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Maximization of Technical Efficiency of a Normal-Half Normal Stochastic Production Frontier Model

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Abstract

The main aspect of stochastic frontier analysis is to estimate the technical efficiency based on the choice and price of inputs and the level of outputs produced. Further the rank of the firms in terms of efficiency can be obtained. The derivation of the Stochastic Production Frontier Model plays a vital role in measuring technical efficiency. The current study focuses on the derivation and maximization of technical efficiency of Normal- Half Normal Stochastic Production Frontier Model.

Keywords: Technical Efficiency, Normal-Half Normal distribution, Stochastic Production Frontier Model, Maximization.

INTRODUCTION

In 2005 Coelli et al assumed a functional form for the relationship between inputs and an output in the Stochastic Frontier Analysis method of frontier estimation. The stochastic production function model was proposed independently by Aigner, Lovel, Schmidt Meeusen and van den Broeck in the year 1977 proposed the stochastic production function model. Kumbhakar, Ghosh and McGuckin [1991] and Huang and Liu [1994] designed stochastic production models for the parametric estimation of both the stochastic frontier and the inefficiency functions. Aigner and Lovell (1976) worked on the production frontier model $y_i = f(x_i, \beta) + \varepsilon_i$ where $\varepsilon_i = v_i - u_i$

The key perspective of stochastic frontier analysis is the introduction of the composite error term which contains two components : a technical inefficiency component and a noise component. A firm is said to be efficient or inefficient with respect to its own production frontier based on the composite error term. The Maximum Likelihood Method or Method of Moments can be employed to obtain the technical efficiency of the firms with the distributional assumptions based on the error terms. Various distributional assumptions have been used in the literature for the two error terms.

In this paper in the derivation of Normal-Half Normal Stochastic Production Frontier Model the following distributional assumptions were made.

(i) The error term represent the statistical noise $v_i \sim iid\ N(0, \sigma_v{}^2)$

(ii) The error term representing the technical efficiency $u_i \sim iid\ N^+(0, \sigma_u{}^2)$

(i.e non-negative half normal).

(iii) v_i and u_i are distributed independently of each other and of the regressors.

Parameters like α, σ_s, μ are estimated using the method of maximum likelihood.

This paper involves four major sections namely:

Section I: Derivation of the Normal-Half Normal Stochastic Production Frontier Model.

Section II: Estimation of the parameters of the Normal-Half Normal Stochastic Production Frontier Model.

Section III: Measurement of the technical efficiency of the Normal-Half Normal Stochastic Production Frontier Model.

Section IV: Maximization of the technical efficiency of the Normal-Half Normal Stochastic Production Frontier Model.

I. THE NORMAL-HALF NORMAL STOCHASTIC PRODUCTION FRONTIER MODEL

The Probability density function of u is given by

$$f(u) = \frac{2}{\sqrt{2\pi}\sigma_u} exp\left\{\frac{-u^2}{2\sigma_u{}^2}\right\} \quad (1)$$

The Probability density function of v is given by

$$f(u) = \frac{1}{\sqrt{2\pi}\sigma_v} exp\left\{\frac{-v^2}{2\sigma_v{}^2}\right\} \quad (2)$$

Since u and v are independently distributed, the joint density function of u and v is the product of their individual probability density functions

$$f(u,v) = f(u).f(v) = \frac{2}{2\pi\sigma_u\sigma_v} exp\left\{\frac{-u^2}{2\sigma_u{}^2} + \frac{-v^2}{2\sigma_v{}^2}\right\} \quad (3)$$

Using the transformation, $\varepsilon = v - u$, the joint density function of u and ε is

$$f(u,\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v} exp\left\{-\frac{u^2}{2\sigma_u{}^2} - \frac{(u+\varepsilon)^2}{2\sigma_v{}^2}\right\} \quad (4)$$

The marginal density function of ε is given by

$$f(\varepsilon) = \int_0^\infty f(u,\varepsilon)du \quad (5)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty exp\left\{-\frac{u^2}{2\sigma_u{}^2} - \frac{(u^2+\varepsilon^2+2u\varepsilon)}{2\sigma_v{}^2}\right\} du \quad (6)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty exp\left\{\frac{u^2\sigma_v{}^2 - u^2\sigma_u{}^2 - \varepsilon^2\sigma_u{}^2 - 2u\varepsilon\sigma_u{}^2}{2\sigma_u{}^2\sigma_v{}^2}\right\} du \quad (7)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty exp\left[-\frac{1}{2}\left\{\frac{u^2(\sigma_v{}^2+\sigma_u{}^2)+\varepsilon^2\sigma_u{}^2+2u\varepsilon\sigma_u{}^2}{\sigma_u{}^2\sigma_v{}^2}\right\}\right] du \quad (8)$$

