Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 10 (2017), pp. 7291-7300

http://www.ripublication.com

Pathway Fractional Integral Operator and its Composition with Special Functions

Neeti Ghiya[1], N. Shivakumar[2] and Vidya Patil[3]

[1]*Department of Mathematics, R. V. College of Engineering, Mysore Road, Bangalore-560059, Karnataka, India.*

[2]*Department of Mathematics, R. V. College of Engineering, Mysore Road, Bangalore-560059, Karnataka, India.*

[3]*Department of Mathematics, R. V. College of Engineering, Mysore Road, Bangalore-560059, Karnataka, India.*

Abstract

Fractional integral operators are extensively used in a large number of areas of mathematical analysis. This paper provides the images of the products of multivariable H-function and generalized polynomial under the pathway fractional integral operator and its composition. The main results are quite general in nature. Further, some interesting special cases are given. These results can be used to evaluate Laplace transform of various special functions.

2010 Mathematical Subject Classification: 26A33, 33E20, 16R50

Keywords: fractional integral operator, Multivariable H-function, Srivastava polynomial, and pathway model.

1. INTRODUCTION

Pathway fractional integral operator is based on well-known Riemann-Liouville (R-L) fractional integral operator and pathway model. Pathway operator is introduced in the paper of Nair [2] and defined as follows:

If $f(x) \in L(a, b)$, $\gamma \in C$, $R(\gamma) > 0$, $a > 0$ and $\beta < 1$, then

$$\left(P_{0_+}^{(\gamma,\ \beta)} f\right)(x) = x^{\gamma} \int_0^{\left[\frac{x}{a(1-\beta)}\right]} \left[1 - \frac{a(1-\beta)t}{x}\right]^{\frac{\gamma}{1-\beta}} f(t)\, dt, \tag{1.1}$$

where β is called pathway parameter. For pathway model, we refer to Mathai [5], Mathai and Haubold ([3], [4]). The pathway model transforms into three different types of densities, type-1 beta, type-2 beta and gamma in statistics. Using the pathway parameter β, this operator can reduce to various fractional integral operators, related to different probability density functions (PDF's) and applications in statistics.

If $\beta \rightarrow 1_-$, $\left[1 - \frac{a(1-\beta)t}{x}\right]^{\frac{\gamma}{1-\beta}} \rightarrow e^{-\frac{a\gamma}{x}t}$ and hence pathway operator switches to the Laplace integral transform of function f with parameter $\frac{a\gamma}{x}$:

$$\left(P_{0_+}^{(\gamma,\ 1)} f\right)(x) = x^{\gamma} \int_0^{\infty} e^{-\frac{a\gamma}{x}t} f(t)\, dt = x^{\gamma} L_f\left(\frac{a\gamma}{x}\right). \tag{1.2}$$

Taking $\beta = 0$ and $a = 1$, then replacing γ by γ-1 in pathway operator (1.1), it transforms to the Riemann-Liouville (R-L) fractional integral operator.

The multivariable H-function was defined by H. M. Srivastava and Panda [6] and it is given as:

$$H_{p,\,q\,:\,p_1,\,q_1\,;\,\cdots;\,p_r,\,q_r}^{0,\,n:\,m_1,\,n_1\,;\,\cdots\,;m_r,\,n_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} \left(a_j;\psi_j',\cdots,\psi_j^{(r)}\right)_{1,p} : \left(c_j',\tau_j'\right)_{1,\,p_1} ;\ldots; \left(c_j^{(r)},\tau_j^{(r)}\right)_{1,\,p_r} \\ \left(b_j;\mu_j',\cdots,\mu_j^{(r)}\right)_{1,q} : \left(d_j',\chi_j'\right)_{1,\,q_1} ;\ldots; \left(d_j^{(r)},\chi_j^{(r)}\right)_{1,\,q_r} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \int_{L_r} \psi_1(\delta_1) \cdots \psi_r(\delta_r)\ \theta(\delta_1,\ldots,\delta_r) z_1^{\delta_1} \cdots z_r^{\delta_r}\, d\delta_1 \cdots d\delta_r, \text{ where } \omega = \sqrt{-1} \tag{1.3}$$

$$\psi_i(\delta_i) = \frac{\prod_{j=1}^{m_i} \Gamma\left(d_j^{(i)} - \chi_j^{(i)}\delta_i\right) \prod_{j=1}^{n_i} \Gamma\left(1 - c_j^{(i)} + \tau_j^{(i)}\delta_i\right)}{\prod_{j=m_i+1}^{q_i} \Gamma\left(1 - d_j^{(i)} + \chi_j^{(i)}\delta_i\right) \prod_{j=n_i+1}^{p_i} \Gamma\left(c_j^{(i)} - \tau_j^{(i)}\delta_i\right)}, \tag{1.4}$$

$$\theta(\delta_1,\cdots,\delta_r) = \frac{\prod_{j=1}^{n} \Gamma\left(1 - a_j + \sum_{i=1}^{r} + \psi_j^{(i)}\delta_i\right)}{\prod_{j=n+1}^{p} \Gamma\left(a_j - \sum_{i=1}^{r} \psi_j^{(i)}\delta_i\right) \prod_{j=1}^{q} \Gamma\left(1 - b_j + \sum_{i=1}^{r} \mu_j^{(i)}\delta_i\right)}. \tag{1.5}$$

Here, we refer Srivastava et al. ([8, pp.252-253, eqns.(C.4)-(C.8)] for the existence and convergence conditions of the multivariable H-function. Throughout this paper it is assumed that this function satisfies the above cited conditions.

