complete statistic.

$$f(x;\theta) = Exp\left[\frac{-3}{2}\ln\theta + \ln\sqrt{\frac{2}{\pi}} + \ln x^2 - \frac{x^2}{2\theta}\right]$$

$$a(\theta) = \frac{-3}{2}\ln\theta + \ln\sqrt{\frac{2}{\pi}} \quad , b(x) = \ln x^2 \quad , c(\theta) = \frac{1}{2\theta} \quad , d(x) = x^2$$

Then; the statistic

$T = \sum_{i=1}^{n} d(x_i) = \sum_{i=1}^{n} x_i^2$ is complete sufficient statistic for θ

3-Minimum Variance Unbiased Estimator

$$L(\theta; x_1, x_2, x_3, \dots, x_n) = \prod_{i=1}^{n} \frac{\sqrt{2}x_i^2}{\sqrt{\pi}\theta^{3/2}} e^{-x^2/2\theta}$$

$$L(\theta; x_1, x_2, x_3, \dots, x_n) = (2)^{\frac{n}{2}}(\pi)^{-\frac{n}{2}} \prod_{i=1}^{n} x_i^2\, \theta^{-\frac{3}{2}n} e^{-\frac{\sum_{i=1}^{n} x_i^2}{2\theta}}$$

Taking the natural logarithm for the $L(\theta; x_1, x_2, x_3, \dots, x_n)$ so we get function:

$$\ln L(\theta; x_i) = \frac{n}{2}\ln 2 - \frac{n}{2}\ln\pi + 2\sum_{i=1}^{n}\ln x_i - \frac{3}{2}n\ln\theta - \frac{\sum_{i=1}^{n} x_i^2}{2\theta}$$

The partial derivative for $\ln L(\theta; x_i)$ with respect to unknown parameter θ is:

$$\frac{\partial \ln L(\theta; x_i)}{\partial\theta} = \frac{-3n}{2\theta} + \frac{\sum_{i=1}^{n} x_i^2}{2\theta^2}$$

$$\frac{\partial \ln L(\theta; x_i)}{\partial\theta} = \frac{3n}{2\theta^2}\left[\frac{\sum_{i=1}^{n} x_i^2}{3n} - \theta\right]$$

$$\frac{\partial \ln L(\theta; x_i)}{\partial\theta} = A(\theta)[T(x) - g(\theta)]$$

$$A(\theta) = \frac{3n}{2\theta^2} \quad , \quad T(x) = \frac{\sum_{i=1}^{n} x_i^2}{3n} \quad , \quad g(\theta) = \theta$$

$\frac{\sum_{i=1}^{n} x_i^2}{3n}$ is MVUE for θ

3.2. Extension Jeffrey Bayesian Estimator Method(EJBEM)[9]**:**

Using extension of Jeffrey's prior in the following form

$g_1(\theta) \propto \left(I(\theta)\right)^K$ or $g_1(\theta) \propto \left(I(\theta)\right)^{2K}$ for some constant k

where $I(\theta)$ is called fisher's information and

$$I(\theta) = -nE\left(\frac{\partial^2 lnf(x;\theta)}{\partial\theta^2}\right) = \frac{3n}{2\theta^2}$$

$$g_1(\theta) = \left(\frac{3n}{2\theta^2}\right)^k \text{ or } g_1(\theta) = \left(\frac{3n}{2\theta^2}\right)^{2k}$$

To find the posterior distribution as follows:

$$f(\theta; x_1, x_2, \dots, x_n) = \frac{\prod_{i=1}^{n} f\,(x_i;\theta) g_1(\theta)}{\int_0^\infty \prod_{i=1}^{n} f\,(x_i;\theta)\, g_1(\theta) d\theta}$$

$$f(\theta; x_i) = \frac{\prod_{i=1}^{n} \frac{\sqrt{2}}{\sqrt{\pi}} \theta^{-\frac{3}{2}} x_i^2 e^{-\frac{x_i^2}{2\theta}} \left(\frac{3n}{2\theta^2}\right)^k}{\int_0^\infty \prod_{i=1}^{n} \frac{\sqrt{2}}{\sqrt{\pi}} \theta^{-\frac{3}{2}} x_i^2\, e^{-\frac{x_i^2}{2\theta}} \left(\frac{3n}{2\theta^2}\right)^k d\theta}$$

