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ON*WI*-IDEALS OF LATTICE WAJSBERG ALGEBRAS

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Abstract

In this paper, we introduce the notionofWajsberg implicative ideal(*WI*-ideal)of lattice Wajsberg algebra.Further, we definelattice *H*-Wajsberg algebra, implication homomorphismand lattice implication homomorphism oflattice Wajsberg algebra. Finally, we give kernel of implication homomorphism and obtain some oftheir properties.

Keywords: Wajsbergalgebra, Lattice Wajsberg algebra, *WI*-ideal, Lattice*H*-Wajsberg algebra, Implication homomorphism, Latticeimplication homomorphism, Kernel.

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1. INTRODUCTION

Many valued logic provide an interesting alternative to the classical logic for modelling and reasoning about system,by allowing additional truth value they support explicit modelling of uncertainity and disagreement. Łukasiewicz logic is a nonclassical, many valued logic. It was originally defined in the early 20th century by Jan Łukasiewicz[8] as a three valued logic. It was later generalized to *n*-valued(for all finite) as well as infinitely many valued ($\aleph_0$-valued) variants.

MordchajWajsberg[10] proposed the concept of Wajsberg algebra in 1935.A.Rose et al.[9]published the proof of Wajsberg algebra in 1958. In the same year, C.C.Chang[4] published his MV-algebra,it is faithful model only for the $\aleph_0$-valued ŁukasiewiczTarki logic. Lattice valued logic is becoming a research field which strongly influences development of algebraic logic, computer science and artificial intelligence technology. In 1984,Font etal.[5] extendedWajsberg algebra as an alternative model for the infinite valued Łukasiewicz logic and introduced lattice structure of Wajsberg algebra and discussed their properties. Further development of Wajsberg algebra, they defined implicative filters and family of implicative filters in lattice Wajsberg algebra.BasheerAhamed and Ibrahim[1,2] introduced the definitions of fuzzy implicative and an antifuzzy implicative filters of lattice Wajsberg algebras and investigated some properties.

In the present paper, we introduce the notion of Wajsberg implicative ideal(*WI*-ideal) of lattice Wajsberg algebra,and discuss some of their properties with examples. We show that, intersection of two *WI*-ideals of lattice Wajsberg algebra is a *WI*-ideal of lattice Wajsberg algebra. Also, we generalize this idea asintersection of family of *WI*-ideal of lattice Wajsberg algebra is a *WI*-ideal. Further, we define the implication homomorphism and kernel of latticeWajsberg algebra.We obtain the Quotient structure by using *WI*-ideal, and investigate the properties of *WI*-ideals related to implication homomorphism.

2. PRELIMINARIES

In this section,We recall some basic definitions and their properties which are helpful to develop our main results.

Definition 2.1[5]. Let $(A,\rightarrow,*,1)$ be an algebra with quasi complement "*"and a binary operation"$\rightarrow$" is called a Wajsberg algebra if and only if it satisfies the following axioms for all $x,y,z\in A$,

(i) $1\rightarrow x = x$

(ii) $(x\rightarrow y)\rightarrow((y\rightarrow z)\rightarrow(x\rightarrow z)) = 1$

(iii) $(x\rightarrow y)\rightarrow y = (y\rightarrow x)\rightarrow x$

(iv) $(x^*\rightarrow y^*)\rightarrow(y\rightarrow x) = 1.$

Proposition2.2[5].A Wajsberg algebra $(A,\to,*,1)$satisfies the following axioms for all $x,y,z\in A$,

(i) $x\to x=1$

(ii) If $(x\to y)=(y\to x)=1$then $x=y$

(iii) $x\to 1=1$

(iv) $(x\to(y\to x))=1$

(v) If $(x\to y)=(y\to z)=1$then $x\to z=1$

(vi) $(x\to y)\to((z\to x)\to(z\to y))=1$

(vii) $x\to(y\to z)=y\to(x\to z)$

(viii) $x\to 0=x\to 1^{*}=x^{*}$

(ix) $(x^{*})^{*}=x$

(x) $(x^{*}\to y^{*})=y\to x$.

