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Weighted Composition Operators on Cesàro Function Spaces

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Abstract

In this paper, an attempt has been made to study the weighted composition operators on Cesàro function spaces. Various properties like boundedness compactness and closed range of weighted composition operators has been discussed on this space.

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1. INTRODUCTION

Let (X,Σ,μ) be a σ-finite measure space. A transformation *T* is said to be measurable if $T^{-1}(A)\in\Sigma$ for $A\in\Sigma$. A measurable transformation *T* is said to be non-singular if $\mu(T^{-1}(B))=0$ whenever $\mu(B)=0$ for every $B\in\Sigma$.

If *T* is non-singular, then we say that μT^{-1} is absolutely continuous with respect to μ. Hence Radon –Nikodym theorem assures the existence of a unique non-negative measurable function *h* such that

$$(\mu T^{-1})(B)=\int_B h\,d\mu \;\; for\ B\in\Sigma .$$

h is called the Radon –Nikodym derivative and is denoted by $\frac{d\mu T^{-}1}{d\mu}$. We always assume that *h* is almost everywhere finite-valued or equivalently is a sub-sigma finite algebra. Let L^0 denote the set of all equivalence classes of complex valued measurable functions defined on *X*, where $X=[0,1]$ or $X=[0,\infty]$. Then for $1<p<\infty$, the Césaro function space is denoted by $Ces_p(X)$ and is defined as

$$Ces_p(X)=\{f\in L^0(X):(\frac{1}{x}\int_0^x|f(t)d\mu(t))\}.$$

The Cesàro function space is a Banach space under the norm given by

$$\|f\|=(\frac{1}{x}\int_0^x|f(t)|d\mu(t))^{\frac{1}{p}}.$$

For more details on this space, one can refer to [1, 5, 7]. Let ϕ be a complex valued measurable function then the linear transformation multiplication operator M_ϕ on the $Ces_p(X)$ space is defined as

$$M_\phi f=\phi.f$$

for all $x\in X$ and $f\in Ces_p(X)$. Let *T* be a measurable non-singular transformation. Then the linear transformation composition operator C_T on $Ces_p(X)$ space is defined as

$$C_T f=f\circ T$$

for all $x\in X$ and $f\in Ces_p(X)$. Also the generalized linear transformation weighted composition operator $W_{(\phi,T)}$ on the $Ces_p(X)$ space is defined as

$$W_{(\phi,T)}f(x)=\phi(T(x)).f(T(x))$$

for all $x\in X$ and $f\in Ces_p(X)$. In [4], authors have discussed about the composition operator on Cesàro function space. The study of composition operators on various Banach spaces has been done by many authors[2,3,6]. The main aim of this paper is to discuss various properties like boundedness, compactness and closed range of the weighed composition operators on Cesàro function spaces.

2. BOUNDEDNESS

Theorem 2.1. *Let* $\phi:X\to C$ *be a measurable function such that* $\phi\in L^\infty(\mu)$ *and* $T:X\to X$ *be a non-singular measurable transformation. Then* $W_{(\phi,T)}$ *is bounded on* $Ces_p(X)$ *if and only if there exist* $K>0$ *such that*

$$\mu T^{-1}(A) \le K\mu(A).$$

Proof. Suppose that the given condition holds. For $f \in Ces_p(X)$,

$$\| W_{(\phi,T)} f \|^p = \int_X (\frac{1}{x}\int_0^x | \phi \circ T . f \circ T \mid d\mu(t))^p d\mu(x)$$

$$= \int_X (\frac{1}{x}\int_0^x | \phi . f \mid d\mu T^{-1}(t))^p d\mu(x)$$

$$\le \int_X (\frac{1}{x}\int_0^x | \phi . f \mid hd\mu(t))^p d\mu(x)$$

$$\le K^p \int_X (\frac{1}{x}\int_0^x | \phi . f \mid d\mu(t))^p d\mu(x)$$

$$\le (K \| \phi \|_\infty)^p \int_X (\frac{1}{x}\int_0^x | f \mid d\mu(t))^p d\mu(x).$$

Therefore, $\| W_{(\phi,T)} f \| \le K \| \phi \|_\infty \| f \|$.

Conversely, suppose that $W_{(\phi,T)}$ is a weighted composition operator on $Ces_p(X)$.