$$f(\theta; x_i) = \frac{\left(\frac{\sum_{i=1}^{n} x_i^2}{2}\right)^{\frac{3}{2}n+2k-1} \theta^{-(\frac{3}{2}n+2k)} e^{-\frac{\sum_{i=1}^{n} x_i^2}{2\theta}}}{\Gamma\left(\frac{3}{2}n + 2k - 1\right)}$$

by using squared error loss function $L(\hat{\theta},\theta)=(\hat{\theta}-\theta)^2$, the risk function is:

$$R(\theta)=\int_0^\infty L(\hat{\theta},\theta)f(\theta;x_i)d\theta$$

$$R(\theta)=\int_0^\infty (\hat{\theta}-\theta)^2\frac{\left(\frac{\sum_{i=1}^n x_i^2}{2}\right)^{\frac{3}{2}n+2k-1}\theta^{-(\frac{3}{2}n+2k)}e^{-\frac{\sum_{i=1}^n x_i^2}{2\theta}}}{\Gamma\left(\frac{3}{2}n+2k-1\right)}d\theta$$

$$R(\theta)=\hat{\theta}^2-\frac{\hat{\theta}\sum_{i=1}^n x_i^2}{\left(\frac{3}{2}n+2k-2\right)}+h$$

The partial derivative for $R(\theta)$ with respect to $\hat{\theta}$ we get

$$\frac{\partial R(\theta)}{\partial\hat{\theta}}=2\hat{\theta}-\frac{\sum_{i=1}^n x_i^2}{\frac{3}{2}n+2k-2}+zero$$

the Bayes estimator $\hat{\theta}$ is the solution of equation $\frac{\partial R(\theta)}{\partial\hat{\theta}}=0$, which results in

$\hat{\theta}=\frac{\sum_{i=1}^n x_i^2}{3n+4k-4}$ is EJBE for θ

3.3. Informative Bayesian prior Estimator method(IBEM):

We based on Improper(a,b) distribution as informative prior to derive the posterior distribution, which is as follows:

$$g_2(\theta)=\begin{cases}\theta^{-(a+1)}e^{-\left(\frac{b}{\theta}\right)}, & \theta>0, \quad -\infty<a<\infty, b>0\\ 0, & o.w\end{cases}$$

To find the posterior distribution of θ as follows

$$f(\theta;x_1,x_2,\dots,x_n)=\frac{\prod_{i=1}^n f(x_i;\theta)g_2(\theta)}{\int_0^\infty\prod_{i=1}^n f(x_i;\theta)\,g_2(\theta)d\theta}$$

$$f(\theta;x_i)=\frac{\left(\frac{\sqrt{2}}{\sqrt{\pi}}\right)^n\theta^{-\frac{3}{2}n}\prod_{i=1}^n x_i^2\,e^{-\frac{\sum_{i=1}^n x_i^2}{2\theta}}\theta^{-(a+1)}e^{-\left(\frac{b}{\theta}\right)}}{\int_0^\infty\left(\frac{\sqrt{2}}{\sqrt{\pi}}\right)^n\theta^{-\frac{3}{2}n}\prod_{i=1}^n x_i^2\,e^{-\frac{\sum_{i=1}^n x_i^2}{2\theta}}\theta^{-(a+1)}e^{-\left(\frac{b}{\theta}\right)}d\theta}$$

$$f(\theta;x_i)=\frac{\left(\frac{\sum_{i=1}^{n}x_i^2+2b}{2}\right)^{\frac{3}{2}n+a}\theta^{-\left(\frac{3}{2}n+a+1\right)}e^{-\frac{\left(\sum_{i=1}^{n}x_i^2+2b\right)}{2\theta}}}{\Gamma\left(\frac{3}{2}n+a\right)}$$