Proposition 2.3[5].A Wajsberg algebra $(A,\to,*,1)$ satisfies the following axioms for all $x,y,z\in A$,

(i) If $x\le y$then $x\to z\ge y\to z$ and $z\to x\le z\to y$

(ii) $x\le y\to z$if and only if $y\le x\to z$

(iii) $(x\vee y)^{*}=(x^{*}\wedge y^{*})$

(iv) $(x\wedge y)^{*}=(x^{*}\vee y^{*})$

(v) $(x\vee y)\to z=(x\to z)\wedge(y\to z)$

(vi) $x\to(y\wedge z)=(x\to y)\wedge(x\to z)$

(vii) $(x\to y)\vee(y\to x)=1$

(viii) $x\to(y\vee z)=(x\to y)\vee(x\to z)$

(ix) $(x\wedge y)\to z=(x\to z)\vee(y\to z)$

(x) $(x\wedge y)\vee z=(x\vee z)\wedge(y\vee z)$

(xi) $(x\wedge y)\to z=(x\to y)\to(x\to z)$.

Definition 2.4[5]. Wajsberg algebra A is called a lattice Wajsberg algebra, if it satisfies the following conditions for all $x, y \in A$,

(i) The Partial ordering $\leq$ on a lattice Wajsberg algebra A,such that $x \leq y$ if and only if $x \to y = 1$

(ii) $(x \vee y) = (x \to y) \to y$

(iii) $(x \wedge y) = ((x^* \to y^*) \to y^*)^*$.Thus($A, \vee, \wedge$,*****,0,1)is a lattice Wajsberg algebra with lower bound 0and upper bound 1.

Definition 2.5[5]. Let A be aWajsberg algebra, a subset F of A is called an implicative filter of A, if it satisfies the following axioms for all $x, y \in A$,

(i) $1 \in F$

(ii) $x \in F$ and $x \to y \in F$ imply $y \in F$.

Definition 2.6[5]. Let L be a lattice. An ideal I of L is a nonempty subset of L,such that

(i) $x \in I$, $y \in L$ and $y \leq x$ imply $y \in I$

(ii) $x, y \in I$ implies $x \vee y \in I$.

We call this as lattice ideal.

Definition 2.7[5].A binary relation "$\sim$"on lattice Wajsberg algebra A as follows,$x \sim y$ if and only if $x \to y$ and $y \to x$ for all $x, y \in A$.

Theorem 2.8[5]. Let A be a lattice Wajsberg algebra and $\phi \neq I \subseteq A$.Then $\langle I \rangle = \{x \in A \mid \exists a_1, ..., a_n \in A, n \in \mathbb{N} / a_1 \to (... \to (a_n \to x)...) = 1\}$ and for any $a \in A$, $\langle a \rangle = \{x \in A \mid \exists\, n \in \mathbb{N} / a^n \to x = 1\}$ where, $a^n \to x = \underbrace{a \to (a \to (... \to (a}_{n\ times} \to x) ...))$ and $a^0 \to x = x$.

3. MAIN RESULTS

3.1. Wajsberg implicative ideal (*WI*-ideal)

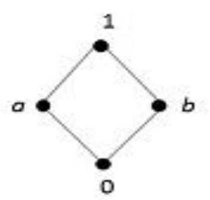
In this section, we define *WI*-ideal in lattice Wajsberg algebra and obtain some useful results with illustrations.

Definition 3.1.1. Let *A* be a lattice Wajsberg algebra. Let *I* be a nonempty subset of *A*, then *I* is called *WI*-idealof lattice Wajsberg algebra *A* satisfies,

(i) $0 \in I$

(ii) $(x \to y)^* \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in A$.

Example 3.1.2. Let $A = \{0, a, b, 1\}$ be a set with Figure (1) as a partial ordering. Define a quasi complement "$*$" and a binary operation "$\to$" on *A* as in tables (1) and (2).



Figure(1)

x	x^*
0	1
a	b
b	a
1	0

Table(1)

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	0	a	1	1
1	0	a	b	1

Table(2)

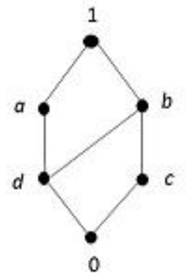
Define $\vee$ and $\wedge$ operations on *A* as follows :

$(x \vee y) = (x \to y) \to y$,

$(x \wedge y) = ((x^* \to y^*) \to y^*)^*$ for all $x, y \in A$.