Let $\sigma_s{}^2 = \sigma_u{}^2 + \sigma_v{}^2$ and $\mu = {}^{\sigma_u}/_{\sigma_v}$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty exp\left[-\frac{1}{2}\left\{\frac{u^2(\sigma_s{}^2)+\varepsilon^2\sigma_u{}^2+2u\varepsilon\sigma_u{}^2}{\sigma_u{}^2\sigma_v{}^2}\right\}\right] du \quad (9)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty exp\left[-\frac{1}{2\sigma_u{}^2\sigma_v{}^2}\left\{\sigma_s{}^2\left(u^2 + \frac{\varepsilon^2\sigma_u{}^2+2u\varepsilon\sigma_u{}^2}{\sigma_s{}^2}\right)\right\}\right] du \quad (10)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty exp\left[-\frac{\sigma_s{}^2}{2\sigma_u{}^2\sigma_v{}^2}\left\{\left(u^2 + \frac{\varepsilon^2\sigma_u{}^2}{\sigma_s{}^2} + \frac{2u\varepsilon\sigma_u{}^2}{\sigma_s{}^2}\right)\right\}\right] du \quad (11)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty exp\left[-\frac{\sigma_s{}^2}{2\sigma_u{}^2\sigma_v{}^2}\left\{\left(u^2 + \frac{\varepsilon^2\sigma_u{}^2}{\sigma_s{}^2} + \frac{2u\varepsilon\sigma_u{}^2}{\sigma_s{}^2} + \frac{\varepsilon^2\sigma_u{}^4}{\sigma_s{}^4} - \frac{\varepsilon^2\sigma_u{}^4}{\sigma_s{}^4}\right)\right\}\right] du \quad (12)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty exp\left[-\frac{\sigma_s{}^2}{2\sigma_u{}^2\sigma_v{}^2}\left\{\left(u^2 + \frac{2u\varepsilon\sigma_u{}^2}{\sigma_s{}^2} + \frac{\varepsilon^2\sigma_u{}^4}{\sigma_s{}^4}\right) + \frac{\varepsilon^2\sigma_u{}^2}{\sigma_s{}^2} - \frac{\varepsilon^2\sigma_u{}^4}{\sigma_s{}^4}\right\}\right] du \quad (13)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty exp\left[-\frac{\sigma_s{}^2}{2\sigma_u{}^2\sigma_v{}^2}\left\{\left(u + \frac{\varepsilon\sigma_u{}^2}{\sigma_s{}^2}\right)^2 + \frac{\sigma_s{}^2\varepsilon^2\sigma_u{}^2 - \varepsilon^2\sigma_u{}^4}{\sigma_s{}^4}\right\}\right] du \quad (14)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty exp\left[-\frac{\sigma_s{}^2}{2\sigma_u{}^2\sigma_v{}^2}\left(u + \frac{\varepsilon\sigma_u{}^2}{\sigma_s{}^2}\right)^2\right] .exp\left[-\frac{\sigma_s{}^2}{2\sigma_u{}^2\sigma_v{}^2}\left\{\frac{\sigma_s{}^2\varepsilon^2\sigma_u{}^2 - \varepsilon^2\sigma_u{}^4}{\sigma_s{}^4}\right\}\right] du \quad (15)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty exp\left[-\frac{\sigma_s{}^2}{2\sigma_u{}^2\sigma_v{}^2}\left(u + \frac{\varepsilon\sigma_u{}^2}{\sigma_s{}^2}\right)^2\right] .exp\left[-\frac{\sigma_s{}^2}{2\sigma_u{}^2\sigma_v{}^2}\left\{\frac{\sigma_u{}^2(\sigma_s{}^2\varepsilon^2 - \varepsilon^2\sigma_u{}^2)}{\sigma_s{}^4}\right\}\right] du \quad ((16)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty exp\left[-\frac{\sigma_s{}^2}{2\sigma_u{}^2\sigma_v{}^2}\left(u + \frac{\varepsilon\sigma_u{}^2}{\sigma_s{}^2}\right)^2\right] .exp\left[-\frac{1}{2\sigma_v{}^2}\left\{\frac{\sigma_s{}^2\varepsilon^2 - \varepsilon^2\sigma_u{}^2}{\sigma_s{}^2}\right\}\right] du \quad (17)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty exp\left[-\frac{\sigma_s{}^2}{2\sigma_u{}^2\sigma_v{}^2}\left(u + \frac{\varepsilon\sigma_u{}^2}{\sigma_s{}^2}\right)^2\right] .exp\left[-\frac{\varepsilon^2}{2\sigma_v{}^2}\left\{\frac{\sigma_s{}^2 - \sigma_u{}^2}{\sigma_s{}^2}\right\}\right] du \quad (18)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty exp\left[-\frac{\sigma_s{}^2}{2\sigma_u{}^2\sigma_v{}^2}\left(u+\frac{\varepsilon\sigma_u{}^2}{\sigma_s{}^2}\right)^2\right].exp\left[-\frac{\varepsilon^2}{2\sigma_v{}^2}\left\{\frac{\sigma_u{}^2+\sigma_v{}^2-\sigma_u{}^2}{\sigma_s{}^2}\right\}\right]du \quad (19)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty exp\left[-\frac{\sigma_s{}^2}{2\sigma_u{}^2\sigma_v{}^2}\left(u+\frac{\varepsilon\sigma_u{}^2}{\sigma_s{}^2}\right)^2\right].exp\left[-\frac{\varepsilon^2}{2\sigma_v{}^2}\left\{\frac{\sigma_v{}^2}{\sigma_s{}^2}\right\}\right]du \quad (20)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty exp\left[-\frac{\sigma_s{}^2}{2\sigma_u{}^2\sigma_v{}^2}\left(u+\frac{\varepsilon\sigma_u{}^2}{\sigma_s{}^2}\right)^2\right].exp\left[-\frac{\varepsilon^2}{2\sigma_s{}^2}\right]du \quad (21)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}exp\left[-\frac{\varepsilon^2}{2\sigma_s{}^2}\right]\int_0^\infty exp\left[-\frac{\sigma_s{}^2}{2\sigma_u{}^2\sigma_v{}^2}\left(u+\frac{\varepsilon\sigma_u{}^2}{\sigma_s{}^2}\right)^2\right]du \quad (22)$$