Generalized polynomial is introduced by Srivastava [7] in the following manner:

$$S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}[t_1,\ldots,t_s] = \sum_{k_1=0}^{[n_1/m_1]},\ldots,\sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1k_1}}{k_1!}\ldots\frac{(-n_s)_{m_sk_s}}{k_s!} A[n_1,k_1;\ldots;n_s,k_s]t_1^{k_1},\ldots,t_s^{k_s}, \tag{1.6}$$

where $n_i(\forall i=1,\ldots,s) = 0,1,2,\ldots; m_1,\ldots,m_s$ are arbitrary positive integers and the coefficients $A[n_1,k_1;\ldots;n_s,k_s]$ are real or complex arbitrary constants. Various polynomials like Hermite, Jacobi, Laguerre and Bessel's etc. can be obtained from (1.6) by suitably specializing the coefficients.

2. MAIN RESULTS

Theorem 1: Let $\gamma, \delta, r_i, \eta_i, h_j, \phi_j \in C$ $(i=1,\cdots,s)$ $(j=1,\cdots,r)$, $R\left(1+\frac{\gamma}{1-\beta}\right)>0$, $h, c \geq 0$, $\beta < 1$, $Z_i, M_i, k \in R$, $R\left(\gamma, \delta, r_i, \eta_i, p_j, \phi_j\right) > 0$ and the coefficients $A[n_1,k_1;\ldots;n_s,k_s]$ are arbitrary constants, real or complex, then

$$P_{0+}^{(\gamma,\beta)}\left[x^{\delta-1}(x^h+c)^{-\eta} S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\left[Z_1x^{r_1}(x^h+c)^{-\eta_1},\ldots,Z_sx^{r_s}(x^h+c)^{-\eta_s}\right] H_{p,\,q:\,p_1,\,q_1;\cdots;p_r,\,q_r}^{0,\,n:\,m_1,\,n_1;\cdots;m_r,\,n_r}\left[M_1x^{h_1}(x^h+c)^{-\phi_1},\cdots,M_rx^{h_r}(x^h+c)^{-\phi_r}\right]\right]$$

$$= \frac{x^{\gamma+\delta}}{c^{\eta}[a(1-\beta)]^{\delta}}\,\Gamma\left(1+\frac{\gamma}{1-\beta}\right)\sum_{k_1}^{[n_1/m_1]}\cdots\sum_{k_s}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_s)_{m_sk_s}}{k_s!}A[n_1,k_1;\ldots;n_s,k_s]\frac{Z_1^{k_1}x^{r_1k_1}}{c^{\eta_1k_1}[a(1-\beta)]^{r_1k_1}}$$

$$\cdots\frac{Z_s^{k_s}x^{rk_s}}{c^{\eta_sk_s}[a(1-\beta)]^{r_sk_s}}\; H_{p+2,\,q+2:\,p_1,\,q_1;\cdots;p_r,\,q_r;\,0\,1}^{0,\,n+2\,:\,m_1,\,n_1;\cdots;m_r,\,n_r;\,1\,0}\left[\begin{matrix}\frac{M_1x^{h_1}}{c^{\phi_1}[a(1-\beta)]^{h_1}}\\ \vdots\\ \frac{M_rx^{h_r}}{c^{\phi_r}[a(1-\beta)]^{h_r}}\\ \frac{x^k}{c\,[a(1-\beta)]^k}\end{matrix}\,\middle|\,\begin{matrix}(a_j:\psi_j',\cdots,\psi_j^{(r)},0),\\ (b_j:\mu_j',\cdots,\mu_j^{(r)},0),\end{matrix}\right.$$

$$\left(1-\eta-\textstyle\sum_{i=1}^{s}\eta_ik_i:\ \phi_1,\cdots,\phi_r,\ 1\right),\ \left(1-\delta-\textstyle\sum_{i=1}^{s}r_ik_i;\ h_1,\cdots,h_r,\ k\right)\ :$$

$$\left(1-\eta-\textstyle\sum_{i=1}^{s}\eta_ik_i:\ \phi_1,\cdots,\phi_r,\ 0\right),\ \left(1-\frac{\gamma}{1-\beta}-\delta-\textstyle\sum_{i=1}^{s}r_ik_i;\ h_1,\cdots,h_r,\ k\right)\ :$$

$$\left.\begin{matrix}(c_j',\tau_j')_{1,p_1};\ \ldots;\ (c_j^{(r)},\tau_j^{(r)})_{1,p_r};\ -\\ (d_j',\chi_j')_{1,q_1};\ \ldots;\ (d_j^{(r)},\chi_j^{(r)})_{1,q_r};\ (0,1)\end{matrix}\right] \tag{2.1}$$