Then, *A* is a lattice Wajsberg algebra. It is easy to verify that, $I_1 = \{0, a\}$ is a *WI*-ideal of *A*.

Example 3.1.3. Let $A = \{0, a, b, c, d, 1\}$ be a set with Figure (2) as a partial ordering. Define a quasi complement "$*$" and a binary operation "$\rightarrow$" on A as in tables (3) and (4).



x	x^*
0	1
a	c
b	d
c	a
d	b
1	0

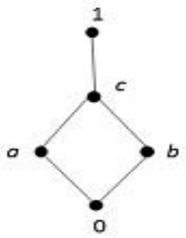
$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Figure (2) Table (3) Table (4)

Define $\vee$ and $\wedge$ operations on A here $(A, \vee, \wedge, 0, 1)$ is a lattice Wajsberg algebra.It is easy to verify that $I_1 = \{0, a, c\}$ is a *WI*-ideal of A. But, $I_2 = \{0, b, c\}$ is not a *WI*-ideal of A,Since $(b \rightarrow c)^* = b^* = d \notin I_2$.

Example 3.1.4. Let $A = \{0, a, b, c, 1\}$ be a set with Figure (3) as a partial ordering.

Define a quasi complement "$*$" and a binary operation "$\rightarrow$" on A as in tables (5) and (6).



x	x^*
0	1
a	b
b	a
c	c
1	0

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	c	1	1
b	a	c	1	1	1
c	c	c	c	1	1
1	0	a	b	c	1

Figure (3) Table(5) Table (6)

Define $\vee$ and $\wedge$ operations on A here $(A, \vee, \wedge, 0, 1)$ is a lattice Wajsbergalgebra. It is easy to check that, $I_1 = \{0, b, c\}$ is a *WI*-ideal of *A*, $I_2 = \{0, a, 1\}$ is also a *WI*-ideal of *A*. But $I_3 = \{0, a, b\}$ is not a *WI*-ideal of *A*Since $(a \to b)^* = c^* = c \notin I_3$.

Definition 3.1.5. The lattice Wajsberg algebra *A* is called a lattice *H*-Wajsberg algebra,

if it satisfies $x \vee y \vee ((x \wedge y) \to z) = 1$ for all $x, y, z \in A$.

In a lattice *H*-Wajsberg algebra *A*, the following hold,

(i) $x \to (x \to y) = (x \to y)$

(ii) $x \to (y \to z) = (x \to y) \to (x \to z)$.

Proposition 3.1.6. Let *I* be a *WI*-idealof a lattice Wajsberg algebra *A* andlet $x \in I$,

If $y \le x$, then $y \in I$ for all $y \in A$.

Proof. If *I* is a *WI*-ideal, then $0 \in I$,

$(x \to y)^* \in I$ and $y \in I$ imply, $x \in I$ for all $x, y \in A$ (1)

If $y \le x$, Since *A* is a lattice Wajsberg algebra $(y \to x)^* = 1^* = 0 \in I$.

Therefore, we get $y \in I$. [by (1)] ■

Proposition 3.1.7. Let *A* be a lattice Wajsberg algebra. Every *WI*-idealof *A* is a lattice ideal.

Proof. Let *I* be a *WI*-idealof *A*. Proposition 3.1.6shows that *I* satisfies (i) of definition 2.6.Let $x, y \in A$, then $((x \vee y) \to y)^* = (((x \to y) \to y) \to y)^* = (x \to y)^* \le (x^*)^* = x$.

If $x, y \in A$, then $((x \vee y) \to y)^* \in I$ and hence, $x \vee y \in I$ from (ii) of definition 3.1.1.

Therefore, *I* is a lattice ideal. ■

Remark 3.1.8. The converse of Proposition 3.1.7 may not be true. In Example 3.1.4, $M = \{b, c\}$ is a lattice ideal. But it is not a *WI*-ideal,Since $(b \to c)^* = 1^* = 0 \notin M$.

