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Variants of Chebyshev Method with Ninth-Order Convergence for Solving Nonlinear Equations

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Abstract

This paper develops variants Chebyshev method by applying Hermite interpolation and finite difference to eliminate the second derivative appearing in the Chebyshev method. We prove that the this method has a ninth-order convergence. The effeciency index of this method is $9^{\frac{1}{5}} \approx 1.5518$. Some numerical example ilustrate that new method are more efficient and perform better than other methods.

AMS subject classification: 65G50, 65H04.
Keywords: Chebyshev method, efficiency index, finite differences, Hermite interpolation, order of convergence.

1. Introduction

One of the most frequently occurring problems in scientific work is to find the roots of nonlinear equations of the form

$$f(x) = 0. \tag{1.1}$$

Analytical method for solving such equation (1.1) are almost non-existent and therefore, it is only possible to obtain aproximate solution by relying on numerical methods based on iterative procedure. One of the best known and probably the most used method for solving equation (1.1) is the Newton method. The Newton method is given as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots, \text{ and } \quad f'(x_n) \neq 0. \tag{1.2}$$

The order of convergence of the equation (1.2) is quadratic for simple roots [2]. To accelarate the convergence of (1.2) many authors have modified it as we can see in [5, 8, 10, 12, 16, 17].

The Chebyshev method is another well-known iterative method, which is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \frac{1}{2}\frac{f(x_n)f''(x_n)}{(f'(x_n))^2}\right), \tag{1.3}$$

which has third-order convergence [1, 4]. Kou et al. in [11] proposed a method which is free from second-derivative by approximating $f''(x_n)$ in (1.3) by a finite difference so we get two-step iterative method with the same order but free second derivative.The other methods modified from (1.3) having a four-order convergence can be seen in [3, 7, 9].

In this paper, we present the combination of Newton method and the Chebyshev method into three-step iteration method. We also incorporate finite difference to approximate the second derivative in second step and Herimite interpolation to approximate the first derivative in the third step. The discussion of the new method and their convergence and analysis are carried out in Section 2. Then, in Section 3 we perform numerical simulations using some test functions, and compare the new method with some other methods.

2. Proposed Methods

In this section, we define new three-step method combination from equation (1.2) and (1.3). The new method is given is

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2.1}$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)}\left(1 + \frac{1}{2}\frac{f(y_n)f''(y_n)}{(f'(y_n))^2}\right), \tag{2.2}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \tag{2.3}$$

First we replace $f''(y_n)$ from (2.2) with a finite difference [13, 14] scheme in what follows:

$$f''(y_n) \approx \frac{f'(y_n) - f'(x_n)}{y_n - x_n}. \tag{2.4}$$

Furthermore, we approximate $f'(z_n)$ by a derivative of Hermite interpolation polynomial $H_3(x)$ so as to meet the interpolation conditions $H_3(x_n) = f(x_n)$, $H_3(y_n) =$

$f(y_n)$, $H_3(z_n) = f(z_n)$, and $H_3'(x_n) = f'(x_n)$. That is

$$\begin{aligned} H_3'(z_n) = & -\frac{3x_n - 2y_n - z_n}{(x_n - y_n)^2(x_n - z_n)} f(x_n) + \frac{(x_n - z_n)^2}{(y_n - x_n)^2(y_n - z_n)} f(y_n) \\ & - \frac{x_n + 2y_n - 3z_n)}{(z_n - x_n)(z_n - y_n)} f(z_n) - \frac{y_n - z_n}{y_n - x_n} f'(x_n). \end{aligned} \tag{2.5}$$

Simplifying equation(2.5) yields

$$H_3'(z_n) = 2f[x_n, z_n] + f[y_n, z_n] - 2f[x_n, y_n] + (y_n - z_n) f[y_n, x_n, x_n]. \tag{2.6}$$

where

$$f[x_n, z_n] = \frac{f(z_n) - f(x_n)}{z_n - x_n}, \tag{2.7}$$

$$f[y_n, z_n] = \frac{f(z_n) - f(y_n)}{z_n - y_n}, \tag{2.8}$$

$$f[x_n, y_n] = \frac{f(y_n) - f(z_n)}{y_n - x_n}, \tag{2.9}$$

$$f[y_n, x_n, x_n] = \frac{f[y_n, x_n] - f'(x_n)}{y_n - x_n}. \tag{2.10}$$

Let $f'(z_n) \approx H_3(z_n)$, and substituting equation (2.4) into (2.2) and (2.6) into (2.3), we obtain