Proof: To establish result (2.1), we use

$$f(t) = t^{\delta-1}(t^h+c)^{-\eta}S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\left[Z_1t^{r_1}(t^h+c)^{-\eta_1},\ldots,Z_st^{r_s}(t^h+c)^{-\eta_s}\right] H_{p,\,q:\,p_1,\,q_1;\cdots;p_r,\,q_r}^{0,\,n:\,m_1,\,n_1;\cdots;m_r,\,n_r}\left[M_1t^{h_1}(t^h+c)^{-\phi_1},\cdots,M_rt^{h_r}(t^h+c)^{-\phi_r}\right]$$

in pathway fractional integral operator (1.1). Express multivariable H-function and generalized polynomial with the help of the equations (1.3) and (1.6) respectively. Further, express $\left(t^h + c\right)^{-\eta - \sum_{i=1}^{s} \eta_i k_i - \sum_{i=1}^{s} \phi_i \delta_i}$ in the form of the Mellin-Barnes type contour integral by Srivastava et al. [8]. Interchange the order of integration under the permissible conditions and evaluate the t – integral using the following integral:

$$\int_0^{\frac{x}{a(1-\beta)}} \left[1 - \frac{a(1-\beta)\,t}{x}\right]^{\frac{\gamma}{1-\beta}} t^{\sigma-1}\, dt = \frac{x^{\sigma}}{[a(1-\beta)]^{\sigma}} \frac{\Gamma(\sigma)\,\Gamma\left(1+\frac{\gamma}{1-\beta}\right)}{\Gamma\left(\frac{\gamma}{1-\beta}+\sigma+1\right)},$$

$\beta<1$, $R(\gamma)>0$, $R(\sigma)>0$.

Then interpreting the contour integrals thus obtained in terms of the H-function of r +1 variable, we easily get the RHS of the result (2.1) after a little simplification.

Theorem 2: Let $\gamma, \gamma', \delta, \delta', \eta, \eta'\ r_i, \eta_i,\ p_j, \phi_j \in C$ $(i = 1, \cdots, s)$ $(j=1, \cdots, r)$, $R\left(1+\frac{\gamma}{1-\beta}\right)>0$, $R\left(1+\frac{\gamma'}{1-\beta'}\right)>0$, $h, c \geq 0$, $h', c' \geq 0$, $\beta < 1$, $\beta' < 1$ $Z_i, k, k' \in R$, $M_i \in R$, $R\left(\gamma, \delta, r_i, \eta_i, p_j, \phi_j\right) > 0$ and the coefficients $A[n_1, k_1; \cdots; n_s, k_s]$ are arbitrary constants, real or complex, then

$$P_{0_+\ y}^{(\gamma',\ \beta')}\left[P_{0_+\ x}^{(\gamma,\ \beta)}\left\{\begin{matrix} x^{\delta-1} y^{\delta'-1}\left(x^h+c\right)^{-\eta}\left(y^{h'}+c'\right)^{-\eta'} S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\left[\begin{matrix} Z_1 x^{r_1} y^{r_1'}\left(x^h+c\right)^{-\eta_1}\left(y^{h'}+c'\right)^{-\eta_1'} \\ ,\ldots, Z_s x^{r_s} y^{r_s'}\left(x^h+c\right)^{-\eta_s}\left(y^{h'}+c'\right)^{-\eta_s'} \end{matrix}\right] \\ H_{p,\ q:\ p_1,\ q_1;\cdots;p_r,\ q_r}^{0,\ n:\ m_1,\ n_1;\cdots;m_r,\ n_r}\left[\begin{matrix} M_1 x^{h_1} y^{h_1'}\left(x^h+c\right)^{-\phi_1}\left(y^{h'}+c'\right)^{-\phi_1'}, \\ \cdots, M_r x^{h_r} y^{h_r'}\left(x^h+c\right)^{-\phi_r}\left(y^{h'}+c'\right)^{-\phi_r'} \end{matrix}\right] \end{matrix}\right\}\right]$$

$$= \frac{x^{\gamma+\delta}}{c^{\eta}\,[a(1-\beta)]^{\delta}}\,\Gamma\left(1+\frac{\gamma}{1-\beta}\right) \frac{y^{\gamma'+\delta'}}{c^{\eta'}\left[a'(1-\beta')\right]^{\delta'}}\,\Gamma\left(1+\frac{\gamma'}{1-\beta'}\right) \sum_{k_1}^{\left[\frac{n_1}{m_1}\right]} \cdots \sum_{k_s}^{\left[\frac{n_s}{m_s}\right]} \frac{(-n_1)_{m_1 k_1}}{k_1!} \cdots \frac{(-n_s)_{m_s k_s}}{k_s!}$$

$$A[n_1, k_1; \ldots; n_s, k_s] \frac{Z_1^{k_1} x^{r_1 k_1} y^{r_1' k_1}}{c^{\eta_1 k_1} c^{\eta_1' k_1} [a(1-\beta)]^{r_1 k_1} \left[a'(1-\beta')\right]^{r_1' k_1}}, \cdots, \frac{Z_s^{k_s} x^{r_s k_s} y^{r_s' k_s}}{c^{\eta_s k_s} c^{\eta_s' k_s} [a(1-\beta)]^{r_s k_s} \left[a'(1-\beta')\right]^{r_s' k_s}}$$

