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TD-Polynomials of Paths and Cycles - A New Approach

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Abstract

A *hypergraph* is an ordered pair $H = (V, E)$ where V is a finite nonempty set called vertices and E is a collection of subsets of V, called *hyper edges* or simply *edges.* For a graph $G = (V, E)$, the *open neighbourhood hypergraph* of G, denoted by $ONH(G)$, is the hypergraph with vertex set V and edge set $\{N_G(x)|x \in V\}$. A *vertex cover* in $ONH(G)$ is a set of vertices intersecting every edge of $ONH(G)$, which is equivalent to a *total dominating set* in G. Using the interplay between total dominating sets and vertex cover in hypergraphs, we determine the total domination polynomial of paths and cycles.

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1. Introduction

A *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a finite non-empty set and $E(G)$ is a collection of 2-point subsets of V. The sets $V(G)$ and $E(G)$ are called the vertex set and edge set of G respectively. The *open neighbourhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V|uv \in E(G)\}$. If the graph G is clear from the context, we write $N(v)$ rather than $N_G(v)$. All graphs considered in this paper are connected and contain no parallel edges, but may have loops. Notations and definitions not given here can be found in [1], [2], [4] and [6]. A *hypergraph* $H = (V, E)$ is a finite nonempty set $V = V(H)$ of elements called *vertices*, together with a finite multi set $E = E(H)$ of subsets of V, called *hyper edges* or simply *edges.* The *order* and *size* of H are $|V|$ and $|E|$, respectively. A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover*) if T has a nonempty intersection with every edge of H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. For further information on hypergraphs refer [2], [9]. Let $\mathcal{C}(H, i)$ be the family of vertex covering sets of H with cardinality i and let $c(H, i) = |\mathcal{C}(H, i)|$. The polynomial $\mathcal{C}(H, x) = \sum_{i=\tau(H)}^{|V(H)|} c(H, i)x^i$ is defined as *vertex cover polynomial* of H. For a graph $G = (V, E)$, the *ONH(G)* or H_G is the *open neighbourhood hypergraph* of G; $H_G = (V, C)$ is the hypergraph with vertex set $V(H_G) = V$ and with edge set $E(H_G) = C = \{N_G(x)|x \in V\}$, consisting of the open neighbourhoods of vertices of V in G. A *total dominating set,* abbreviated TD-set, of a graph $G = (V, E)$ with no isolated vertex is set S of vertices of G such that every vertex of G is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. Let $\mathcal{D}_t(G, i)$ be the family of total dominating sets of G with cardinality i and let $d_t(G, i) = |\mathcal{D}_t(G, i)|$. The polynomial $\mathcal{D}_t(G, x) = \sum_{i=\gamma_t(G)}^{|V(G)|} d_t(G, i)x^i$ is defined as *total domination polynomial* of G [8].

We need the following theorems to prove the main results of this paper.

Theorem 1.1. [5] The ONH of a connected bipartite graph consists of two components, while the ONH of a connected graph that is not bipartite is connected.

Theorem 1.2. [6] If G is a graph with no isolated vertex and H_G is the ONH of G, then $\gamma_t(G) = \tau(H_G)$.

Theorem 1.3. [3] Let G be a graph and $L = \{x \in V(G)|\, xx \in E(G)\}$. Then $\mathcal{C}(G, x) = x^{|L|}\mathcal{C}(G - L, x)$.

Theorem 1.4. [3] Let $G = G_1 \cup G_2$ be the union of two graphs G_1 and G_2. Then $\mathcal{C}(G, x) = \mathcal{C}(G_1, x)\mathcal{C}(G_2, x)$.