$$Let\ t = \frac{\sigma_s}{\sigma_u\sigma_v}\left(u+\frac{\varepsilon\sigma_u{}^2}{\sigma_s{}^2}\right)$$

$$dt = \frac{\sigma_s}{\sigma_u\sigma_v}du$$

$$As\ u \to 0, t \to \frac{\varepsilon\mu}{\sigma_s}\ and\ as\ u \to \infty, t \to \infty$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v}exp\left[-\frac{\varepsilon^2}{2\sigma_s{}^2}\right]\int_{\frac{\varepsilon\mu}{\sigma_s}}^\infty exp\left(-\frac{t^2}{2}\right)\frac{\sigma_u\sigma_v}{\sigma_s}dt \quad (23)$$

$$f(\varepsilon) = \frac{1}{\pi\sigma_s}exp\left[-\frac{\varepsilon^2}{2\sigma_s{}^2}\right]\int_{\frac{\varepsilon\mu}{\sigma_s}}^\infty exp\left(-\frac{t^2}{2}\right)dt \quad (24)$$

$$f(\varepsilon) = \frac{2}{2\pi\sigma_s}exp\left[-\frac{\varepsilon^2}{2\sigma_s{}^2}\right]\int_{\frac{\varepsilon\mu}{\sigma_s}}^\infty exp\left(-\frac{t^2}{2}\right)dt \quad (25)$$

$$f(\varepsilon) = \frac{2}{\left(\sqrt{2\pi}\right)^2\sigma_s}exp\left[-\frac{\varepsilon^2}{2\sigma_s{}^2}\right]\int_{\frac{\varepsilon\mu}{\sigma_s}}^\infty exp\left(-\frac{t^2}{2}\right)dt \quad (26)$$

$$f(\varepsilon) = \frac{2}{\sqrt{2\pi}\sigma_s}exp\left[-\frac{\varepsilon^2}{2\sigma_s{}^2}\right]\frac{1}{\sqrt{2\pi}}\int_{\frac{\varepsilon\mu}{\sigma_s}}^\infty exp\left(-\frac{t^2}{2}\right)dt \quad (27)$$

$$f(\varepsilon) = \frac{2}{\sqrt{2\pi}\sigma_s}exp\left[-\frac{\varepsilon^2}{2\sigma_s{}^2}\right]\left[1-\Phi\left(\frac{\varepsilon\mu}{\sigma_s}\right)\right] \quad (28)$$

$$f(\varepsilon) = \frac{2}{\sigma_s}\phi\left(\frac{\varepsilon}{\sigma_s}\right)\Phi\left(-\frac{\varepsilon\mu}{\sigma_s}\right) \quad (29)$$

Where ϕ is the density function and Φ is the standard normal cumulative distribution.

$$E(\varepsilon) = E(v-u) = E(v) - E(u) = 0 - E(u) = -E(u) \quad (30)$$

$$E(\varepsilon) = -\int_0^\infty u\,f(u)\,du = -\int_0^\infty u\frac{2}{\sqrt{2\pi}\sigma_u}exp\left(-\frac{u^2}{2\sigma_u{}^2}\right)du \quad (31)$$

$$Put\ t = \frac{u^2}{2\sigma_u{}^2}$$

$$dt = \frac{2u\,du}{2\sigma_u{}^2}$$

$u\, du = \sigma_u{}^2\, dt$

$As\ \ u \to 0, t \to 0\ and\ as\ u \to \infty, t \to \infty$

$$E(\varepsilon) = -\int_0^\infty \frac{2\sigma_u{}^2}{\sqrt{2\pi}\sigma_u} exp(-t)\, dt \tag{32}$$

$$E(\varepsilon) = -\sqrt{\frac{2}{\pi}}\sigma_u \tag{33}$$

$$V(\varepsilon) = V(v) - V(u) \tag{34}$$

$$V(u) = E(u^2) - [E(u)]^2 \tag{35}$$

$$E(u^2) = \int_0^\infty u^2\, f(u)\, du \tag{36}$$

$$E(u^2) = \int_0^\infty u^2\ \frac{2}{\sqrt{2\pi}\sigma_u} exp\left(-\frac{u^2}{2\sigma_u{}^2}\right)\ \ du \tag{37}$$