$$H^{0,\ n+4\ :\ m_1,\ n_1;\cdots;m_r,\ n_r;\ 1\ 0;\ 1\ 0}_{p+4,\ q+4\ :\ p_1,\ q_1;\cdots;p_r,\ q_r;\ 0\ 1;\ 0\ 1}\left[\begin{array}{c} \frac{M_1 x^{h_1} y^{h_1'}}{c^{\phi_1} c'^{\phi_1'}[a(1-\beta)]^{h_1}[a'(1-\beta')]^{h_1'}} \\ \vdots \\ \frac{M_r x^{h_r} y^{h_r'}}{c^{\phi_r} c'^{\phi_r'}[a(1-\beta)]^{h_r}[a'(1-\beta')]^{h_r'}} \\ \frac{x^k}{c[a(1-\beta)]^k} \\ \frac{y^{k'}}{c'[a'(1-\beta')]^{k'}} \end{array}\right| \begin{array}{l} (a_j;\psi_j',\cdots,\psi_j^{(r)},0),(1-\eta-\sum_{i=1}^s \eta_i k_i;\ \phi_1,\cdots,\phi_r,\ 1,0), \\ (b_j;\mu_j',\cdots,\mu_j^{(r)},0),(1-\eta-\sum_{i=1}^s \eta_i k_i;\ \phi_1,\cdots,\phi_r,\ 0,\ 0), \end{array}$$

$$\left(1-\delta-\sum_{i=1}^s r_i k_i,\ h_1,\cdots,h_r,\ k,0\right),\qquad \left(1-\delta'-\sum_{i=1}^s r_i' k_i;\ h_1',\cdots,h_r'\ 0,k'\right),$$

$$\left(-\frac{\gamma}{1-\beta}-\delta-\sum_{i=1}^s r_i k_i;\ h_1,\cdots,h_r,\ k,0\right),\quad \left(-\frac{\gamma'}{1-\beta'}-\delta'-\sum_{i=1}^s r_i' k_i;\ h_1',\cdots,h_r',0,k'\right),$$

$$\left.\begin{array}{l} \left(1-\eta'-\sum_{i=1}^s \eta_i' k_i;\ \phi_1',\phi_2'\cdots,\phi_r',\ 0,1\right)\ :\ (c_j',\tau_j')_{1,p_1};\ \ldots;\ (c_j^{(r)},\tau_j^{(r)})_{1,p_r};\ -\ ;\ - \\ \left(1-\eta'-\sum_{i=1}^s \eta_i' k_i;\ \phi_1',\cdots,\phi_r',\ 0,0\right)\ :\ (d_j',\chi_j')_{1,q_1};\ \ldots;\ (d_j^{(r)},\chi_j^{(r)})_{1,q_r};\ (0,1);\ (0,1) \end{array}\right]. \qquad (2.2)$$

This result can be obtained on the similar lines of the proof of the theorem 1.

3. SPECIAL CASES

(I) If $n=p$, $m_i=1$, $n_i=p_i$, $q_i=q_i+1$, $i=1,\cdots,r$ in the main result (2.1) multivariable H-function transforms to the generalized Lauricella function of several complex variables due to Srivastava and Daoust [9], we have

$$P_{0_+}^{(\gamma,\beta)}\left[x^{\delta-1}(x^h+c)^{-\eta} S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\left[Z_1 x^{r_1}(x^h+c)^{-\eta_1},\ldots,Z_s x^{r_s}(x^h+c)^{-\eta_s}\right]\right.$$

$$F^{p:\,p_1;\cdots;p_r}_{q:\,q_1;\cdots;q_r}\left[\begin{array}{l} \left[\left(1-a_j:\psi_j^{(1)},\cdots,\psi_j^{(r)}\right)\right]_{1,p}\ :\ \left[\left(1-c_j^{(1)},\ \tau_j^{(1)}\right)\right]_{1,p_1};\cdots \\ \left[\left(1-b_j:\mu_j^{(1)},\cdots,\mu_j^{(r)}\right)\right]_{1,q}\ :\ \left[\left(1-d_j^{(1)},\chi_j^{(1)}\right)\right]_{1,q_1};\cdots \end{array}\right.$$

$$\left.\left.\begin{array}{l} ;\ \left[\left(1-c_j^{(r)};\tau_j^{(r)}\right)\right]_{1,p_r}; \\ ;\ \left[\left(1-d_j^{(r)}\right);\chi_j^{(r)}\right]_{1,q_r}; \end{array}\ -M_1 x^{h_1}(x^h+c)^{-\phi_1},\cdots,\ -M_r x^{h_r}(x^h+c)^{-\phi_r}\right]\right]$$

$$=\frac{x^{\gamma+\delta}}{c^{\eta}[a(1-\beta)]^{\delta}}\,\Gamma\left(1+\frac{\gamma}{1-\beta}\right)\sum_{k_1}^{[n_1/m_1]}\ldots\sum_{k_s}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_s)_{m_sk_s}}{k_s!}A[n_1,k_1;\ldots;n_s,k_s]$$

$$\frac{Z_1^{k_1}x^{r_1k_1}}{c^{\eta_1k_1}[a(1-\beta)]^{r_1k_1}}\cdots\frac{Z_s^{k_s}x^{r_sk_s}}{c^{\eta_sk_s}[a(1-\beta)]^{r_sk_s}}\ \frac{\Gamma(\delta+\sum_{i=1}^s r_i k_i)}{\Gamma\left(\frac{\gamma}{1-\beta}+\delta+\sum_{i=1}^s r_i k_i\right)}\ \frac{\Gamma(\eta+\sum_{i=1}^s \eta_i k_i)}{\Gamma(\eta+\sum_{i=1}^s \eta_i k_i)}$$