By using squared error loss function $L(\hat{\theta},\theta)=(\hat{\theta}-\theta)^2$,the risk function is:

$$R(\theta)=\int_0^{\infty}L(\hat{\theta},\theta)f(\theta;x_i)d\theta$$

$$R(\theta)=\int_0^{\infty}(\hat{\theta}-\theta)^2\frac{\left(\frac{\sum_{i=1}^{n}x_i^2+2b}{2}\right)^{\frac{3}{2}n+a}\theta^{-\left(\frac{3}{2}n+a+1\right)}e^{-\frac{\left(\sum_{i=1}^{n}x_i^2+2b\right)}{2\theta}}}{\Gamma\left(\frac{3}{2}n+a\right)}d\theta$$

$$R(\theta)=\hat{\theta}^2-\frac{\hat{\theta}(\sum_{i=1}^{n}x_i^2+2b)}{\left(\frac{3}{2}n+a-1\right)}+k$$

The partial derivative for $R(\theta)$ with respect to $\hat{\theta}$ we get

$$\frac{\partial R(\theta)}{\partial\hat{\theta}}=2\hat{\theta}-\frac{\sum_{i=1}^{n}x_i^2+2b}{\frac{3}{2}n+a-1}+zero$$

the Bayes estimator $\hat{\theta}$ is the solution of equation $\frac{\partial R(\theta)}{\partial\hat{\theta}}=0$, which results in

$$2\hat{\theta}-\frac{\sum_{i=1}^{n}x_i^2+2b}{\frac{3}{2}n+a-1}=0$$

$\hat{\theta}=\frac{\sum_{i=1}^{n}x_i^2+2b}{3n+2a-2}$ is IBE for θ

3.4. Shrinkage Estimator Method(SEM)[10]**:**

In many problems there is some prior information about the parameter θ and this prior information is created as initial values symbolled by θ_0,Then the estimated method for this case is called Shrinkage estimation method.

The Shrinkage estimator is a linear combination between initial value θ_0 and estimated value $\hat{\theta}$ based on Shrinkage weight function which is denoted by $\psi(\hat{\theta})$.

There are two types of Shrinkage weight function which are as follows:

a-Constant Shrinkage weight function K:

The Shrinkage estimation of θ is:

$$\tilde{\theta} = K\hat{\theta} + (1-K)\theta_0$$

where K is constant, $K \in [0,1]$, K is confidence quantity for $\hat{\theta}$ and $(1-K)$ is confidence quantity for θ_0.

b-Variable Shrinkage weight function $\psi(\hat{\theta})$:

The Shrinkage estimation of θ is :

$$\tilde{\theta} = \psi(\hat{\theta})\hat{\theta} + [1-\psi(\hat{\theta})]\theta_0$$

where $\psi(\hat{\theta})$ is Shrinkage weight function depend on $\hat{\theta}$, $\psi(\hat{\theta}) \in [0,1]$,

and $\psi(\hat{\theta})$ is a confidence quantity for$\hat{\theta}$,$[1-\psi(\hat{\theta})]$ is a confidence quantity for θ_0.

Now, for Maxwell-Boltzmann distribution, we can find the Shrinkage estimator for the scale parameter θ as follows:

$\tilde{\theta} = \psi(\hat{\theta})\hat{\theta} + [1-\psi(\hat{\theta})]\theta_0$

$$\psi(\hat{\theta}) = \frac{b}{10var(\hat{\theta})}$$

$\hat{\theta}$ is MLE of θ,θ_0 is initial value which is prior information, b is any constant and 0<$b \leq 1$, $var(\hat{\theta})$ get it from fisher information matrix for MLE method.

The Shrinkage estimator for θ is became as following.