$Put\ t = \frac{u^2}{2\sigma_u{}^2}$

$dt = \frac{2u\, du}{2\sigma_u{}^2}$

$u\, du = \sigma_u{}^2\, dt$

$u^2 = 2t\sigma_u{}^2$

$u = \sqrt{2t}\sigma_u$

$$E(u^2) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sqrt{2t}\sigma_u}{\sigma_u} e^{-t}\sigma_u{}^2\, dt \tag{38}$$

$$E(u^2) = \frac{2}{\sqrt{\pi}}\sigma_u{}^2 \int_0^\infty \sqrt{t}\, e^{-t}\, dt \tag{39}$$

$$E(u^2) = \frac{2}{\sqrt{\pi}}\sigma_u{}^2 \int_0^\infty t^{\left(\frac{3}{2}-1\right)} e^{-t}\, dt \tag{40}$$

$$E(u^2) = \frac{2}{\sqrt{\pi}}\sigma_u{}^2 \Gamma\left(\frac{1}{2}+1\right) \tag{41}$$

$$E(u^2) = \frac{2}{\sqrt{\pi}}\sigma_u{}^2 \Gamma\left(\frac{1}{2}\right) \tag{42}$$

$$E(u^2) = \frac{2}{\sqrt{\pi}}\sigma_u{}^2 \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \tag{43}$$

$$E(u^2) = \frac{1}{\sqrt{\pi}}\sigma_u{}^2\sqrt{\pi} \tag{44}$$

$$E(u^2) = \sigma_u{}^2 \tag{45}$$

$$V(u) = E(u^2) - [E(u)]^2 \tag{46}$$

$$V(u) = {\sigma_u}^2 - \left[-\sqrt{\frac{2}{\pi}}\sigma_u\right]^2 \tag{47}$$

$$V(u) = {\sigma_u}^2 - \frac{2}{\pi}{\sigma_u}^2 \tag{48}$$

$$V(u) = \left(\frac{\pi-2}{\pi}\right){\sigma_u}^2 \tag{49}$$

$$V(\varepsilon) = {\sigma_v}^2 - \left(\frac{\pi-2}{\pi}\right){\sigma_u}^2 \tag{50}$$

II. ESTIMATION OF THE PARAMETERS

The likelihood function of the sample is given by

$$L = \prod_{i=1}^{N} f(\varepsilon_i) \tag{51}$$

The log likelihood function for $\varepsilon_i = y_i - m(x_i, \alpha)$ is

$$\ln L = constant - N \ln \sigma + \sum_{i=1}^{N}\left[ln\, \Phi\left(-\frac{\varepsilon_i \mu}{\sigma_s}\right) - \frac{1}{2{\sigma_s}^2}\sum_{i=1}^{N}{\varepsilon_i}^2\right] \tag{52}$$

$$\ln L = -\left(\frac{N}{2}\right)(ln 2\pi + ln{\sigma_s}^2) + \sum_{i=1}^{N}\left[ln\, \Phi\left(-\frac{\varepsilon_i \mu}{\sigma_s}\right) - \frac{{\varepsilon_i}^2}{2{\sigma_s}^2}\right] \tag{53}$$

$$L[\alpha, \mu, {\sigma_s}^2] = ln\ L - \left(\frac{N}{2}\right)ln 2\pi - \left(\frac{N}{2}\right)ln{\sigma_s}^2 - \frac{1}{2}\sum_{i=1}^{N}\left(\frac{y_i - x_i'\alpha}{\sigma_s}\right)^2 + \sum_{i=1}^{N} ln\Phi\left[\frac{(y_i - x_i'\alpha)\mu}{\sigma_s}\right] \tag{54}$$

The first order partial derivatives with respect to $\alpha, \mu, {\sigma_s}^2$ are obtained by differentiating (54) partially and equating them to zero as below

$$P_\alpha' = \frac{\partial lnL}{\partial \alpha} = \frac{1}{{\sigma_s}^2}\sum_{i=1}^{N}(y_i - x_i'\alpha)x_i' + \sum_{i=1}^{N}\left\{\frac{\phi\left[\frac{(y_i - x_i'\alpha)\mu}{\sigma_s}\right]}{\Phi\left[\frac{(y_i - x_i'\alpha)\mu}{\sigma_s}\right]}\left(-\frac{x_i'\mu}{\sigma_s}\right)\right\} \tag{55}$$

$$P_\mu' = \frac{\partial lnL}{\partial \mu} = \sum_{i=1}^{N}\left\{\frac{\phi\left[\frac{(y_i - x_i'\alpha)\mu}{\sigma_s}\right]}{\Phi\left[\frac{(y_i - x_i'\alpha)\mu}{\sigma_s}\right]}\left(\frac{y_i - x_i'\alpha}{\sigma_s}\right)\right\} \tag{56}$$