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2.11}$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)}\left(1 - \frac{f'(y_n)f'(x_n)f(y_n) - f(y_n)(f'(x_n))^2}{2f(x_n)(f'(y_n))^2}\right), \tag{2.12}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{2f[x_n, z_n] + f[y_n, z_n] - 2f[x_n, y_n] + (y_n - z_n) f[x_n, x_n, y_n]}. \tag{2.13}$$

We prove the following convergence theorem for new method presented by scheme (2.11), (2.12), and (2.13).

Theorem 2.1. Assume that functions f is sufficiently differentiable and f has a simple root $\alpha \in I$. If the initial point x_0 is sufficiently close to α, then the method of iteration in equations (2.11)–(2.13) has ninth-order convergence and satisfying the following error equation:

$$e_{n+1} = (-\frac{3}{2} c_2^3 c_3 c_4) e_n^9 + \mathcal{O}(e_n^{10}),$$

where $e_n = x_n - \alpha$ and $c_k = \dfrac{f^{(k)}(\alpha)}{k! f'(\alpha)}$ for $k = 2, 3, 4, \ldots, 9$.

Proof. Let α be simple root of the equation $f(x) = 0$, then $f'(\alpha) \neq 0$. Furthermore, using Taylor expansion of the $f(x_n)$ about $x_n = \alpha$, we get

$$f(x_n) = f(\alpha) + f'(\alpha)(x_n - \alpha) + \frac{f''(\alpha)}{2!}(x_n - \alpha)^2 + \frac{f^{(3)}(\alpha)}{3!}(x_n - \alpha)^3 + \frac{f^{(4)}(\alpha)}{4!}(x_n - \alpha)^4 + \cdots + \mathcal{O}(x_n - \alpha)^{10}. \quad (2.14)$$

Because $f(\alpha) = 0$ so that equation (2.14) can be rewritten in the form of

$$f(x_n) = f'(\alpha)e_n + \frac{1}{2!}f''(\alpha)e_n^2 + \frac{1}{3!}f^{(3)}(\alpha)e_n^3 + \frac{1}{4!}f^{(4)}(\alpha)e_n^4 + \cdots + \mathcal{O}(e_n^{10}), \quad (2.15)$$

where $c_k = \dfrac{f^{(k)}(\alpha)}{k!f'(\alpha)}$.

Furthermore, in the same way the Taylor expansion again for $f'(x_n)$ about $x_n = \alpha$, so after simplification, we obtain

$$f'(x_n) = f'(\alpha)\big(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots + \mathcal{O}(e_n^{10})\big). \quad (2.16)$$

From equation (2.15) and (2.16), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + \cdots + \mathcal{O}(e_n^{10}). \quad (2.17)$$

Using equation (2.17) in (2.11), we obtain

$$y_n = \alpha + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + \cdots + \mathcal{O}(e_n^{10}). \quad (2.18)$$

Expanding $f(y_n)$ and $f'(y_n)$ about $y_n = \alpha$ and from (2.18), we have

$$f(y_n) = f'(\alpha)(c_2e_n^2 + 2(c_3 + c_2^2)e_n^3 + (5c_2^4 - 7c_2c_3 + 3c_4)e_n^4 + \cdots + \mathcal{O}(e_n^{10})). \quad (2.19)$$

and

$$f'(y_n) = f'(\alpha)\big(1 + 2c_2^2e_n^2 + 4(c_2c_3 - c_2^3)e_n^3 + \cdots + \mathcal{O}(e_n^9)\big). \quad (2.20)$$

Dividing (2.19) by (2.20) gives us

$$\frac{f(y_n)}{f'(y_n)} = c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (3c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + \cdots + \mathcal{O}(e_n^{10}). \quad (2.21)$$