Proposition 3.1.9. In a lattice *H*-Wajsbergalgebra *A*, every lattice ideal is a *WI*-ideal.

Proof.Let *I*be a lattice ideal of *A*. Assume that $(x \to y)^* \in I$ and $y \in I$

Now, $y \vee (x \to y)^* = (y \to (x \to y)^*) \to (x \to y)^*$ [from (ii) of definition 2.4]

$= ((x \to y) \to y^*) \to (x \to y)^*$ [from (x) and (ix) of Proposition 2.2]

$= (x \to y) \to (y^*)^*$

$= (x \to y) \to y$ [from (ix) of Proposition 2.2]

$= x \vee y$ [from (ii) of definition 2.4]

By (ii) of definition 2.6, $x \vee y = y \vee (x \to y)^* \in I$, since $x \leq x \vee y$ and from(i) of definition 2.6 we have, $x \in I$ .Clearly, $0 \in I$. Hence, we get *I* is a *WI*-idealof *A*. ■

Definition 3.1.10. Let *I* be a nonempty subset of a lattice Wajsberg algebra *A*, we define a complement subset I^* of I in A is $I^* = \{a^* / a \in I\}$.

Note: It is clear that, every nonempty subset *I* is not an implicative filter. Similarly, the set I^* for every subset *I* is not a *WI*-ideal in general. In fact,the dual concept of an implicative filter is one of a *WI*-idealin a lattice Wajsberg algebra.

Proposition 3.1.11. Let *I* be a nonempty subset of a lattice Wajsberg algebra *A*. Then, *I* is an implicative filter of *A*,if and only if I^* is a *WI*-idealof *A*.

Proof.Let *I* be an implicative filter of *A*. Then $1 \in I$ and so $0 = 1^* \in I^*$.

Let $(x \to y)^* \in I^*$ and $y \in I^*$ for all $x, y \in A$.Then $(x \to y)^* = u^*$ and $y = v^*$ for some $u, v \in I^*$.Thus, $v \to x^* = x \to v^* = x \to y = ((x \to y)^*)^* = (u^*)^* = u \in I$

Since *I* is an implicativefilter, we have $x^* \in I$ and so $x = (x^*)^* \in I^*$.

This proves that, I^* is a *WI*-ideal of *A*.

Conversely, if I^* is a *WI*-ideal of *A*. Then, $1 \in I$ Since $1^* = 0 \in I^*$.

Let $x, y \in A$ be such that $x \in I$ and $x \to y \in I$.

Then,we get $x^* \in I^*$ and $(y^* \to x^*)^* = (x \to y)^* \in I^*$.

As I^* is a *WI*-ideal from(ii) of definition3.1.1that $y^* \in I^*$ or $y \in I$.

Hence, we have I is an implicative filter of *A*. ■

Proposition 3.1.12. Intersection of *WI*-ideals of a lattice Wajsberg algebra *A* is a *WI*-ideal of *A*.

Proof. Let I_1 and I_2 be two *WI*-ideals of a lattice Wajsberg algebra *A*. Since, $0 \in I_1$, $0 \in I_2$ imply $0 \in I_1 \cap I_2$.Therefore, $I_1 \cap I_2$ is nonempty. If $(x \to y)^* \in I_1 \cap I_2$ and $y \in I_1 \cap I_2$.

Then, we have

$(x \to y)^* \in I_1$ and $y \in I_1$ (2)

$(x \to y)^* \in I_2$ and $y \in I_2$ (3)

Since I_1 and I_2 are *WI*-ideals of *A*, from (2) and (3) $x \in I_1$ and $x \in I_2$.

Thus, we have $x \in I_1 \cap I_2$. Hence, $I_1 \cap I_2$ is a *WI*-ideals of a lattice Wajsberg algebra *A*. ■

Proposition3.1.13. If R is a nonempty family of*WI*-ideals of a lattice Wajsberg algebra*A*. Then, $I = \cap R$ is also a *WI*-ideal of *A*.

Proof. It is obvious from Proposition 3.1.12. ■

Note. Let I be a subset of a lattice Wajsberg algebra *A*. Then, the least *WI*-ideal containing $I$is called *WI*-ideal generated by Iwritten $\langle I \rangle$ of I and is well-defined.