$$F^{p+2:\, p_1;\cdots;p_r;\, 0}_{q+2:\, q_1;\cdots;q_r;\, 1}\left[\begin{matrix}\frac{-M_1 x^{h_1}}{c^{\phi_1}[a(1-\beta)]^{h_1}}\\ \vdots \\ \frac{-M_r x^{h_r}}{c^{\phi_r}[a(1-\beta)]^{h_r}} \\ \frac{-x^k}{c\,[a(1-\beta)]^k}\end{matrix}\right| \begin{matrix}[\delta+\sum_{i=1}^{s} r_i k_i;\ h_1,\cdots,h_r,\ k\ ],\ [\eta+\sum_{i=1}^{s}\eta_i k_i;\ \phi_1,\cdots,\phi_r,\ 1], \\ \left[\frac{\gamma}{1-\beta}+\delta+\sum_{i=1}^{s} r_i k_i;\ h_1,\cdots,h_r,\ k\right],\ [\eta+\sum_{i=1}^{s}\eta_i k_i;\ \phi_1,\cdots,\phi_r,\ 0],\end{matrix}$$

$$\left.\begin{matrix}[1-a_j:\psi_j^{(1)},\cdots,\psi_j^{(r)},\ 0]:\left[\left(1-c_j^{(1)},\ \tau_j^{(1)}\right)\right]_{1,p_1};\cdots;\left[\left(1-c_j^{(r)};\tau_j^{(r)}\right)\right]_{1,p_r};\ - \\ [1-b_j:\mu_j^{(1)},\cdots,\mu_j^{(r)},\ 0]:\left[\left(1-d_j^{(1)},\chi_j^{(1)}\right)\right]_{1,q_1};\cdots;\left[\left(1-d_j^{(r)}\right);\chi_j^{(r)}\right]_{1,q_r};(1,1)\end{matrix}\right]. \quad (3.1)$$

(II) If $\psi_j^{'},\cdots,\psi_j^{(r)}=\mu_j^{'},\cdots,\mu_j^{(r)}=\tau_j^{'},\cdots,\tau_j^{(r)}=\chi_j^{'},\cdots,\chi_j^{(r)}=\varepsilon$ in the main result (2.1), the multivariable H-function reduces to the multivariable G-function due to Saxena [10], we get

$$P_{0+}^{(\gamma,\ \beta)}\left[x^{\delta-1}\,(x^h+c)^{-\eta}S^{m_1,\ldots,m_s}_{n_1,\ldots,n_s}\left[Z_1x^{r_1}(x^h+c)^{-\eta_1},\ldots,Z_sx^{r_s}(x^h+c)^{-\eta_s}\right]\right.$$
$$\left. G^{0,\ n:\ m_1,\ n_1;\cdots;m_r,\ n_r}_{p,\ q:\ p_1,\ q_1;\cdots;p_r,\ q_r}\left[\begin{matrix}\left[M_1x^{h_1}(x^h+c)^{-\phi_1}\right]^{\frac{1}{\epsilon}}\\ \vdots \\ \left[M_rx^{h_r}(x^h+c)^{-\phi_r}\right]^{\frac{1}{\epsilon}}\end{matrix}\right|\begin{matrix}(a_j)_{1,\ p}:(c_j^{'})_{1,\ p_1};\ \ldots;(c_j^{(r)})_{1,\ p_r}\\ (b_j)_{1,\ q}:(d_j^{'})_{1,\ q_1};\ \ldots;(d_j^{(r)})_{1,\ q_r}\end{matrix}\right]\right]$$

$$=\frac{x^{\gamma+\delta}}{c^{\eta}\,[a(1-\beta)]^{\delta}}\ \Gamma\left(1+\frac{\gamma}{1-\beta}\right)\sum_{k_1}^{\left[\frac{n_1}{m_1}\right]}\cdots\sum_{k_s}^{\left[\frac{n_s}{m_s}\right]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_s)_{m_sk_s}}{k_s!}A[n_1,k_1;\ldots;n_s,k_s]\ \frac{Z_1^{k_1}x^{r_1k_1}}{c^{\eta_1k_1}[a(1-\beta)]^{r_1k_1}}$$

$$\cdots\frac{Z_s^{k_s}x^{r_sk_s}}{c^{\eta_sk_s}[a(1-\beta)]^{r_sk_s}}\ G^{0,\ \ n+2\ :\ m_1,\ n_1;\cdots;m_r,\ n_r;\ 1\ 0}_{p+2,\ q+2\ :\ p_1,\ q_1;\cdots;p_r,\ q_r;\ 0\ 1}\left[\begin{matrix}\left[M_1x^{h_1}(x^h+c)^{-\phi_1}\right]^{\frac{1}{\epsilon}}\\ \vdots \\ \left[M_rx^{h_r}(x^h+c)^{-\phi_r}\right]^{\frac{1}{\epsilon}}\\ \left[\frac{x^k}{c\,[a(1-\beta)]^k}\right]^{\frac{1}{\epsilon}}\end{matrix}\right|\begin{matrix}(a_j)_{1,\ p}:\\ (b_j)_{1,\ q}:\end{matrix}$$