Hence, from (2.15), (2.16), (2.19), and (2.20), we have

$$f'(y_n)f'(x_n)f(y_n) - (f'(x_n))^2f(y_n) = -2c_2^2e_n^3 + (2c_2^3 - 7c_2c_3)e_n^4 + \cdots + \mathcal{O}(e_n^{10}), \quad (2.22)$$

and

$$2f(x_n)(f'(y_n))^2 = 2e_n + 2c_2^2e_n^2 + 2(4c_2^2 + c_3)e_n^3 + \cdots + \mathcal{O}(e_n^{10}). \quad (2.23)$$

Similarly, we obtain

$$\frac{f'(y_n)f'(x_n)f(y_n)-(f'(x_n))^2f(y_n)}{2f(x_n)(f'(y_n))^2}=-2c_2^2e_n2+(2c_2^3-\frac{7}{2}c_2c_3)e_n^3+\cdots+\mathcal{O}(e_n^{10}). \tag{2.24}$$

Substituting equation (2.18), (2.21), and (2.24) into (2.12) give us

$$z_n=\alpha-\frac{3}{2}c_2^2c_3e_n^5+\Big(2c_2^5+\frac{13}{2}c_2^3c_3-6c_2c_3^2-2c_2^2c_4\Big)e_n^6+\cdots+\mathcal{O}(e_n^{10}), \tag{2.25}$$

Applying Taylor expansion of $f(z_n)$ about $z_n=\alpha$, we get

$$f(z_n)=f'(\alpha)\Big(-\frac{3}{2}c_2^2c_3e_n^5+\Big(2c_2^5-6c_2c_3^2-2c_2^2c_4+\frac{13}{2}c_2^3c_3\Big)e_n^6+\cdots+\mathcal{O}(e_n^{10})\Big). \tag{2.26}$$

From equation (2.15), (2.19), (2.18), (2.25), and (2.26), we have

$$f[x_n,z_n]=f'(\alpha)\big(1+c_2e_n+c_3e_n^2+c_4e_n^3+c_5e_n^4+\cdots+\mathcal{O}(e_n^{10})\big), \tag{2.27}$$

$$f[y_n,z_n]=f'(\alpha)\big(1+c_2^2e_n^2+\big(2c_2c_3-2c_2^3\big)e_n^3+\cdots+\mathcal{O}(e_n^{10})\big), \tag{2.28}$$

$$f[x_n,y_n]=f'(\alpha)\big(1+c_2e_n+(c_2^2+c_3)e_n^2+(-2c_2^3+3c_2c_3+c_4)e_n^3+\cdots+\mathcal{O}(e_n^{10})\big), \tag{2.29}$$

and

$$f[x_n,x_n,y_n]=f'(\alpha)(c_2+2c_3e_n+(c_2c_3+3c_4)e_n^2+\cdots+\mathcal{O}(e_n^{10})). \tag{2.30}$$

From equation (2.18), (2.25), (2.27), (2.28), (2.29), and (2.30), we get

$$2f[x_n,z_n]+f[y_n,z_n]-2f[x_n,y_n]+(y_n-z_n)f[x_n,x_n,y_n]=f'(\alpha)\Big(1+\Big(\frac{64c_2^3}{c_2^2}-96c_3^2+c_2c_4-14c_2^4+64c_2^2c_3-\frac{16c_2^4}{c_2^4}\Big)e_n^4+\cdots+\mathcal{O}(e_n^{10})\Big) \tag{2.31}$$

Dividing (2.26) by (2.31), we have

$$\frac{f(z_n)}{2f[x_n,z_n]+f[y_n,z_n]-2f[x_n,y_n]+(y_n-z_n)f[x_n,x_n,y_n]}=-\frac{3}{2}c_2^2c_3e_n^5+\cdots+\mathcal{O}(e_n^{10}) \tag{2.32}$$

Substituting equation (2.25) and (2.32) into (2.13), we obtain

$$x_{n+1}=\alpha-\Big(\frac{3}{2}c_2^3c_3c_4\Big)e_n^9+\mathcal{O}(e_n^{10}). \tag{2.33}$$

Therefore $e_{n+1}=x_{n+1}-\alpha$, then from (2.33) we get

$$e_{n+1}=\Big(-\frac{3}{2}c_2^3c_3c_4\Big)e_n^9+\mathcal{O}(e_n^{10}). \tag{2.34}$$