Theorem 1.5. [3] For the path graph P_n, where $n > 1$, we have

$$\mathcal{C}(P_n, x) = \sum_{i=0}^{n} \binom{i+1}{n-i} x^i.$$

Theorem 1.6. [3] For the cycle graph C_n, where $n \geq 3$, we have

$$\mathcal{C}(C_n, x) = \sum_{i=1}^{n} \frac{n}{i} \binom{i}{n-i} x^i.$$

Theorem 1.7. **[7]** The total domination polynomial of a connected bipartite graph G is the product of the vertex cover polynomials of the two components of H_G, while the total domination polynomial of a connected graph that is not bipartite is the vertex cover polynomial of H_G.

2. Main Results

Lemma 2.1. Let $P_n^{'}$ be the graph shown in Figure 1.

Then,

$$\mathcal{C}(P_n^{'}, x) = x\mathcal{C}(P_{n-1}, x) = x\sum_{i=0}^{n-1}\binom{i+1}{n-(i+1)}x^i.$$

Proof. The proof follows immediately from Theorem 1.5. ■

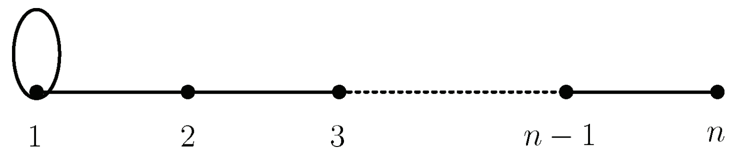

Figure 1: The Graph $P_n^{'}$

Lemma 2.2. Let $P_n^{''}$ be the graph shown in Figure 2.

Then,

$$\mathcal{C}(P_n^{''}, x) = x^2\mathcal{C}(P_{n-2}, x) = x^2\sum_{i=0}^{n-2}\binom{i+1}{n-(i+2)}x^i.$$

Proof. The proof follows from Theorem 1.5. ■

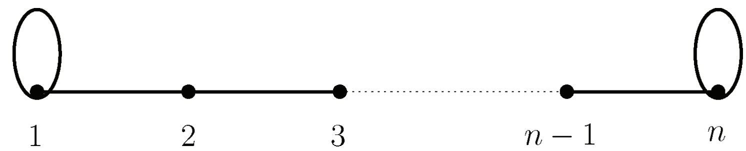

Figure 2: The Graph $P_n^{''}$

Using the hypergraph terminology, we can easily find the total domination polynomials of paths and cycles.

Theorem 2.3. For $n \geq 1$, $D_t(P_{2n}, x) = \left[\mathcal{C}(P_n^{'}, x)\right]^2$.

Proof. Since P_{2n} is bipartite, the open neighbourhood hypergraph of P_{2n} has two components say G_1 and G_2. Let $V(P_{2n}) = \{1, 2, 3, \dots, 2n\text{-}1, 2n\}$. Then the edge sets of $ONH(P_{2n})$ are $E(G_1) = \{\{1, 3\}, \{3, 5\}, \{5, 7\}, \dots, \{2n\text{-}3, 2n\text{-}1\}, \{2n\text{-}1\}\}$ and $E(G_2) = \{\{2\}, \{2, 4\}, \{4, 6\}, \dots, \{2n\text{-}2, 2n\}\}$. Clearly G_1 is isomorphic to G_2. Using the terminology in [2], the graph G_2 can be drawn as shown in Figure 3.

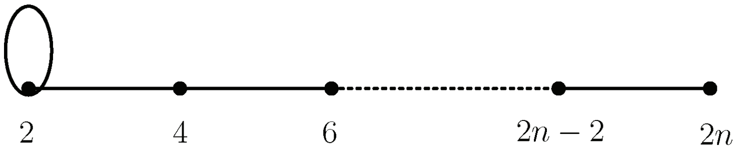

Figure 3: The Graph G_2

Since G_2 is isomorphic to $P_n^{'}$ and G_1 is isomorphic to G_2, the proof follows from Theorem 1.4 and 1.7. ■

Theorem 2.4. For $n \geq 1$, the total domination polynomial of path P_{2n} is,

$$D_t(P_{2n}, x) = x^2 \left[\sum_{i=0}^{n-1} \binom{i+1}{n-(i+1)} x^i \right]^2 .$$

Proof. The proof follows from Lemma 2.1 and Theorem 2.3. ■

Theorem 2.5. For $n \geq 1$, $D_t(P_{2n+1}, x) = \mathcal{C}(P_{n+1}, x)\mathcal{C}(P_n^{''}, x)$.