The next Proposition 3.1.14 shows a description of elements of $\langle I \rangle$.

Proposition 3.1.14. If *I* is a nonempty subset of a lattice Wajsberg algebra *A*. Then, $\langle I \rangle = \{x \in A / a_n^* \to (...(a_1^* \to x^*)...) = 1$ for some $a_1,..., a_n \in I\}$.

Proof. Let $U = \{x \in A / a_n^* \to (...(a_1^* \to x^*)...) = 1$ for some $a_1,..., a_n \in I\}$.

Since, *I* is nonempty, there exists $a \in I$.

Here $0 \in U$. Because, $a^* \to 0^* = a^* \to 1 = 1$.

Let $(x \to y)^* \in U$ and $y \in U$. Then, there exist $a_i \in I (i = 1,2,..., n)$ and $b_j \in I\, (j = 1,2,..., m)$ such that,

$$a_n^* \to (...(a_1^* \to ((x \to y)^*)^* ...) = 1 \qquad \text{(I)}$$

$$b_m^* \to (...(b_1^* \to y^*)...) = 1 \qquad \text{(II)}$$

Since " * " is quasi complement, it follows from (x) of Proposition 2.2 that,

(I) is equivalent to,

$$a_n^* \to (...(a_1^* \to ((y^* \to x^*))^* ...)) = 1 \qquad \text{(III)}$$

which implies from (vii) of Proposition 2.2,

$$y^* \le a_n^* \to (...(a_1^* \to x^*)...) \qquad \text{(IV)}$$

Combining (II), (IV) and (i) of Proposition 2.3, we get

$1 = b_m^* \to (...(b_1^* \to y^*)...)$

$\le b_m^* \to (...(b_1^* \to (a_n^* \to (...(a_1^* \to x^*)...))...)$ and

$$b_m^* \to (...(b_1^* \to (a_n^* \to (...(a_1^* \to x^*)...)))...) = 1$$

This shows that, $x \in U$. Therefore, *U* is a *WI*-ideal of *A* containing *I*.

Let *V* be any *WI*-ideal Containing *I* and let $x \in U$.

Then $a_n^* \to (...(a_1^* \to x^*)...) = 1$, for some $a_1,..., a_n \in I$.

Thus, $1 = a_n^* \to (a_{n-1}^* \to (...(a_1^* \to x^*)...))$

$\quad = a_n^* \to ((a_{n-1}^* \to (...(a_1^* \to x^*)...))^*)^*$

$= ((a_{n-1}^* \to (...(a_1^* \to x^*)...))^* \to a_n)$ [from (x) of Proposition 2.2]

which implies that,

$((a_{n-1}^* \to (...(a_1^* \to x^*)...))^* \to a_n)^* = 1^* = 0 \in V$

Noticing that, $a_n \in I \subseteq V$ and V is a *WI*-ideal,we have

$$(a_{n-1}^* \to (...(a_1^* \to x^*)...))^* \in V$$

Now, $(a_{n-1}^* \to (...(a_1^* \to x^*)...))^* = (a_{n-1}^* \to (a_{n-2}^* \to (...(a_1^* \to x^*)...)))^*$

$= (a_{n-1}^* \to ((a_{n-2}^* \to (...(a_1^* \to x^*)...))^*)^*)^*$

$= ((a_{n-2}^* \to (...(a_1^* \to x^*)...))^* \to a_{n-1})^*$ [from (x) of Proposition 2.2]

Since, $a_{n-1} \in I \subseteq V$, it followsfrom (ii) of definition 3.1.1 that

$(a_{n-2}^* \to (...(a_1^* \to x^*)...))^* \in V$

By repeating the above argument,we saythat $x = (x^*)^* \in V$.

This proves that $U \subseteq V$. Hence, we get $U = \langle I \rangle$. ■

Note. (i) For any natural number *n*, we define $n(x) \to y$ recursively as follows, $1(x) \to y = x \to y$ and $(n+1)\ (x) \to y = x \to (n(x) \to y)$.

(ii) Any element '*a*'of a lattice Wajsberg algebra *A*, we have $\langle a \rangle = \{x \in A / n(a^*) \to x^* = 1$, for some natural number n$\}$.