If $A \in \Sigma$ such that $\mu(A) < \infty$, then ${}_{XA} \in Ces_p(X)$ and

$$\mu T^{-1}(A) = \| C_{T\,XA} \|^p \le \| W_{(\phi,T)XA} \|^p \le \| W_{(\phi,T)} \|^p \|_{XA}\|^p = \| W_{(\phi,T)} \|^p \ \mu(A).$$

Let $K = \| W_{(\phi,T)} \|^p$. Then $\mu T^{-1}(A) \le K\mu(A)$.

3. CLOSED RANGE

Theorem 3.1. Let $T : X \to X$ be a non-singular measurable transformation and $\phi : X \to C$ be a measurable function such that $W_{(\phi,T)}$ is bounded. Also, $h \in L^\infty(\mu)$ and is bounded away from zero.

Then $W_{(\phi,T)}$ has closed range if and only if there exist $\delta > 0$ such that $|\phi(x)| \ge \delta$ a.e. on S where

$$S = (x \in X : \phi(x) \ne 0\}$$

is support of ϕ.

Proof. Suppose that $|\phi(x)| \geq \delta$ a. e. on S and $h > k$ a.e. for some $k > 0.$ Then

$$\| W_{(\phi,T)} f \|^p = \int_X (\frac{1}{x}\int_0^x |\phi \circ T. f \circ T \,|\, d\mu(t))^p d\mu(x)$$

$$= \int_X (\frac{1}{x}\int_x^0 |\phi. f \,|\, d\mu T^{-1}(t))^p d\mu(x)$$

$$\geq k\int_X (\frac{1}{x}\int_0^x |\phi. f \,|\, d\mu(t))^p d\mu(x).$$

Hence

$$\| W_{(\phi,T)} \; f \| \geq k\delta \| f \|$$

Let $f_n, f \in Ces_p(X)$ such that

$$W_{(\phi,T)} f_n \to f,$$

For $n, m \in N$

$$\| W_{(\phi,T)} (f_n - f_m) \| \to 0 \text{ as n, m} \to \infty.$$

This gives $\| f_n - f_m \| \to 0$ as n, m $\to \infty$. Since $Ces_p(S)$ is complete, so we can find $g \in Ces_p(S)$ such that $f_n \to g$. Using the continuity of $W_{(\phi,T)}$, we get the desire result.

Conversely, let $W_{(\phi,T)}$, has closed range then there exists a $\varepsilon > 0$ such that $\| W_{(\phi,T)} f \| \leq \varepsilon \| f \|$

for all $f \in Ces_p(S)$. Let if possible the set given by

$$A = \{x \in X : |\phi(x)| < \delta\}$$

has positive measure then ${}_{XA} \in Ces_p(S)$. Also

$$\| W_{(\phi,T)} f \| \leq k^{\frac{1}{p}} \| M_{\phi XA} \|$$

$$\leq k^{\frac{1}{p}} \delta \|_{XA} \|$$

which is a contradiction. Thus

$$|\phi(x)| \geq \delta \text{ a. e. on } S$$

4. COMPACTNESS

Compactness of weighted composition operators play an important role in the study of the structure of weighted composition operators.

Theorem 4.1. *Let T be a non-singular measurable transformation from X to itself such that h is essentially bounded and bounded away from zero. Let $\phi : X \to C$ be a measurable function such that $W_{(\phi,T)}$ is bounded on $Ces_p(X)$. Then the following are equivalent:*

(1) $W_{(\phi,T)}$ is compact.

(2) M_ϕ is compact.

(3) $Ces_p(\phi,\delta)$ are finite dimensional for which $\delta > 0$ where the symbol (ϕ,δ)

means the set $\{x \in X : |\phi(x)| > \delta\}$.

Proof. $1 \Leftrightarrow 2.$ Since $h \in L^\infty(\mu)$ then for $f \in Ces_p(\phi,\delta)$ for $t > 0$,

$$\| W_{(\phi,T)} f \| = \int_X \left(\frac{1}{x}\int_0^x | \phi \circ T . f \circ T \,|\, d\mu(t)^p \, d\mu(x)\right.$$

$$= \int_X \left(\frac{1}{x}\int_0^x | \phi . f \,|\, d\mu T^{-1}(t)\right)^p d\mu(x)$$

$$\leq \int_X \left(\frac{1}{x}\int_0^x | \phi . f \,|\, h d\mu(t)\right)^p d\mu(x)$$