$$\tilde{\theta} = \frac{b}{10var(\hat{\theta})}\hat{\theta} + \left[1 - \frac{b}{10var(\hat{\theta})}\right]\theta_0$$

the mean square error for Shrinkage estimator for θ is as follows

$$MSE(\tilde{\theta}) = E(\tilde{\theta} - \theta)^2$$

$$MSE(\tilde{\theta}) = E\left[\frac{b}{10var(\hat{\theta})}\hat{\theta} + \left(1 - \frac{b}{10var(\hat{\theta})}\right)\theta_0 - \theta\right]^2$$

$$MSE(\tilde{\theta}) = \frac{b^2}{100var^2(\hat{\theta})}E(\hat{\theta} - \theta_0)^2 - \frac{2b}{10var(\hat{\theta})}(\theta - \theta_0)E(\hat{\theta} - \theta_0) + (\theta - \theta_0)^2$$

to get the value of b we must minimize the mean square error then

$$\frac{\partial MSE(\tilde{\theta})}{\partial b} = \frac{2b}{100var^2(\hat{\theta})} E(\hat{\theta} - \theta_0)^2 - \frac{2}{10var(\hat{\theta})}(\theta - \theta_0)E(\hat{\theta} - \theta_0)$$

$$\frac{\partial MSE(\tilde{\theta})}{\partial b} = 0$$

$$\frac{bE(\hat{\theta} - \theta_0)^2}{100var^2(\hat{\theta})} = \frac{(\theta - \theta_0)E(\hat{\theta} - \theta_0)}{10var(\hat{\theta})}$$

$$b = \frac{10var(\hat{\theta})(\theta - \theta_0)E(\hat{\theta} - \theta_0)}{E(\hat{\theta} - \theta_0)^2}$$

under the assumption that $E(\hat{\theta}) = \theta$ since $\hat{\theta}$ is MLE of θ and $\hat{\theta}$ is unbiased

$$b = \frac{10var(\hat{\theta})(\theta - \theta_0)^2}{E(\hat{\theta} - \theta_0)^2}$$

$$var(\hat{\theta} - \theta_0) = E(\hat{\theta} - \theta_0)^2 - [E(\hat{\theta} - \theta_0)]^2$$

$$E(\hat{\theta} - \theta_0)^2 = var(\hat{\theta}) + (\theta - \theta_0)^2$$

$$\hat{b} = \frac{10var(\hat{\theta})(\hat{\theta} - \theta_0)^2}{var(\hat{\theta}) + (\hat{\theta} - \theta_0)^2}$$

Recall equation of Shrinkage estimator, with compensate equation of b:

$$\tilde{\theta} = \frac{10var(\hat{\theta})(\hat{\theta} - \theta_0)^2}{[var(\hat{\theta}) + (\hat{\theta} - \theta_0)^2]} \frac{\hat{\theta}_{MLE}}{10var(\hat{\theta})} + \left[1 - \frac{10var(\hat{\theta})(\hat{\theta} - \theta_0)^2}{[var(\hat{\theta}) + (\hat{\theta} - \theta_0)^2]} \frac{1}{10var(\hat{\theta})}\right]\theta_0$$

$$\tilde{\theta} = \frac{(\hat{\theta} - \theta_0)^2 \sum_{i=1}^n x_i^2}{3n\left[var(\hat{\theta}) + (\hat{\theta} - \theta_0)^2\right]} + \left[1 - \frac{(\hat{\theta} - \theta_0)^2}{var(\hat{\theta}) + (\hat{\theta} - \theta_0)^2}\right]\theta_0$$

$\hat{\theta} = \frac{\sum_{i=1}^n x_i^2}{3n}$, $var(\hat{\theta}) = \frac{1}{I(\theta)}$, $var(\hat{\theta}) = \frac{2\theta^2}{3n}$

$$\tilde{\theta} = \frac{\left(\frac{\sum_{i=1}^n x_i^2}{3n} - \theta_0\right)^2 \sum_{i=1}^n x_i^2}{3n\left[\frac{2\theta^2}{3n} + \left(\frac{\sum_{i=1}^n x_i^2}{3n} - \theta_0\right)^2\right]} + \left[1 - \frac{\left(\frac{\sum_{i=1}^n x_i^2}{3n} - \theta_0\right)^2}{\frac{2\theta^2}{3n} + \left(\frac{\sum_{i=1}^n x_i^2}{3n} - \theta_0\right)^2}\right]\theta_0$$

$$\tilde{\theta} = \frac{\left(\sum_{i=1}^n x_i^2 - 3n\theta_0\right)^2 \sum_{i=1}^n x_i^2}{3n\left[6n\theta^2 + \left(\sum_{i=1}^n x_i^2 - 3n\theta_0\right)^2\right]} + \left[1 - \frac{\left(\sum_{i=1}^n x_i^2 - 3n\theta_0\right)^2}{\left[6n\theta^2 + \left(\sum_{i=1}^n x_i^2 - 3n\theta_0\right)^2\right]}\right]\theta_0 \text{ is SE for } \theta$$