$$\left.\begin{matrix}(1-\eta-\sum_{i=1}^{s}\eta_ik_i),\quad \left(1-\delta-\sum_{i=1}^{s}r_ik_i\right):(c_j')_{1,p_1};\ \ldots;(c_j^{(r)})_{1,\ p_r};\ - \\ (1-\eta-\sum_{i=1}^{s}\eta_ik_i),\left(1-\frac{\gamma}{1-\beta}-\delta-\sum_{i=1}^{s}r_ik_i\right):(d_j')_{1,q_1};\ \ldots;(d_j^{(r)})_{1,q_r};(0)\end{matrix}\right]. \quad (3.2)$$

(III) If n = p = q = 0 in the main result (2.1), the multivariable H-function reduces to the product of 'r' Fox's H-function, we obtain

$$P_{0+}^{(\gamma,\beta)}\left[\begin{array}{l} x^{\delta-1}\left(x^h+c\right)^{-\eta} S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\left[Z_1 x^{r_1}\left(x^h+c\right)^{-\eta_1},\ldots,Z_s x^{r_s}\left(x^h+c\right)^{-\eta_s}\right] \\ \prod_{i=1}^{r}\left\{H_{p_i,\,q_i}^{m_i,\,n_i}\left[M_i\, x^{h_i}\left(x^h+c\right)^{-\phi_i}\left|\begin{array}{c}\left(c_j^{(i)},\ \tau_j^{(i)}\right)_{1,\ p_i} \\ \left(d_j^{(i)},\chi_j^{(i)}\right)_{1,\ q_i}\end{array}\right.\right]\right\}\end{array}\right]$$

$$= \frac{x^{\gamma+\delta}}{c^{\eta}\,[a(1-\beta)]^{\delta}}\ \Gamma\left(1+\frac{\gamma}{1-\beta}\right)\sum_{k_1}^{\left[\frac{n_1}{m_1}\right]}\cdots\sum_{k_s}^{\left[\frac{n_s}{m_s}\right]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_s)_{m_sk_s}}{k_s!}A\left[n_1,k_1;\ldots;n_s,k_s\right]\frac{{Z_1}^{k_1}\,x^{r_1k_1}}{c^{\eta_1k_1}[a(1-\beta)]^{r_1k_1}},$$

$$\ldots,\frac{{Z_s}^{k_s}\,x^{rk_s}}{c^{\eta_sk_s}[a(1-\beta)]^{r_sk_s}}\quad H_{2,\ 2\ :\ p_1,\ q_1\ ;\cdots;\ p_r,\ q_r\ ;\ 0,\ 1}^{0,\ 2\ :\ m_1,\ n_1\ ;\cdots;\ m_r,\ n_r;\ 1,\ 0}\left[\begin{array}{c}\frac{M_1\,x^{h_1}}{c^{\phi_1}[a(1-\beta)]^{h_1}}\\ \vdots \\ \frac{M_r\,x^{h_r}}{c^{\phi_r}[a(1-\beta)]^{h_r}} \\ \frac{x^k}{c\,[a(1-\beta)]^k}\end{array}\right|\begin{array}{l}\left(1-\eta-\sum_{i=1}^{s}\eta_ik_i;\ \phi_1,\cdots,\phi_r,\ 1\right), \\ \left(1-\eta-\sum_{i=1}^{s}\eta_ik_i;\ \phi_1,\cdots,\phi_r,\ 0\right),\end{array}$$

$$\left.\begin{array}{l}\left(1-\delta-\sum_{i=1}^{s}r_ik_i;\ h_1,\cdots,h_r,\ k\right)\quad :\left(c_j',\ \tau_j'\right)_{1,\ p_1};\ \ldots\ ;\left(c_j^{(r)},\ \tau_j^{(r)}\right)_{1,\ p_r}\ ;\ - \\ \left(1-\frac{\gamma}{1-\beta}-\delta-\sum_{i=1}^{s}r_ik_i;\ h_1,\cdots,h_r,\ k\right)\ :\left(d_j',\ \chi_j'\right)_{1,\ q_1};\ \ldots\ ;\left(d_j^{(r)},\ \chi_j^{(r)}\right)_{1,\ q_r}\ ;\ (0,1)\end{array}\right]. \tag{3.3}$$

(IV) By giving suitable values to the parameters in the main results (2.1) and (2.2), we obtain the known results established by Chaurasia and Ghiya [1].

A number of other special cases can be obtained from the main results (2.1) and (2.2) by specializing the parameters.

4. APPLICATIONS

In view of the importance of the main results, we mention some interesting applications.