$$P_{\sigma^2}' = \frac{\partial lnL}{\partial \sigma^2} = -\frac{N}{2{\sigma_s}^2} + \frac{1}{2{\sigma_s}^4}\sum_{i=1}^{N}(y_i - x_i'\alpha)^2 + \frac{\mu}{2{\sigma_s}^3}\sum_{i=1}^{N}\left\{\frac{\phi\left[\frac{(y_i - x_i'\alpha)\mu}{\sigma_s}\right]}{\Phi\left[\frac{(y_i - x_i'\alpha)\mu}{\sigma_s}\right]}(y_i - x_i'\alpha)\right\} \tag{57}$$

$$P_\mu' = 0 \Rightarrow \sum_{i=1}^{N}\left\{\frac{\phi\left[\frac{(y_i - x_i'\alpha)\mu}{\sigma_s}\right]}{\Phi\left[\frac{(y_i - x_i'\alpha)\mu}{\sigma_s}\right]}\left(\frac{y_i - x_i'\alpha}{\sigma_s}\right)\right\} = 0 \tag{58}$$

$$-\frac{N}{2{\sigma_s}^2} + \frac{1}{2{\sigma_s}^4}\sum_{i=1}^{N}(y_i - x_i'\alpha)^2 = 0 \tag{59}$$

$$\frac{N}{2{\sigma_s}^2} = \frac{1}{2{\sigma_s}^4}\sum_{i=1}^{N}(y_i - {x_i}'\alpha)^2 \tag{60}$$

$$N = \frac{1}{{\sigma_s}^2}\sum_{i=1}^{N}(y_i - {x_i}'\alpha)^2 \tag{61}$$

Where ${x_i}'$ is a $m \times 1$ column vector.

The likelihood estimator of ${\sigma_s}^2$ is obtained as

$${\sigma_s}^2 = \frac{1}{N}\sum_{i=1}^{N}(y_i - {x_i}'\alpha)^2 \tag{62}$$

Let $A - N \times m\ data\ matrix$

$B - N \times 1\ data\ vector$

$\varepsilon - N \times 1\ vector$

$$\beta = \frac{\phi\left[\frac{(y_i - {x_i}'\alpha)\mu}{\sigma_s}\right]}{\Phi\left[\frac{(y_i - {x_i}'\alpha)\mu}{\sigma_s}\right]}$$

Using the above assumptions in (55),(56)and (57) we have

$${P_\alpha}' = \frac{1}{{\sigma_s}^2}(A'y - A'A\alpha) = \frac{\mu A'\beta}{\sigma_s} = 0 \tag{63}$$

$${P_\mu}' = \frac{1}{\sigma_s}\varepsilon'\beta = 0 \tag{64}$$

$${P_{\sigma^2}}' = -\frac{N}{2{\sigma_s}^2} + \frac{1}{2{\sigma_s}^4}\varepsilon\varepsilon' + \frac{\mu\varepsilon'\beta}{2{\sigma_s}^3} = 0 \tag{65}$$

Multiplying (63) by ${\sigma_s}^2(AA')^{-1}$

$$(AA')^{-1}(A'y - A'A\alpha) - \sigma_s(AA')^{-1}\mu A'\beta = 0 \tag{66}$$

$$(AA')^{-1}A'y - \alpha - \sigma_s(AA')^{-1}\mu A'\beta = 0 \tag{67}$$

$$\Rightarrow \alpha = (AA')^{-1}A' - [y - \sigma_s\mu\beta] \tag{68}$$

The likelihood estimator of α is

$$\alpha' = p - q \tag{69}$$

Define slope vector of the OLS estimate $p = (AA')^{-1}A'y$

OLS estimate $q = \sigma_s(AA')^{-1}\mu A'\beta$

From equation (63)

$$\frac{1}{{\sigma_s}^2}A'\varepsilon - \frac{\mu A'\beta}{\sigma_s} = 0 \tag{70}$$

Multiplying (70) by $\sigma_s\alpha'$

$$\frac{\alpha' A'\varepsilon}{\sigma_s} - \alpha'\mu A'\beta = 0 \tag{71}$$

The likelihood estimator of μ is

$$\mu = \frac{\alpha' A' \varepsilon}{\sigma_s \alpha' A' \beta} \tag{72}$$

$If\ u_i \sim N(0, \sigma_u{}^2)$, the conditional distribution of u given ε is

$$f(u/\varepsilon) = \frac{f(u,\varepsilon)}{f(\varepsilon)} \tag{73}$$

$$f(u/\varepsilon) = \frac{\frac{1}{\pi\sigma_u\sigma_v} exp\left\{-\frac{u^2}{2\sigma_u{}^2} - \frac{(u+\varepsilon)^2}{2\sigma_v{}^2}\right\}}{\frac{2}{\sqrt{2\pi}\sigma_s} exp\left[-\frac{\varepsilon^2}{2\sigma_s{}^2}\right]\left[1-\Phi\left(\frac{\varepsilon\mu}{\sigma_s}\right)\right]} \tag{74}$$

