Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 10 (2017), pp. 7401–7426

http://www.ripublication.com/gjpam.htm

L_p Convergence of Higher order Hermite or Hermite-Fejér Interpolation polynomials with exponential-type weights (II)

Hee Sun Jung[1]

Department of Mathematics Education,
Sungkyunkwan University, Seoul 03063, Republic of Korea.

Dong Hyun Ko

Department of Mathematics Education,
Sungkyunkwan University, Seoul 03063, Republic of Korea.

Ryozi Sakai

Department of Mathematics,
Meijo University, Nagoya 468-8502, Japan.

Abstract

Let $Q \in C^1(\mathbb{R}) : \mathbb{R} \to \mathbb{R}^+ = [0, \infty)$ be an even function. We consider the weight $w_\rho(x) = |x|^\rho e^{-Q(x)}$, $\rho \geq 0$, $x \in \mathbb{R}$. In this paper, we obtain the L_p-convergence theorems with respect to the higher order Hermite-Fejér interpolation polynomials based at the zeros of the orthonormal polynomials for $w_\rho^2(x)$.

AMS subject classification: 41A05, 41A10.
Keywords: higher order Hermite-Fejér interpolation polynomials.

[1]Corresponding author.

1. Introduction

Let ν be a positive integer. Let ℓ be a non-negative integer with $\ell \le \nu - 1$. For a function $f : \mathbb{R} \to \mathbb{R}$ and a set $\{x_{k,n}\}_{k=1}^{n}$, $n \ge 1$ with $x_{n,n} < x_{n-1,n} < \cdots < x_{2,n} < x_{1,n}$, we define the higher order Hermite-Fejér interpolation polynomial $L_n(\ell, \nu, f; x) \in \mathcal{P}_{\nu n-1}$ as follows:

$$L_n^{(j)}(\ell, \nu, f; x_{k,n}) = f^{(j)}(x_{k,n}), \quad j = 0, 1, 2, \ldots, \ell,$$
$$L_n^{(j)}(\ell, \nu, f; x_{k,n}) = 0, \quad j = \ell + 1, \ell + 2, \ldots, \nu - 1,$$

where $f^{(0)}(x) := f(x)$ and $\mathcal{P}_n$ is the class of polynomial of degree at most n. Then for each $P \in \mathcal{P}_{\nu n-1}$ we see $L_n(\nu - 1, \nu, P; x) = P(x)$. $L_n(0, 1, f; x)$ is the Lagrange interpolation polynomial and $L_n(0, 2, f; x)$ is the ordinary Hermite-Fejér interpolation polynomial. Here, we are interested in the higher order Hermite-Fejér interpolation polynomial $L_n(\ell, \nu, f; x)$ for the zeros of a sequence of orthogonal polynomials for polynomial exponential-type weights. In [6], it was investigated for uniform convergence and divergence theorems of the higher order Hermite-Fejér interpolation polynomial based at the zeros of a sequence of orthogonal polynomials for polynomial exponential-type weights. For even integers ν, [7] showed L_p-convergence theorems with respect to $L_n(\ell, \nu, f; x)$ for the zeros of a sequence of orthogonal polynomials for polynomial exponential-type weights. In this paper, for an integer $\nu \ge 2$, we will prove L_p-convergence theorems with respect to $L_n(\ell, \nu, f; x)$ for polynomial exponential-type weights.

To be precise, we will consider polynomial exponential-type weights and orthogonal polynomials associated with them. For an even function $Q \in C^1(\mathbb{R}) : \mathbb{R} \to \mathbb{R}^+ = [0, \infty)$, we consider the weight $w(x) = \exp(-Q(x))$ and define for $\rho > -1/2$,

$$w_\rho(x) := |x|^\rho w(x), \quad x \in \mathbb{R}.$$

Suppose that $\int_0^\infty x^n w_\rho^2(x)dx < \infty$ for all $n = 0, 1, 2, \ldots$. Then we can construct the orthonormal polynomials $p_{n,\rho}(x) = p_n(w_\rho^2; x)$ of degree n for $w_\rho^2(x)$, that is,

$$\int_{-\infty}^{\infty} p_{n,\rho}(x) p_{m,\rho}(x) w_\rho^2(x)dx = \delta_{mn} \quad \text{(Kronecker delta)},$$

where $p_{n,\rho}(x) = \gamma_n x^n + \ldots$, $\gamma_n = \gamma_{n,\rho} > 0$, and the zeros of $p_{n,\rho}(x)$ by

$$-\infty < x_{n,n,\rho} < x_{n-1,n,\rho} < \ldots < x_{2,n,\rho} < x_{1,n,\rho} < \infty.$$