(I) When $\beta \to 1_-$, $\frac{\gamma}{1-\beta} \to \infty$ then main result (2.1) yields the Laplace transform result.

$$\lim_{\beta\to 1} P_{0+}^{(\gamma,\ \beta)}\left[\begin{array}{l} x^{\delta-1}\left(x^h+c\right)^{-\eta} S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\left[Z_1 x^{r_1}\left(x^h+c\right)^{-\eta_1},\ldots,Z_s x^{r_s}\left(x^h+c\right)^{-\eta_s}\right] \\ H_{p,\ q:\ p_1,\ q_1\ ;\cdots;\ p_r,\ q_r}^{0,\ n:\ m_1,\ n_1\ ;\cdots;\ m_r,\ n_r}\left[M_1x^{h_1}\left(x^h+c\right)^{-\phi_1},\cdots,M_rx^{h_r}\left(x^h+c\right)^{-\phi_r}\right]\end{array}\right]$$

$$= \frac{x^{\gamma+\delta}}{c^{\eta}\,[a\gamma]^{\delta}}\ \sum_{k_1}^{\left[\frac{n_1}{m_1}\right]}\cdots\sum_{k_s}^{\left[\frac{n_s}{m_s}\right]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_s)_{m_sk_s}}{k_s!}A\left[n_1,k_1;\ldots;n_s,k_s\right]\frac{{Z_1}^{k_1}\,x^{r_1k_1}}{c^{\eta_1k_1}[a\gamma]^{r_1k_1}},\cdots,\frac{{Z_s}^{k_s}\,x^{r_sk_s}}{c^{\eta_sk_s}[a\gamma]^{r_sk_s}}$$

$$H^{0,\ n+2\ :\ m_1,\ n_1;\cdots;m_r,\ n_r;\ 1\ 0}_{p+2,\ q+1\ :\ p_1,\ q_1;\cdots;p_r,\ q_r;\ 0\ 1}\left[\begin{matrix}\frac{M_1 x^{h_1}}{c^{\phi_1}[a\gamma]^{h_1}}\\ \vdots \\ \frac{M_r x^{h_r}}{c^{\phi_r}[a\gamma]^{h_r}}\\ \frac{x^k}{c\,[a\gamma]^k}\end{matrix}\middle| \begin{matrix}\left(a_j;\psi'_j,\cdots,\psi^{(r)}_j,0\right),\\ \left(b_j;\mu'_j,\cdots,\mu^{(r)}_j,0\right),\end{matrix}\right.$$

$$\begin{matrix}\left(1-\eta-\sum_{i=1}^s\eta_i k_i;\ \phi_1,\cdots,\phi_r,\ 1\right), & \left(1-\delta-\sum_{i=1}^s r_i k_i;\ h_1,\cdots,h_r,\ k\right) & : \\ \left(1-\eta-\sum_{i=1}^s\eta_i k_i;\ \phi_1,\cdots,\phi_r,\ 0\right), & - & : \end{matrix}$$

$$\left.\begin{matrix}\left(c'_j,\ \tau'_j\right)_{1,\ p_1};\ \ldots;\ \left(c^{(r)}_j,\ \tau^{(r)}_j\right)_{1,\ p_r};\ - \\ \left(d'_j,\ \chi'_j\right)_{1,\ q_1};\ \ldots;\ \left(d^{(r)}_j,\ \chi^{(r)}_j\right)_{1,\ q_r};\ (0,1)\end{matrix}\right]. \tag{4.1}$$

(II) When $\beta\to 1_-$, $\beta'\to 1_-$ result (2.2) yields the Laplace transform of two variables result.

$$\lim_{\substack{\beta\to1\\ \beta'\to1}} P^{(\gamma',\ \beta')}_{0_+\ y}\left[P^{(\gamma,\ \beta)}_{0_+\ x}\left\{\begin{matrix}x^{\delta-1}y^{\delta'-1}\left(x^h+c\right)^{-\eta}\left(y^{h'}+c'\right)^{-\eta'}\ S^{m_1,\ldots,m_s}_{n_1,\ldots,n_s}\left[\begin{matrix}Z_1 x^{r_1}y^{r'_1}\left(x^h+c\right)^{-\eta_1}\left(y^{h'}+c'\right)^{-\eta'_1},\\ \ldots,Z_s x^{r_s}y^{r'_s}\left(x^h+c\right)^{-\eta_s}\left(y^{h'}+c'\right)^{-\eta'_s}\end{matrix}\right]\\ H^{0,\ n:\ m_1,\ n_1;\cdots;m_r,\ n_r}_{p,\ q:\ p_1,\ q_1;\cdots;p_r,\ q_r}\left[\begin{matrix}M_1 x^{h_1}y^{h'_1}\left(x^h+c\right)^{-\phi_1}\left(y^{h'}+c'\right)^{-\phi'_1},\\ \cdots,M_r x^{h_r}y^{h'_r}\left(x^h+c\right)^{-\phi_r}\left(y^{h'}+c'\right)^{-\phi'_r}\end{matrix}\right]\end{matrix}\right\}\right]$$

$$=\frac{x^{\gamma+\delta}}{c^{\eta}[a\gamma]^{\delta}}\ \frac{y^{\gamma'+\delta'}}{c'^{\eta'}[a'\gamma']^{\delta'}}\ \sum_{k_1}^{\left[\frac{n_1}{m_1}\right]}\ldots\sum_{k_s}^{\left[\frac{n_s}{m_s}\right]}\frac{(-n_1)_{m_1k_1}}{k_1!}\ldots\frac{(-n_s)_{m_sk_s}}{k_s!}A[n_1,k_1;\ldots;n_s,k_s]\frac{Z_1^{k_1}x^{r_1k_1}y^{r'_1k_1}}{c^{\eta_1k_1}c^{\eta'_1k_1}[a\gamma]^{r_1k_1}[a'\gamma']^{r'_1k_1}}$$