3.2 Homomorphism and Kernel

In this section, we define the notion of Wajsberg homomorphism and kernel of lattice Wajsberg algebra. Further, we investigate some properties with illustrations.

Definition 3.2.1. Let A_1 and A_2 be lattice Wajsberg algebras, $f : A_1 \to A_2$ be a mapping from A_1 to A_2, if for any $x, y \in A_1$ $f(x \to y) = f(x) \to f(y)$ holds, then *f* is called an implication homomorphism from A_1 to A_2. If *f* is an implication homomorphism andsatisfies

(i) $f(x \vee y) = f(x) \vee f(y)$

(ii) $f(x \wedge y) = f(x) \wedge f(y)$

(iii) $f(x^*) = (f(x))^*$ for all $x, y \in A_1$.

Then, *f* is called alattice implication homomorphism from A_1 to A_2.

Definition 3.2.2. Let $f : A_1 \to A_2$ be an implication homomorphism, the kernelof f written *Ker(f)* is defined as *Ker(f)* $= \{x \in A_1 / f(x) = 0\}$.

Proposition 3.2.3. If an implication homomorphism $f : A_1 \to A_2$ satisfies *f*(0) = 0 Then *f* is a lattice implication homomorphism.

Proof.Let *f* satisfies *f*(0) = 0 $x, y \in A$.

$$f(x^*) = (f(x \to 0)) = f(x) \to f(0) = f(x) \to 0 = (f(x))^*$$

$$f(x \wedge y) = f((x^* \vee y^*)^*) = (f(x^*) \vee f(y^*))^* = (f(x^*)^* \wedge f(y^*)^*) = (f(x) \wedge f(y))$$

$$f(x \vee y) = f((x^* \wedge y^*)^*) = (f(x^*) \wedge f(y^*))^* = (f(x^*)^* \vee f(y^*)^*) = (f(x) \vee f(y))$$

Hence, *f* is a lattice implication homomorphism. ■

Proposition 3.2.4. Let $f : A_1 \to A_2$ be an implication homomorphism of lattice implication algebras. If *Ker(f)* $\neq \phi$, then $0 \in$ *Ker(f)*.

Proof. If *Ker(f)* $\neq \phi$ .Then there exist $x \in A_1$ such that *f*(x) = 0 and hence, *f*(0) = 0 that is, $0 \in$ *Ker(f).*■

Definition 3.2.5. Let *I* be a*WI*-ideal of a lattice Wajsberg algebra *A*, a binary relation "∼" on *A* as follows,$x \sim y$ if and only if $(x \to y)^* \in I$ and $(y \to x)^* \in I$ for all $x, y \in A$

The above definition is verified by the example 3.2.6 as shown below.

Example 3.2.6. From the example 3.1.14,consider a *WI*-ideal I_1 = {0, *b*, *c*} of *A*, we define abinary relation "∼" on I_1 as follows:

If 0 ~ b then $(0 \to b)^* = 1^* = 0 \in I_1$ and if b ~ 0 then $(b \to 0)^* = a^* = b \in I_1$,Similarly for b~ $c$and$c$ ~ 0.

Propositon3.2.7. A binary relation"~" is an equivalence relation on A.

Proof. From the definitions of 2.7 and 3.2.5,a binary relation"~" is both reflexive and symmetric.

To prove.Binary relation"~"is transitive.

If $x \sim y$ and $y \sim z$ for all $x, y, z \in A$.

Then, we have $(x \to y)^* \in I$, $(y \to z)^* \in I$, $(y \to x)^* \in I$, $(z \to y)^* \in I$.

Since $((x \to z)^* \to (x \to y)^*)^* \le ((x \to y) \to (x \to z))^* \le (y \to z)^*$

From the Proposition 3.1.6,

we have $((x \to z)^* \to (x \to y)^*)^* \in I$,So that $(x \to z)^* \in I$.

Because $(x \to y)^* \in I$ and I is a WI-ideal of A. Similarly, we have $(z \to x)^* \in I$.

Thus, x~ z. Hence,we have"~" is transitive. ■

Proposition 3.2.8. If $x \sim u$ and $y \sim v$then $(x \to y) \sim (u \to v)$.