4. NUMERICAL RESULTS AND COMMENTS:

In this section; simulation technique used to generate many various of samples by using Monte Carlo method, to compare between the methods of estimation which are mentioned in previous section

First: Generation of a sample from Maxwell distribution, we followed an algorithm suggested by Krishna and Malik (2009) [4] the following steps:

a-Generate two random numbers $X_1$and $X_2$from uniform distribution U (0, 1).
b-Obtain two standard normal variates Y_1andY_2 using the transformation

$$Y_1 = \sqrt{-2\log(X_1)}\, cos2\pi(X_2) \quad , \quad Y_2 = \sqrt{-2\log(X_1)}\, sin2\pi(X_2) ,$$

and find $Z = \frac{Y_1+Y_2}{\sqrt{2}}$ which is $N(0,1)$.
c-Repeating steps 1 and 2 three times generate a chi-square X_3^2 variate using
$T = \sum_{i=1}^{3} z_i^2$ which is gamma $G\left(\frac{3}{2},\frac{1}{\theta}\right)$ variate.
d- Using the transformation $V = \sqrt{\frac{T\theta}{2}}$ we get a number generated from Maxwell variate.

Second: To generate x which distributed as Maxwell-Boltzmann function, we must choose many values to the parameter θ as well as we must choose many various samples sizes which are as follows:

$\theta = 0.5 , 1.0 , 1.5 , 2.0 , 2.5$

$n = 10, 50 , 100 , 200$

Considered many different initial values

$\theta_0 = 0.75 ,1 ,1.25 ,2$
Assumed many values of $b, a,$ and k as follow

$$a = -1, 1, 2 , \qquad b = 1,2,3 , \qquad k = 2,3$$

And we replication the data of experiment (500) times, then the number of all generating experiment is (5) times.

Third: By MAT LAB program, we have got the following estimated values for the scale parameter to the Maxwell-Boltzmann distribution and numerical results scheduled in table (1)

Fourth: Computing the Mean Squares Error measure (MSE) for all situations studied in this paper, and scheduled in the tables (2)

Table 1: The estimate values for parameters θ in several methods

When $n = 10$

Methods			θ 0.5	1.0	1.5	2.0	2.5
MVUE			0.5046	1.0148	1.5168	1.9906	2.4615
EJBE		**k=2**	0.4453	0.8955	1.3384	1.7564	2.1719
		k=3	0.3984	0.8012	1.1975	1.5715	1.9433
IBE	$a = -1$	$b = 1$	0.7361	1.3248	1.9040	2.4506	2.9941
		$b = 2$	0.8900	1.4787	2.0579	2.6045	3.1479
		$b = 3$	0.8921	1.4823	2.0611	2.6122	3.1554
	$a = 1$	$b = 1$	0.6380	1.1482	1.6502	2.1239	2.5949
		$b = 2$	0.7713	1.2815	1.7835	2.2572	2.7282
		$b = 3$	0.9046	1.4148	1.9168	2.3906	2.8615
	$a = 2$	$b = 1$	0.5981	1.0764	1.5470	1.9911	2.4327
		$b = 2$	0.7231	1.2014	1.6720	2.1161	2.5577
		$b = 3$	0.8481	1.3264	1.7970	2.2411	2.6827
SE	θ_0	**0.75**	1.0442	1.4320	1.5105	1.5875	1.7340
		1.00	1.0256	1.6903	1.9267	2.0076	2.0934
		1.25	0.9668	1.7746	2.2313	2.4201	2.5099
		2.00	0.8175	1.6486	2.4215	3.0376	3.4678