$$\cdots\frac{Z_s^{k_s}x^{rk_s}y^{r'_sk_s}}{c^{\eta_sk_s}c^{\eta'_sk_s}[a\gamma]^{r_sk_s}[a'\gamma']^{r'_sk_s}}$$

$$H^{0,\ n+4\ :\ m_1,\ n_1;\cdots;m_r,\ n_r;\ 1,0;\ 1,\ 0}_{p+2,\ q+2\ :\ p_1,\ q_1;\cdots;p_r,\ q_r;\ 0,\ 1;0,\ 1}\left[\begin{matrix}\frac{M_1 x^{h_1}y^{h'_1}}{c^{\phi_1}c^{\phi'_1}[a\gamma]^{h_1}[a'\gamma']^{h'_1}}\\ \vdots\\ \frac{M_r x^{h_r}y^{h'_r}}{c^{\phi_r}c^{\phi'_r}[a\gamma]^{h_r}[a'\gamma']^{h'_r}}\\ \frac{x^k}{c\,[a\gamma]^k}\\ \frac{y^{k'}}{c'\,[a'\gamma']^{k'}}\end{matrix}\middle|\begin{matrix}\left(a_j;\psi'_j,\cdots,\psi^{(r)}_j,0\right),\\ \left(b_j;\mu'_j,\cdots,\mu^{(r)}_j,0\right),\end{matrix}\right.$$

$$\begin{matrix}\left(1-\eta-\sum_{i=1}^s\eta_i k_i;\ \phi_1,\cdots,\phi_r,\ 1,\ 0\right),\left(1-\delta-\sum_{i=1}^s r_i k_i;\ h_1,\cdots,h_r,\ k,0\right),\left(1-\delta'-\sum_{i=1}^s r'_i k_i;\ h'_1,\cdots,h'_r\ 0,k'\right),\\ \left(1-\eta-\sum_{i=1}^s\eta_i k_i;\ \phi_1,\cdots,\phi_r,\ 0,0\right),\left(1-\eta'-\sum_{i=1}^s\eta'_i k_i;\ \phi'_1,\cdots,\phi'_r,\ 0,0\right),\qquad -\ ,\end{matrix}$$

$$\left.\begin{matrix} \left(1-\eta'-\sum_{i=1}^{s}\eta_i'k_i;\ \phi_1',\cdots,\phi_r',\ 0,1\right):\left(c_j',\ \tau_j'\right)_{1,\ p_1};\ \ldots;\ \left(c_j^{(r)},\ \tau_j^{(r)}\right)_{1,\ p_r};\ -;\ - \\ -\qquad :\ \left(d_j',\ \chi_j'\right)_{1,\ q_1};\ \ldots;\ \left(d_j^{(r)},\ \chi_j^{(r)}\right)_{1,\ q_r};\ (0,1);\ \ (0,1) \end{matrix}\right] \tag{4.2}$$

5. CONCLUSION

In this article we provide the composition formulas of pathway fractional integral operator with generalized polynomial and H-function of r variable which is expressed in H-function of r +1 variables. Pathway operator is related to pathway model and various fractional integral operators, generalized polynomial and multivariable H-function are general in nature. As result, many new and known results can be obtained from our findings.

REFERENCES

[1] **Chaurasia, V. B. L., and Ghiya, Neeti,** 2010, "Pathway fractional integral operator pertaining to special functions," Global Journal of science Frontier Research, 10, pp. 79-83.

[2] **Nair, Seema S.,** 2009, "Pathway fractional integration operator," Fractional calculus & Applied analysis, 12(3), pp. 237-252.

[3] **Mathai, M., and Haubold, H. J.,** 2008, "On generalized distributions and pathways," Physics letters, 372, pp. 2109-2113.

[4] **Mathai, M., and Haubold, H. J.,** 2007, "Pathway model, Super statistics, Tsalle's statistics and a generalized measure of entropy," Physica A, 375, pp. 110-122.

[5] **Mathai, M.,** 2005, "A pathway to matrix-variate gamma and normal densities", Linear Algebra and its Applications," 396, pp. 317-328.

[6] **Srivastava, H. M., and Panda, R.,** 1996, "Some bilateral generating functions for a class of generalized hyper geometric polynomials," J. Raine Angew.Math., 283/284, pp. 265-274.

[7] **Srivastava, H. M.,** 1985, "A multilinear generating function for the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials," Pacific J. Math, 117, pp. 183-191.

[8] **Srivastava, H. M., Gupta K. C., and Goyal, S. P.,** 1982, "The H-functions of one and two variables with applications," South Asian publishers, New Delhi, Madras.

[9] **Srivastava, H. M., and Daoust, Martha,** 1969, "Certain generalized

Neumann equations associated with the Kampe de Ferrite function," Nederl. Akad. Wetensch. Proc. Ser. A-72-Indag. Math., 31, pp. 449-457.

[10] **Saxena, R. K.,** 1960, "An integral involving G-function," Proc. Nat. Inst. Sci. India, 26A, pp. 661-664.