Proof.Assume that $x \sim u$ and $y \sim v$.

Then, we have $(x \to u)^* \in I$, $(u \to x)^* \in I$, $(y \to v)^* \in I$, $(v \to y)^* \in I$.

Since, $((x \to y) \to (x \to v))^* \le (y \to v)^*$ and $((x \to v) \to (x \to y))^* \le (v \to y)^*$.

It follows from Proposition 3.1.6 that,

$((x \to y) \to (x \to v))^* \in I$ and $((x \to v) \to (x \to y))^* \in I$

which implies that, $(x \to y) \sim (x \to v)$. Similarly, we get $(x \to v) \sim (u \to v)$.

By the transitivity of a binary relation "~". We conclude that $(x \to y) \sim (u \to v)$. ■

Definition 3.2.9. Let I_x bethe equivalence class containing x and A/I be a set of all equivalence classes of A with respect to a binary relation "∼" then, $I_x = \{y \in A / x \sim y\}$ and $A/I = \{I_x / x \in A\}$.

Remark 3.2.10. It is clear that, $I_0 = I$ and $I_1 = \{y \in A / y^* \in I\}$.Define binary operations "⊔","⊓","$\Rightarrow$" unary operation "*N*" on A/I as follows,

$I_x \sqcup I_y = I_{x\vee y}, I_x \sqcap I_y = I_{x\wedge y}, I_x \Rightarrow I_y = I_{xy}, I_x^N = I_{x^*}$

for all $I_x, I_y \in A/I$. It can be easily verified that,(A/I ,⊔,⊓, I_0, I_1)is a bounded lattice. Moreover, A/I is a lattice Wajsberg algebra, which is called alattice Wajsberg quotient algebra of *A* by the *WI*-idealof *I.*

Proposition 3.2.11. Let $f : A_1 \to A_2$ be an implication homomorphism of latticeWajsberg algebras. Assume that, $Ker(f) \neq \phi$ .Then, $Ker(f)$ is a*WI*-idealof A_1.

Proof.Since $Ker(f) \neq \phi$. It follows from Proposition 3.2.4that $0 \in Ker(f)$.

Let $(x \to y)^* \in Ker(f)$ and $y \in Ker(f)$. Then, $f((x \to y)^*) = 0$ and $f(y) = 0$.

Hence, $0 = f((x \to y)^*) = f((x \to y))^*$

$$= (f(x) \to f(y))^*$$

$$= (f(x) \to 0)^*$$

$$= ((f(x))^*)^*$$

$$= f(x)$$

which implies that, $x \in Ker(f)$. ■

Proposition 3.2.12. Let $f : A \to \{0,1\}$ be an onto implication homomorphism of lattice Wajsberg algebra. Then, the kernel of f is a maximal *WI*-ideal of *A.*

Proof. Since f is onto, $Ker(f) \neq \phi$. By Proposition 3.2.11 $Ker(f) = K$ is a *WI*-idealof *A*. Suppose, *K* is not maximal. Then,there is a proper *WI*-ideal I containing *K*.

Therefore, there exist $x, y \in A$ such that $x \in A/I$ and $y \in I/K$.

Thus, $f(x) = f(y) = 1$ andso $f(x \to y) = f(x) \to f(y) = 1$.

It follows that, $f((x \to y)^*) = (f(x \to y))^* = 1^* = 0$.

So that, $(x \to y)^* \in K \subseteq I$ .Since $y \in I$, from (ii) of definition 3.1.1 we have, $x \in I$. This is a contradiction. Therefore, kernel of f is a maximal *WI*-ideal of *A*. ■

Proposition 3.2.13. Let A_1 and A_2 be lattice Wajsberg algebras and let $f : A_1 \to A_2$ be an onto implication homomorphism. Then, $A_1 / Ker(f)$ is isomorphic to A_2.

Proof. Let $K = Ker(f)$. Since f is onto. K is a *WI*-idealof A_1 by Proposition 3.2.11.

If $f(x) = f(y)$ then $f(x \to y)^* = (f(x \to y))^* = (f(x) \to f(y))^* = 1^* = 0$.