When $n = 50$

Methods			θ 0.5	1.0	1.5	2.0	2.5
MVUE			0.4989	1.0009	1.5043	1.9902	2.5030
EJBE		**k=2**	0.4860	0.9749	1.4653	1.9385	2.4380
		k=3	0.4737	0.9502	1.4282	1.8894	2.3763
IBE	$a = -1$	$b = 1$	0.5400	1.0557	1.5729	2.0721	2.5990
		$b = 2$	0.5674	1.0831	1.6003	2.0995	2.6264
		$b = 3$	0.5686	1.0852	1.6037	2.1032	2.6318
	$a = 1$	$b = 1$	0.5256	1.0276	1.5310	2.0168	2.5297
		$b = 2$	0.5523	1.0542	1.5577	2.0435	2.5563
		$b = 3$	0.5789	1.0809	1.5843	2.0702	2.5830
	$a = 2$	$b = 1$	0.5187	1.0141	1.5109	1.9903	2.4964
		$b = 2$	0.5450	1.0404	1.5372	2.0166	2.5227
		$b = 3$	0.5713	1.0667	1.5635	2.0429	2.5490
SE	θ_0	**0.75**	0.6925	1.3196	1.5021	1.6320	1.9809
		1.00	0.6436	1.3243	1.8508	2.0003	2.1335
		1.25	0.6122	1.2651	1.9169	2.3576	2.5028
		2.00	0.5662	1.1530	1.7636	2.3769	3.0188

When $n = 100$

Methods			θ				
			0.5	**1.0**	**1.5**	**2.0**	**2.5**
MVUE			0.4993	1.0007	1.5064	2.0074	2.4952
EJBE		**k=2**	0.4928	0.9875	1.4866	1.9810	2.4624
		k=3	0.4864	0.9747	1.4673	1.9553	2.4304
IBE	$a = -1$	$b = 1$	0.5196	1.0277	1.5403	2.0481	2.5425
		$b = 2$	0.5331	1.0412	1.5538	2.0616	2.5560
		$b = 3$	0.5342	1.0433	1.5568	2.0661	2.5611
	$a = 1$	$b = 1$	0.5127	1.0140	1.5197	2.0208	2.5086
		$b = 2$	0.5260	1.0274	1.5331	2.0341	2.5219
		$b = 3$	0.5393	1.0407	1.5464	2.0474	2.5352
	$a = 2$	$b = 1$	0.5093	1.0073	1.5097	2.0074	2.4920
		$b = 2$	0.5225	1.0205	1.5229	2.0206	2.5052
		$b = 3$	0.5358	1.0338	1.5362	2.0339	2.5184
SE	θ_0	**0.75**	0.6065	1.2425	1.5013	1.6949	2.1253
		1.00	0.5753	1.1945	1.7891	2.0020	2.1734
		1.25	0.5576	1.1455	1.7715	2.3156	2.5011
		2.00	0.5333	1.0787	1.6431	2.2231	2.8110

When $n = 200$

Methods			θ				
			0.5	**1.0**	**1.5**	**2.0**	**2.5**
MVUE			0.4987	0.9997	1.5010	2.0034	2.5022
EJBE		**k=2**	0.4954	0.9931	1.4911	1.9901	2.4856
		k=3	0.4922	0.9865	1.4813	1.9770	2.4693
IBE	$a = -1$	$b = 1$	0.5088	1.0131	1.5178	2.0235	2.5257
		$b = 2$	0.5155	1.0198	1.5245	2.0302	2.5324
		$b = 3$	0.5166	1.0219	1.5275	2.0346	2.5375
	$a = 1$	$b = 1$	0.5054	1.0064	1.5077	2.0100	2.5089
		$b = 2$	0.5121	1.0130	1.5144	2.0167	2.5155
		$b = 3$	0.5187	1.0197	1.5210	2.0234	2.5222
	$a = 2$	$b = 1$	0.5037	1.0030	1.5027	2.0034	2.5005
		$b = 2$	0.5104	1.0097	1.5094	2.0100	2.5072
		$b = 3$	0.5170	1.0163	1.5160	2.0166	2.5138
SE	θ_0	**0.75**	0.5553	1.1614	1.5006	1.7650	2.2716
		1.00	0.5377	1.1070	1.7074	2.0008	2.2423
		1.25	0.5283	1.0757	1.6541	2.2413	2.4999
		2.00	0.5158	1.0392	1.5711	2.1170	2.6796

From table (1) we can make the following comments:

1-Noting that estimated values of parameter θ are vibrating and have (254) times bigger values comparison with true values of θ,and have (66) times smaller values comparison with true values of θ for all samples sizes of four studied methods.