$$f(u/\varepsilon) = \frac{\sigma_s\left[1-\Phi\left(\frac{\varepsilon\mu}{\sigma_s}\right)\right]^{-1}}{\sqrt{2\pi}\,\sigma_u\sigma_v} exp\left\{-\frac{1}{2}\left[\frac{u^2}{\sigma_u{}^2} + \frac{(u^2+\varepsilon^2+2u\varepsilon)}{\sigma_v{}^2} - \frac{\varepsilon^2}{\sigma_s{}^2}\right]\right\} \tag{75}$$

$$f(u/\varepsilon) = \frac{\sigma_s\left[1-\Phi\left(\frac{\varepsilon\mu}{\sigma_s}\right)\right]^{-1}}{\sqrt{2\pi}\,\sigma_u\sigma_v} exp\left\{-\frac{1}{2}\left[\frac{u^2\sigma_v{}^2+(u^2+\varepsilon^2+2u\varepsilon)\sigma_u{}^2}{\sigma_u{}^2\sigma_v{}^2} - \frac{\varepsilon^2}{\sigma_s{}^2}\right]\right\} \tag{76}$$

$$f(u/\varepsilon) = \frac{\sigma_s\left[1-\Phi\left(\frac{\varepsilon\mu}{\sigma_s}\right)\right]^{-1}}{\sqrt{2\pi}\,\sigma_u\sigma_v} exp\left\{-\frac{1}{2}\left[\frac{u^2(\sigma_u{}^2+\sigma_v{}^2)+(\varepsilon^2\sigma_u{}^2+2u\varepsilon\sigma_u{}^2)}{\sigma_u{}^2\sigma_v{}^2} - \frac{\varepsilon^2}{\sigma_s{}^2}\right]\right\} \tag{77}$$

$$f(u/\varepsilon) = \frac{\sigma_s\left[1-\Phi\left(\frac{\varepsilon\mu}{\sigma_s}\right)\right]^{-1}}{\sqrt{2\pi}\,\sigma_u\sigma_v} exp\left\{-\frac{1}{2}\left[\frac{u^2\sigma_s{}^2+(\varepsilon^2\sigma_u{}^2+2u\varepsilon\sigma_u{}^2)}{\sigma_u{}^2\sigma_v{}^2} - \frac{\varepsilon^2}{\sigma_s{}^2}\right]\right\} \tag{78}$$

$$f(u/\varepsilon) = \frac{\sigma_s\left[1-\Phi\left(\frac{\varepsilon\mu}{\sigma_s}\right)\right]^{-1}}{\sqrt{2\pi}\,\sigma_u\sigma_v} exp\left\{-\frac{1}{2}\left[\frac{\sigma_s{}^2}{\sigma_u{}^2\sigma_v{}^2}\left(u + \frac{\varepsilon\sigma_u{}^2}{\sigma_s{}^2}\right)^2 + \frac{\varepsilon^2}{\sigma_s{}^2} - \frac{\varepsilon^2}{\sigma_s{}^2}\right]\right\} \tag{79}$$

$$f(u/\varepsilon) = \frac{\sigma_s\left[1-\Phi\left(\frac{\varepsilon\mu}{\sigma_s}\right)\right]^{-1}}{\sqrt{2\pi}\,\sigma_u\sigma_v} exp\left\{-\frac{1}{2}\left[\frac{\sigma_s{}^2}{\sigma_u{}^2\sigma_v{}^2}\left(u + \frac{\varepsilon\sigma_u{}^2}{\sigma_s{}^2}\right)^2\right]\right\} \tag{80}$$

$$f(u/\varepsilon) = \frac{\sigma_s\left[1-\Phi\left(\frac{\varepsilon\mu}{\sigma_s}\right)\right]^{-1}}{\sqrt{2\pi}\,\sigma_u\sigma_v} exp\left\{-\frac{1}{2}\left(\frac{u+\frac{\varepsilon\sigma_u{}^2}{\sigma_s{}^2}}{\sigma_u\sigma_v/\sigma_s}\right)^2\right\} \tag{81}$$

Let $\lambda_* = \frac{\varepsilon\sigma_u{}^2}{\sigma_s{}^2}$ and $\sigma_{s_*} = \sigma_u\sigma_v/\sigma_s$

$$f(u/\varepsilon) = \frac{\sigma_s\left[1-\Phi\left(\frac{\lambda_*}{\sigma_{s_*}}\right)\right]^{-1}}{\sqrt{2\pi}\,\sigma_{s_*}} exp\left\{-\frac{1}{2}\left(\frac{u+\lambda_*}{\sigma_{s_*}}\right)^2\right\} \tag{82}$$