The proof is completed. ■

3. Numerical Examples

In this section some numerical simulations are performed to compare Chebyshev-Hermite method with some other methods, such as Newton's method (NM), Chebyshev's method (CM), Behl-Kanwar's method (BKM, [3]), and Noor et al. Method (NeM, [15]). The following examples are used for numerical testing:

$$\begin{aligned}
f_1(x) &= x^2 - (1-x)^5, && \alpha \in (0.0, 0.5),\\
f_2(x) &= \sin^2(x) - x^2 + 1, && \alpha \in (1.0, 1.5),\\
f_3(x) &= 10xe^{-x^2} - 1, && \alpha \in (1.5, 2.0),\\
f_4(x) &= (x+2)e^{x} - 1, && \alpha \in (-1.0, 0.0),\\
f_5(x) &= xe^{x^2}\sin^2(x) + 3\cos(x) + 5, && \alpha \in (-1.5, -1.0),\\
f_6(x) &= x^3 - x + 3, && \alpha \in (-2.0, -1.5).
\end{aligned}$$

And the computational order convergence (COC) can be approximated by using the following equation:

$$COC = \frac{\ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln|(x_n-\alpha)/(x_{n-1}-\alpha)|}. \tag{3.1}$$

Calculations using software with 800 digits of accuracy and tolerance $\epsilon = 1.0 \times 10^{-100}$. The stoping criteria of the iteration are $|x_{n+1} - x_n| < \epsilon$ and $|f(x_{n+1})| < \epsilon$, x_{n+1} is taken as the exact root α computed.

In Table 1, we give initial value (x_0), number of iteration (N), and the computational order of convergence (COC). Table 1 shows a comparison of the number of iterations and COC several methods to resolve the above functions including Newton's method (NM), Chebyshev's method (CM), Behl-Kanwar's method (BKM), Noor et al. Method (NeM) and Chebyshev-Hermite's method (CHM) for some given initial values.

The computational result presented in Table 1 generally shown that the CHM has the number of iterations less when compared to the other methods, such as Newton's method, Chebyshev's method, and Behl-Kanwar's method. This means that CHM has a better efficiency in computing process than the three methods. From Table 1 we observe that the computational order of convergence (COC) perfectly coincides with the theoretical results at Theorem 1. The results presented in Table 1 show that the CHM has higher convergence order compared to the three methods. Several numerical examples are presented and compared with Noor et al. method [15] to ilustrate the efficiency and accuracy of CHM. Based on Table 1, we observe that CHM is comparable with the Noor et al. method [15].

4. Conclusion

In this paper we present the variants of Chebyshev's method by removing the second derivative using finite difference. This method requires three functions and two first

Table 1: Comparison Number of Iteration and *COC*

$f(x)$	x_0	*N*					*COC*				
		NM	CM	BKM	NeM	CHM	NM	CM	BKM	NeM	CHM
$f_1(x)$	-0.5	10	7	5	3	3	2.00	3.00	4.00	8.87	9.00
	0.0	9	5	5	3	3	2.00	3.00	4.00	9.00	9.00
	0.5	7	5	4	3	3	2.00	3.00	4.00	9.00	9.00
	1.0	8	5	5	3	3	2.00	3.00	4.00	9.00	9.00
$f_2(x)$	0.6	10	9	6	12	3	2.00	3.00	4.00	8.99	9.00
	1.0	8	7	5	3	3	2.00	3.00	4.00	9.00	9.00
	1.5	7	5	4	3	2	2.00	3.00	4.00	9.00	8.92
	2.1	8	6	4	3	3	2.00	3.00	4.00	9.00	9.00
$f_3(x)$	1.2	8	6	4	3*	3	2.00	3.00	4.00	9.00	9.00
	1.4	8	5	4	3*	3	2.00	3.00	4.00	9.00	9.00
	1.8	7	5	4	3*	3	2.00	3.00	4.00	9.00	9.00
	2.0	8	6	5	3*	3	2.00	3.00	4.00	9.00	9.00
$f_4(x)$	-2.0	16	–	9	–	5	2.00	3.00	4.00	–	8.99
	-1.0	9	6	5	3	3	2.00	3.00	4.00	9.00	9.00
	0.0	8	5	4	3	3	2.00	3.00	4.00	9.00	9.00
	2.0	8	6	4	3	3	2.00	3.00	4.00	9.00	9.00
$f_5(x)$	-2.0	11	7	6	3	4	2.00	3.00	4.00	9.00	9.00
	-1.4	8	5	4	3	3	2.00	3.00	4.00	9.00	9.00
	-1.0	8	5	4	3	3	2.00	3.00	4.00	9.00	9.00
	-0.8	9	6	5	3	3	2.00	3.00	4.00	9.00	9.00
$f_6(x)$	-1.9	7	5	4	3	2	2.00	3.00	4.00	9.00	9.86
	-1.7	6	4	3	2	2	2.00	3.00	4.00	9.00	9.99
	-0.9	10	19	6	4	3	2.00	3.00	4.00	9.00	9.96
	-0.7	13	21	7	21	4	2.00	3.00	4.00	9.00	9.00