Proof. Let the vertex set of P_{2n+1} is $\{1, 2, 3, \dots, 2n\text{-}1, 2n, 2n\text{+}1\}$. Then the open neighbourhood hypergraph of P_{2n+1} has two components G_1 and G_2 with edge sets $E(G_1) = \{\{1, 3\}, \{3, 5\}, \{5, 7\}, \dots, \{2n\text{-}3, 2n\text{-}1\}, \{2n\text{-}1, 2n\text{+}1\}\}$ and $E(G_2) = \{\{2\}, \{2, 4\}, \{4, 6\}, \dots, \{2n\text{-}2, 2n\}, \{2n\}\}$. The graph G_2 can be drawn as in Figure 4.

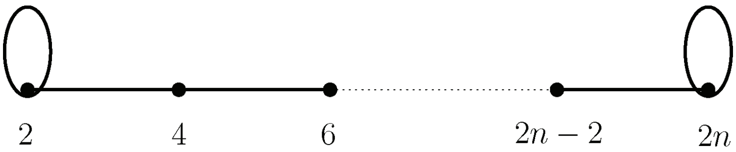

Figure 4: The Graph G_2

Let $\{\{1, 3\}, \{3, 5\}, \{5, 7\}, \dots, \{2n\text{-}3, 2n\text{-}1\}, \{2n\text{-}1, 2n\text{+}1\}\}$ be the edge set of the path P_{n+1}. Since $E(G_1) = E(P_{n+1})$, a set S is a vertex cover of G_1 if and only if S is a vertex cover of P_{n+1}. Since G_2 is isomorphic to $P_n^{''}$, by Theorems 1.4 and 1.7, $D_t(P_{2n+1}, x) = \mathcal{C}(G_1, x)\mathcal{C}(G_2, x) = \mathcal{C}(P_{n+1}, x)\mathcal{C}(P_n^{''}, x)$. This completes the proof. ■

Theorem 2.6.

$$D_t(P_{2n+1}, x) = x^2 \left[\sum_{i=0}^{n+1} \binom{i+1}{n+1-i} x^i \right] \left[\sum_{i=0}^{n-2} \binom{i+1}{n-(2+i)} x^i \right]$$

Proof. The proof follows immediately from 1.4, 1.5, 1.7 and 2.5. ■

Theorem 2.7. $D_t(C_{2n}, x) = [\mathcal{C}(C_n, x)]^2$.

Proof. Let $V(C_{2n}) = \{1, 2, \ldots, 2n\}$. Since C_{2n} is bipartite, its open neighbourhood hypergraph has two cycles, say C' and C'' with edge sets $E(C') = \{\{2, 4\}, \{4, 6\}, \ldots, \{2n\text{-}2, 2n\}, \{2n, 2\}\}$ and $E(C'') = \{\{1, 3\}, \{3, 5\}, \ldots, \{2n\text{-}3, 2n\text{-}1\}, \{2n\text{-}1, 1\}\}$. Since C' and C'' are isomorphic to the cycle C_n, the proof follows by Theorems 1.4 and 1.7. ■

Theorem 2.8.

$$D_t(C_{2n}, x) = \left[\sum_{i=1}^{n} \frac{n}{i} \binom{i}{n-i} x^i\right]^2 .$$

Proof. The proof follows from Theorems 1.6 and 2.7. ■

Theorem 2.9. If n is odd, then

$$D_t(C_n, x) = \sum_{i=1}^{n} \frac{n}{i} \binom{i}{n-i} x^i .$$

Proof. Since the open neighbourhood hypergraph of a cycle of odd order is isomorphic to itself, the proof follows from Theorem 1.6. ■

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