III. MEASUREMENT OF TECHNICAL EFFICIENCY

As $f(u/\varepsilon) \sim N^+(\lambda_*, \sigma_{s_*}{}^2)$, the mean $E(u/\varepsilon)$ can be regarded as the point estimator of u_i

$$E(u/\varepsilon) = \int_0^\infty u\, f(u/\varepsilon)\ du \tag{83}$$

$$E(u/\varepsilon) = \int_0^\infty u \frac{\sigma_s\left[1-\Phi\left(\frac{\lambda_*}{\sigma_{s_*}}\right)\right]^{-1}}{\sqrt{2\pi}\,\sigma_{s_*}} exp\left\{-\frac{1}{2}\left(\frac{u+\lambda_*}{\sigma_{s_*}}\right)^2\right\}\, du \tag{84}$$

Define $t = \frac{u+\lambda_*}{\sigma_{s_*}}$ and $dt = du$

As $u \to 0, t \to \frac{\lambda_*}{\sigma_{s_*}}$ and as $u \to \infty, t \to \infty$

$$E(u/\varepsilon) = \frac{\left[1-\Phi\left(\frac{\lambda_*}{\sigma_{s_*}}\right)\right]^{-1}}{\sqrt{2\pi}\,\sigma_{s_*}} \int_{\frac{\lambda_*}{\sigma_{s_*}}}^\infty (\sigma_{s_*} t - \lambda_*)\, exp\left\{-\frac{1}{2}t^2\right\}\, \sigma_{s_*} dt \tag{85}$$

$$E(u/\varepsilon) = \left[1-\Phi\left(\frac{\lambda_*}{\sigma_{s_*}}\right)\right]^{-1}\left[\frac{{\sigma_{s_*}}^2}{\sqrt{2\pi}\,\sigma_{s_*}} \int_{\frac{\lambda_*}{\sigma_{s_*}}}^\infty t\, exp\left\{-\frac{1}{2}t^2\right\} dt + \right.$$

$$\left.\frac{\lambda_*\sigma_{s_*}}{\sigma_{s_*}\sqrt{2\pi}} \int_{\frac{\lambda_*}{\sigma_{s_*}}}^\infty exp\left\{-\frac{1}{2}t^2\right\}\, dt\right] \tag{86}$$

Let $s = \frac{t^2}{2}$, $ds = tdt$

As $t \to \infty, s \to \infty$ and $t \to \frac{\lambda_*}{\sigma_{s_*}}, s \to \frac{\lambda_*}{2{\sigma_{s_*}}^2}$

$$E(u/\varepsilon) = \left[1-\Phi\left(\frac{\lambda_*}{\sigma_{s_*}}\right)\right]^{-1}\left\{\frac{\sigma_{s_*}}{\sqrt{2\pi}} \int_{\frac{\lambda_*}{2{\sigma_{s_*}}^2}}^\infty e^{-s}\, ds + \lambda_*\left[1-\Phi\left(\frac{\lambda_*}{\sigma_{s_*}}\right)\right]\right\} \tag{87}$$

$$E(u/\varepsilon) = \left[1-\Phi\left(\frac{\lambda_*}{\sigma_{s_*}}\right)\right]^{-1}\left\{\frac{\sigma_{s_*}}{\sqrt{2\pi}} exp\left(-\frac{\lambda_*}{2{\sigma_{s_*}}^2}\right) + \lambda_*\right\} \tag{88}$$

$$E(u/\varepsilon) = \left[1-\Phi\left(\frac{\lambda_*}{\sigma_{s_*}}\right)\right]^{-1}\left\{\sigma_{s_*}\phi\left(\frac{\lambda_*}{\sigma_{s_*}}\right) + \lambda_*\right\} \tag{89}$$

$$E(u/\varepsilon) = \left[1-\Phi\left(\frac{\varepsilon{\sigma_u}^2}{{\sigma_s}^2}.\frac{\sigma_s}{\sigma_u\sigma_v}\right)\right]^{-1}\left\{\sigma_{s_*}\phi\left(\frac{\varepsilon{\sigma_u}^2}{{\sigma_s}^2}.\frac{\sigma_s}{\sigma_u\sigma_v}\right) + \lambda_*\right\} \tag{90}$$

$$E(u/\varepsilon) = \left[1-\Phi\left(\frac{\varepsilon\mu}{\sigma_s}\right)\right]^{-1}\left\{\sigma_{s_*}\phi\left(\frac{\varepsilon\mu}{\sigma_s}\right) + \frac{\varepsilon{\sigma_u}^2}{{\sigma_s}^2}\right\} \tag{91}$$

$$E(u/\varepsilon) = \sigma_{s_*}\left\{\frac{\phi\left(\frac{\varepsilon\mu}{\sigma_s}\right)}{\left[1-\Phi\left(\frac{\varepsilon\mu}{\sigma_s}\right)\right]} + \frac{\varepsilon\mu}{\sigma_s}\right\} \tag{92}$$

$$E(u_i/\varepsilon_i) = \sigma_{s_*}\left\{\frac{\phi\left(\frac{\varepsilon_i\mu}{\sigma_s}\right)}{\left[1-\Phi\left(\frac{\varepsilon_i\mu}{\sigma_s}\right)\right]} + \frac{\varepsilon_i\mu}{\sigma_s}\right\} \tag{93}$$