derivative evaluations per iteration. We have that the order convergence of this method is nine. Analysis of the efficiency shows that this method is better than Newton's method, and Chebyshev's method, and Behl-Kanwar's method. We have that new method defined by (2.11)–(2.13) has the efficiency index $9^{\frac{1}{5}} \approx 1.5518$, wich is much better than the Newton's method $2^{\frac{1}{2}} \approx 1.4142$ and Chebyshev's method $3^{\frac{1}{3}} \approx 1.4422$.

References

[1] S. Amat, S. Busquier, J. M. Guiterres and M. A. Hernandes, On the global convergence of Chebyshev's iterative method, *Journal of Computational and Applied Mathematics*, 220(2008), 17–21.

[2] K. E. Atkinson, 1987. *An introduction to numerical analysis, 2^{nd} ed.*, John Wiley and Sons, New York.

[3] R. Behl and V. Kanwar, Variant of Chebyshev's methods with optimal order convergence, *Tamsui Oxford Journal of Information and Mathematical Sciences*, 29 (2013), 39–53.

[4] V. Candela and A. Marquina, Recurance relations for rational cubic method, *Computing*, 45 (1990), 355–367.

[5] C. Chun and B. Neta, Certain improvement of Newton's method with fourth-order convergence, *Applied Mathematics and Computation*, 215 (2009), 821–823.

[6] R. V. Dukkipati, *Numerical Methods*, New Delhi, New Age International(P) Ltd., New Delhi, 2010.

[7] H. Esmaeili and A. N. Rezaei, A uniparametric family of modifications for Chebyshev's method, *Lectures Matematices*, 33 (2012), 95–106.

[8] L. Fang, G. He and Z. Hu, A cubically convergent Newton-type method under weak condition, *Journal of Computaitional and Applied Mathematics*, 220 (2008), 409–412.

[9] J. Jayakumar and P. Jayasilan, Second derivative free modification with parameter for Chebyshev's method, *International Journal of Computational Engineering Research*, 03 (2013), 38–42.

[10] V. Kanwar, A family of third orde multipoint methods for solving nonlinear equations, *Applied Mathematics and Computation*, 176 (2006), 409–413.

[11] J. Kou, L. Yitian and W. Xiuhua, A uniparametric Chebyshev-type method free from second derivatives, *Applied Mathematics and Computation*. 179(2006), 296–300.

[12] J. Kou, L. Yitian and W. Xiuhua, Third-order modification of Newton's method, *Journal Computational and Applied Mathematics*, 205 (2007), 1–5.

[13] M.N. Muhaijir, M. Imran and M.D.H. Gamal, Variants of Chebyshev's method with eighth-order convergence for solving nonlinear equations, *Applied and Computational Mathematics*, 5(6), 247–251, 2016.

[14] M. A. Noor, W. A. Khan and A. Husain, A new modified Halley method without second derivatives for nonlinear equation, *Applied Mathematics and Computation*, 189 (2007), 1268–1273.

[15] M. A. Noor, W. A. Khan, K. I. Noor and E. Alsaid, Higher-order iterative methods free from second derivetive for solving nonlinear equations, *International Journal of the Physical Sciences*, Vol. 6(8), pp. 1887–1893, 2011.

[16] A. Y. Ozban, Some new variants of Newton's method, *Applied Mathematic Letters* 13 (2004), 87–93.

[17] S. Weerakon and T. G. I. Fernando, A variant of Newton's methods with accelarated third order convergence, *Applied Mathematic Letters*, 13 (2000), 87–93.