Similarly, $f(y \to x)^* = 0$ .Hence, we have $(x \to y)^* \in K$ and $(y \to x)^* \in K$.

Whichmeans that,*x* and *y* belong to the same equivalent class of A_1 / K.

Conversely, if $x \sim y(K)$.Then, $(x \to y)^* \in K$ and $(y \to x)^* \in K$.

It follows that, $(f(x) \to f(y))^* = (f(x \to y)^*) = 0$ and
$(f(y) \to f(x))^* = (f(y \to x))^* = 0$

Hence, $f(x) \to f(y) = 0^* = 1$ and $f(y) \to f(x) = 0^* = 1$.

So, we have $f(x) = f(y)$ [from(ii) of Proposition 2.2].

Therefore, we get $\phi : A_1 / K \to A_2, K_x \to f(x)$ is oneto one correspondence between A_1 / K and A_2. Now, for any $K_x, K_y \in A_1 / K$.

We have, $\phi(K_x \Rightarrow K_y) = \phi(K_{xy}) = f(x \to y) = f(x) \to f(y) = \phi(K_x) \to \phi(K_y)$

Thus, we get ϕ is the required isomorphism. ■

Proposition 3.2.14. Let A_1, A_2 and A_3 be lattice Wajsberg algebras, $h: A_1 \to A_2$ an onto implication homomorphism, and $g: A_1 \to A_3$ implication homomorphism with nonempty kernels. If $Ker(h) \subset Ker(g)$ then there is a unique implication homomorphism $f: A_2 \to A_3$ satisfying $foh = g$.

Proof.For any $y \in A_2$, there exists $x \in A_1$ such that $y = h(x)$ for the element *x*.

Put $z = g(x)$. Then, we show that the function $f: y \to z$ is well-defined and satisfies $foh = g$. Let $y = h(x_1) = h(x_2)$ for $x_1, x_2 \in A_1$.Then, $1 = h(x_1) \to h(x_2) = h(x_1 \to x_2)$ which implies that, $0 = 1^* = (h(x_1 \to x_2))^* = h((x_1 \to x_2)^*)$

Hence, we get $(x_1 \to x_2)^* \in Ker(h)$.

Since $Ker(h) \subset Ker(g)$

we obtain, $0 = g((x_1 \to x_2)^*) = g((x_1 \to x_2))^* = (g(x_1) \to g(x_2))^*$

It follows that, $g(x_1) \to g(x_2) = 0^* = 1$ or $g(x_1) \leq g(x_2)$.

Similarly, we have $g(x_2) \leq g(x_1)$ .Therefore, if $h(x_1) = h(x_2)$ then $g(x_1) = g(x_2)$.

This shows that, $f: y \to z$ is well-defined,and in the above case, we have $g(x) = f(h(x))$ that is $foh = g$.

To prove: *f* is animplication homomorphism.

Let $y_1, y_2 \in A_2$ for any $x_1, x_2 \in A_1$ such that $y_1 = h(x_1)$ and $y_2 = h(x_2)$.

we have

$$\begin{aligned} f(y_1 \to y_2) &= f(h(x_1) \to h(x_2)) \\ &= f(h(x_1 \to x_2)) \\ &= g(x_1 \to x_2) \\ &= g(x_1) \to g(x_2) \\ &= f(h(x_1)) \to f(h(x_2)) \\ &= (f(y_1) \to f(y_2)) \end{aligned}$$

Hence,*f*is animplication homomorphism. The uniqueness of *f* follows directly from the fact that *h* is an onto implication homomorphism. ■

4. CONCLUSION

In this paper, we have introduced the notion of Wajsberg implicative ideal(*WI*-ideal) of lattice Wajsberg algebra, and discussed some of their properties with examples. We have shown that, the intersection of two *WI*-ideals of lattice Wajsberg algebra is a *WI*-ideal and also, we have generalized that idea as intersection of family of *WI*-ideal of lattice Wajsberg algebra is a *WI*-ideal. Further, we have defined the implication homomorphism and kernel of lattice Wajsberg algebra. Then, we have obtained the quotient structure of Wajsberg algebra by using *WI*-ideal, and investigated the properties of *WI*-ideals related to implication homomorphism.

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