2-Showing that the estimated values of parameter θ are converge to the true values of θ (20) times in minimum variance unbiased method, (33) times in extension Jeffrey Bayesian method, (146) times in informative Bayesian prior method, and (31) times in Shrinkage method.

Table 2: The Mean Square Error for parameters θ in several methods When $n = 10$

Methods			θ				
			0.5	**1.0**	**1.5**	**2.0**	**2.5**
MVUE			0.0179	0.0673	0.1789	0.2708	0.4135
EJBE		**k=2**	0.0169	0.0631	0.1652	0.2701	0.4284
		k=3	0.0214	0.0813	0.2028	0.3523	0.5667
IBE	$a = -1$	$b = 1$	0.0795	0.1948	0.4011	0.5635	0.7927
		$b = 2$	0.1758	0.3184	0.5491	0.7258	0.9684
		$b = 3$	0.3195	0.4893	0.7444	0.9354	1.1915
	$a = 1$	$b = 1$	0.0369	0.0890	0.2012	0.2860	0.4211
		$b = 2$	0.0914	0.1463	0.2590	0.3368	0.4641
		$b = 3$	0.1816	0.2391	0.3524	0.4232	0.5428
	$a = 2$	$b = 1$	0.0253	0.0648	0.1592	0.2380	0.3667
		$b = 2$	0.0654	0.0995	0.1866	0.2514	0.3655
		$b = 3$	0.1368	0.1655	0.2452	0.2961	0.3955
SE	θ_0	**0.75**	0.3112	0.1911	0.0042	0.1950	0.6937
		1.00	0.2977	0.5027	0.1911	0.0065	0.2086
		1.25	0.2404	0.6560	0.5781	0.1903	0.0173
		2.00	0.1214	0.5049	1.0515	1.2872	1.0881
Prefer			EJBE *k*=2	EJBE *k*=2	SE $\theta_0 = 0.75$	SE $\theta_0 = 1.0$	SE $\theta_0 = 1.25$

When $n = 50$

Methods			θ				
			0.5	**1.0**	**1.5**	**2.0**	**2.5**
MVUE			0.0034	0.0133	0.0319	0.0525	0.0798
EJBE		**k=2**	0.0034	0.0133	0.0315	0.0535	0.0795
		k=3	0.0037	0.0145	0.0339	0.0595	0.0872
IBE	$a = -1$	$b = 1$	0.0052	0.0172	0.0390	0.0606	0.0940
		$b = 2$	0.0081	0.0210	0.0438	0.0653	0.1002
		$b = 3$	0.0126	0.0263	0.0500	0.0715	0.1078
	$a = 1$	$b = 1$	0.0040	0.0141	0.0329	0.0527	0.0806
		$b = 2$	0.0061	0.0163	0.0352	0.0543	0.0829
		$b = 3$	0.0096	0.0199	0.0390	0.0574	0.0866
	$a = 2$	$b = 1$	0.0037	0.0132	0.0312	0.0512	0.0777
		$b = 2$	0.0053	0.0146	0.0325	0.0514	0.0782
		$b = 3$	0.0084	0.0174	0.0351	0.0529	0.0801
SE	θ_0	**0.75**	0.0413	0.1099	0.0007	0.1520	0.3377
		1.00	0.0246	0.1215	0.1345	0.0013	0.1564
		1.25	0.0163	0.0868	0.2078	0.1433	0.0026
		2.00	0.0079	0.0380	0.1070	0.2075	0.3621
Prefer			MVUE, EJBE($k = 2$)	IBE ($a = 2, b = 1$)	SE $\theta_0 = 0.75$	SE $\theta_0 = 1.0$	SE $\theta_0 = 1.25$