Estimates of u_i can be obtained from

$$TE_i = \exp[-E(u_i/\varepsilon_i)] \tag{94}$$

As proposed by Battese and Coelli(1988) an alternative point estimator for TE_i can be preferred when u_i is not close to zero which is defined as

$$TE_i = E[\exp\{-u_i\}/\varepsilon_i\,] \tag{95}$$

IV. MAXIMIZATION OF TECHNICAL EFFICENCY

The second order partial derivatives are obtained as follows

$$\frac{\partial^2}{\partial\alpha^2} lnL = \frac{1}{{\sigma_s}^2}\sum_{i=1}^{N}(-2x_i') + \sum_{i=1}^{N}\left\{\frac{\phi\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]}{\Phi\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]}\left(\frac{-x_i'}{\sigma_s}\right) + \left(\frac{x_i'\alpha}{\sigma_s}\right)\left[\frac{\Phi\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]\phi'\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]\left(\frac{-x_i'\mu}{\sigma_s}\right)-\phi\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]\Phi'\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]\left(\frac{-x_i'\mu}{\sigma_s}\right)}{\left\{\Phi\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]\right\}^2}\right]\right\} \tag{96}$$

Let $\delta = \dfrac{\phi\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]}{\Phi\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]}$

$$\frac{\partial^2}{\partial\alpha^2} lnL = \frac{1}{{\sigma_s}^2}\sum_{i=1}^{N}(-2x_i') + \sum_{i=1}^{N}\left\{\left(\frac{-x_i'}{\sigma_s}\right)\delta + \left(\frac{x_i'\alpha}{\sigma_s}\right)^2\delta'\right\} \tag{97}$$

$$\frac{\partial^2}{\partial\alpha^2} lnL = \sum_{i=1}^{N}\left[\frac{1}{{\sigma_s}^2}(-2x_i') + \left\{\left(\frac{-x_i'}{\sigma_s}\right)\delta + \left(\frac{x_i'\alpha}{\sigma_s}\right)^2\delta'\right\}\right] \tag{98}$$

$$\frac{\partial^2}{\partial\mu^2} lnL = \sum_{i=1}^{N}\left(\frac{y_i-x_i'\alpha}{\sigma_s}\right)\left[\frac{\Phi\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]\phi'\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]\left(\frac{-x_i'\mu}{\sigma_s}\right)-\phi\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]\Phi'\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]\left(\frac{-x_i'\mu}{\sigma_s}\right)}{\left\{\Phi\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]\right\}^2}\right] \tag{99}$$

$$\frac{\partial^2}{\partial\mu^2} = \sum_{i=1}^{N}\left(\frac{y_i-x_i'\alpha}{\sigma_s}\right)\left(\frac{-x_i'\alpha}{\sigma_s}\right)\delta' \tag{100}$$

$$\frac{\partial^2}{\partial{\sigma_s}^{2^2}} lnL = \frac{N}{2{\sigma_s}^4} + \left(\frac{1}{2}\right)\left(\frac{-2}{{\sigma_s}^6}\right)\sum_{i=1}^{N}(y_i - x_i'\alpha)^2 + \sum_{i=1}^{N}(y_i - x_i'\alpha)\frac{\mu}{2}\left\{\frac{\phi\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]}{\Phi\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]}\left(\frac{-3}{2}\right)({\sigma_s}^2)^{\left(\frac{-3}{2}-1\right)} + \frac{1}{{\sigma_s}^3}\left[\frac{-\Phi\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]\phi'\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right](y_i-x_i'\alpha)\frac{\mu}{2{\sigma_s}^3}+\phi\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]\Phi'\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right](y_i-x_i'\alpha)\frac{\mu}{2{\sigma_s}^3}}{\left\{\Phi\left[\frac{(y_i-x_i'\alpha)\mu}{\sigma_s}\right]\right\}^2}\right]\right\} \tag{101}$$

$$\frac{\partial^2}{\partial{\sigma_s}^{2^2}} lnL = \frac{N}{2{\sigma_s}^4} - \left(\frac{1}{{\sigma_s}^6}\right)\sum_{i=1}^{N}(y_i - x_i'\alpha)^2 + \sum_{i=1}^{N}(y_i - x_i'\alpha)\frac{\mu}{2}\left\{\delta\left(\frac{-3}{2{\sigma_s}^5}\right) + \frac{1}{{\sigma_s}^3}\left[-\delta'(y_i - x_i'\alpha)\frac{\mu}{2{\sigma_s}^3}\right]\right\} \tag{102}$$

The second order partial derivatives calculated above using equations (98), (100) and (102) are estimated at each of its critical points for maximization of the outputs or minimization of the inputs.

CONCLUSION

If the second order partial derivatives are less than zero then the technical efficiency is said to be maximum. Also the inefficiency u_i can be obtained following the estimation of the parameters. If $\varepsilon_i < 0$, then the producer is said to be technically efficient and alternatively if $\varepsilon_i > 0$, then the producer is technically inefficient.

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