$$\leq k \int_X \left(\frac{1}{x}\int_0^x | \phi . f \,|\, d\mu(t)\right)^p d\mu(x)$$

$$= k \| M_\phi f \|.$$

This gives

$$\| W_{(\phi,T)} f \| \leq k \| M_\phi f \|.$$

Also, h is bounded away from zero i.e. $h > \delta$ a. e. for some $\delta > 0$ then

$$\mu T^{-1}(A) = \int_A h d\mu \geq \delta \mu(A)$$

for all $A \in \Sigma.\, A \subseteq S,$ where $S = \{x : \phi(x) \neq 0\}.$ Thus we find that

$$\| W_{(\phi,T)} f \| \geq \delta \| M_\phi f \|.$$

Hence for each $f \in Ces_p(S), 1 < p < \infty$, we have

$$\| W_{(\phi,T)} f \| \approx k \| M_{\phi,} f \|.$$

$2 \Leftrightarrow 3$. Suppose M_ϕ is a compact operator and since restriction of a compact operator to a closed invariant subspace is again a compact operator. Therefore $M_\phi \,|\, Ces_p(S)$ is finite dimensional for each $\delta > 0$. In particular for $\delta = \frac{1}{n}$ where $n \in N$. For each *n*, define

$$\phi_n(x) = \begin{cases} \phi(x), & x \in (\phi, 1/n) \\ 0, & \text{otherwise} \end{cases}$$

Since $\phi \in L^\infty(\mu)$ therefore $\phi_n \in L^\infty(\mu)$. Also, for any $f \in Ces_p(\phi, \delta), t > 0$,

$$\| (M_{\phi_n} - M_\phi) \| \leq \frac{1}{n} \| f \|.$$

This gives that M_{ϕ_n} converges uniformly to M_ϕ. As $Ces_p\left(\phi, \frac{1}{n}\right)$ is finite dimensional so M_ϕ is a finite rank operator and thus M_ϕ being a limit of M_{ϕ_n} becomes a compact operator.

A set $A \in \sum$ is called an atom for μ if $\mu(A) > 0$ and given $(B) \in \sum$ either $\mu(A \cap B)$ is 0 or $\mu(A - B)$ is 0. μ is called purely atomic or atomic if every measurable set of positive measure contains an atom. Otherwise μ is called non-atomic.

Theorem 4.2. [6] *Let* $\phi : X \to C$ *be a measurable function such that* $\phi \in L^\infty(\mu)$ *and* $T : X \to X$ *be a non-singular measurable transformation with* $h \in L^\infty(\mu)$. *Let* (A_n) *be a sequence of atoms of X such that* $\mu(A_n) > 0$ *for each n. Then* $W_{(\phi,T)}$ *is compact on the* $Ces_p(X)$ *if* μ *is purely atomic and* $d_n = \dfrac{\mu T^{-1}(A_n)}{\mu(A_n)} \to 0$

Theorem 4.3. *If* μ *is non-atomic and* $W_{(\phi,T)}$ *is bounded on* $Ces_p(X)$ *then* $W_{(\phi,T)}$ *is compact if and only if* $\phi.h = 0$ *almost everywhere* .

Proof. Proof is along the similar lines as in [2]

REFERENCES

[1] B. D. Hassard and D. A. Hussein, On Cesàro function spaces, Tamkang Journal of Mathematics, 4 (1973), 19 – 25.

[2] A. Gupta and N. Bhatia, Composition and weighted composition operators on Generalized Lorentz- Zygmund spaces, Journal of Mathematical Analysis, 4 (2013), 1 -13

[3] A. Gupta and N. Bhatia, Composition Operators on Lorentz-Karamata Bochner Spaces, American Journal of Mathematical Analysis , 3 (2015), Vol. 3, 21 – 25.

[4] K. Raj, S. Pandoh and S. Jamwal, Composition Cesàro function spaces, Journal of Function spaces, 2014, Article ID 501057, 6 pages https:// dx.doi.org/10.1155/2014/501057.

[5] J. S. Shine, A note on Cesàro function spaces, Tamkang Journal of Mathematics, 1 (1970), 91-95.

[6] R. K. Singh and J. S. Manhas, Composition operators on function spaces, North Holland Math. Studies 179, Amsterdam, (1993) .

[7] P. W. Sy, W. Y. Zhang and P. Y. Lee, The dual of Cesàro function spaces, Glasnik Mathematiciki, 22 (1987), 103 -112.