When $n = 100$

Methods			θ				
			0.5	**1.0**	**1.5**	**2.0**	**2.5**
MVUE			0.0014	0.0060	0.0167	0.0256	0.0418
EJBE		**k=2**	0.0014	0.0060	0.0164	0.0253	0.0421
		k=3	0.0015	0.0064	0.0169	0.0263	0.0445
IBE	$a = -1$	$b = 1$	0.0019	0.0070	0.0188	0.0286	0.0447
		$b = 2$	0.0026	0.0079	0.0200	0.0301	0.0460
		$b = 3$	0.0036	0.0092	0.0217	0.0319	0.0477
	$a = 1$	$b = 1$	0.0016	0.0062	0.0171	0.0260	0.0418
		$b = 2$	0.0021	0.0068	0.0178	0.0267	0.0423
		$b = 3$	0.0030	0.0077	0.0189	0.0278	0.0430

	$a = 2$	$b = 1$	0.0015	0.0060	0.0166	0.0253	0.0413
		$b = 2$	0.0019	0.0064	0.0170	0.0257	0.0413
		$b = 3$	0.0027	0.0071	0.0178	0.0264	0.0416
SE	θ_0	**0.75**	0.0130	0.0646	0.0004	0.1091	0.1887
		1.00	0.0072	0.0454	0.0953	0.0008	0.1275
		1.25	0.0048	0.0282	0.0945	0.1135	0.0013
		2.00	0.0026	0.0125	0.0389	0.0799	0.1490
Prefer			MVUE, EJBE(k=2)	MVU EJB(k=2) IBE $a = 2, b = 1$	SE $\theta_0 = 0.75$	SE $\theta_0 = 1.0$	SE $\theta_0 = 1.25$

When $n = 200$

Methods			θ				
			0.5	**1.0**	**1.5**	**2.0**	**2.5**
MVUE			0.0008	0.0030	0.0079	0.0131	0.0209
EJBE		**k=2**	0.0008	0.0030	0.0079	0.0130	0.0208
		k=3	0.0009	0.0031	0.0080	0.0133	0.0213
IBE	$a = -1$	$b = 1$	0.0009	0.0032	0.0083	0.0138	0.0218
		$b = 2$	0.0011	0.0034	0.0086	0.0142	0.0222
		$b = 3$	0.0013	0.0037	0.0090	0.0146	0.0227
	$a = 1$	$b = 1$	0.0009	0.0030	0.0080	0.0132	0.0209
		$b = 2$	0.0010	0.0031	0.0081	0.0134	0.0211
		$b = 3$	0.0012	0.0034	0.0083	0.0136	0.0214
	$a = 2$	$b = 1$	0.0008	0.0030	0.0079	0.0130	0.0207
		$b = 2$	0.0009	0.0031	0.0079	0.0131	0.0208
		$b = 3$	0.0011	0.0032	0.0081	0.0133	0.0209
SE	θ_0	**0.75**	0.0040	0.0297	0.0001	0.0679	0.0783
		1.00	0.0023	0.0149	0.0516	0.0003	0.0834
		1.25	0.0017	0.0090	0.0334	0.0705	0.0007
		2.00	0.0011	0.0046	0.0134	0.0282	0.0571
Prefer			MVU, EJBE(*k*=2), IBE (*a*=2, *b*=1)	MVUE, EJBE (k=2) IBE (*a*=1,2,*b*=1)	SE $\theta_0 = 0.75$	SE $\theta_0 = 1.0$	SE $\theta_0 = 1.25$

From table (2) we can make the following comments:

1-The values of mean squares error for $\hat{\theta}$ are decreasing where the samples sizes are increasing for all values of θ in all methods.

2-Noting that the values of MSE are vibrating for all increasing value of θ. The smallest values of MSE are (0.0001) when $(\theta = 1.5\,, n = 200)$ for Shrinkage estimator method at $\theta_0 = 0.75$.

5. CONCLUSIONS:

Throughout the estimator parameters for all four methods, we see that all values of estimator parameters are close to the true values of parameters in Maxwell-Boltzmann distribution. Also we can see that the mean squared error procedure for all four methods have a smallest value, specially the Shrinkage estimator method and far away from informative Bayesian prior estimator method.

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