For $f \in \mathrm{C}^{(\ell)}(\mathbb{R})$, we will treat the higher order Hermite-Fejér interpolation polynomial $L_n(\ell, \nu, f; x)$ for the zeros $\left\{x_{k,n,\rho}\right\}_{k=1}^{n}$ of a sequence of orthogonal polynomials $p_n(w_\rho^2; x)$. The fundamental polynomials $h_{s,k,n,\rho}(\ell, \nu; x) \in \mathcal{P}_{\nu n-1}$, $k = 1, 2, \ldots, n$, of $L_n(\ell, \nu, f; x)$ are defined by

$$h_{s,k,n,\rho}(\ell, \nu; x) = l_{k,n,\rho}^{\nu}(x) \sum_{i=s}^{\nu-1} e_{s,i}(\ell, \nu, k, n)(x - x_{k,n,\rho})^i,$$

satisfying $h^{(j)}_{s,k,n,\rho}(\ell, \nu; x_{p,n,\rho}) = \delta_{sj}\delta_{kp}$, for $j, s = 0, 1, \ldots, \nu - 1$, $p = 1, 2, \ldots, n$, where

$$l_{k,n,\rho}(x) = \frac{p_n(w_\rho^2; x)}{(x - x_{k,n,\rho})p_n'(w_\rho^2; x_{k,n,\rho})}.$$

Then the (ℓ, ν)-order Hermite-Fejér interpolation polynomials $L_n(\ell, \nu, f; x)$ can be expressed as

$$L_n(\ell, \nu, f; x) = \sum_{k=1}^{n}\sum_{s=0}^{\ell} f^{(s)}(x_{k,n,\rho})h_{s,k,n,\rho}(\ell, \nu; x).$$

When $\ell = \nu - 1$, for $f \in C^{(\nu-1)}(\mathbb{R})$ we have

$$L_n^{(j)}(\nu - 1, \nu, f; x_{k,n,\rho}) = f^{(j)}(x_{k,n,\rho}), \quad j = 0, 1, \ldots, \nu - 1.$$

In this paper we will give the L_p-convergence theorems for $L_n(\ell, \nu, f; x)$ with all integers $\nu \geq 2$(see [8], [9] for Freud-type weights). In section 2 we give the preliminaries for these studies, and in section 3 we report the direct L_p-convergence theorems. In section 4 we prove the theorems.

2. Preliminaries

In the rest of this paper we write $x_{k,n} := x_{k,n,\rho}$, $h_{k,n}(x) := h_{k,n,\rho}(\nu, x)$, $l_{k,n}(x) := l_{k,n,\rho}(x)$, $h_{s,k,n}(x) := h_{s,k,n,\rho}(\nu, x)$ and $p_n(x) := p_{n,\rho}(x)$ if it does not confuse us. For any nonzero real valued functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ if there exists a constant $C_1, C_2 > 0$ independent of x such that $C_1 g(x) \leq f(x) \leq C_2 g(x)$ for all x. Similarly, for any positive numbers $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{=1}^{\infty}$ we define $c_n \sim d_n$. Throughout this paper $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t or polynomials $P_n(x)$. The same symbol does not necessarily denote the same constant in different occurrences. We say that $f : \mathbb{R} \to \mathbb{R}^+$ is quasi-increasing if there exists $C > 0$ such that $f(x) \leq Cf(y)$, $0 < x < y$. First, we need the following definition from [11].

Definition 2.1. Let $Q : \mathbb{R} \to \mathbb{R}^+$ be a continuous even function and satisfy the following properties:

(a) $Q'(x)$ is continuous in $\mathbb{R}$, with $Q(0) = 0$.

(b) $Q''(x)$ exists and is positive in $\mathbb{R}\backslash\{0\}$.

(c) $\lim_{x\to\infty} Q(x) = \infty$.

(d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0$$

is quasi-increasing in $(0, \infty)$, with

$$T(x) \geq \Lambda > 1, \quad x \in \mathbb{R}\backslash\{0\}.$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \le C_1 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e.} \quad x \in \mathbb{R}\setminus\{0\}.$$

Then, we write $w = \exp(-Q) \in \mathcal{F}(C^2)$. If there also exists a compact subinterval $J(\ni 0)$ in $\mathbb{R}$, and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \ge C_2 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e.} \quad x \in \mathbb{R}\setminus J,$$

then we write $w = \exp(-Q) \in \mathcal{F}(C^2+)$.

For some typical examples and their properties, we refer to [7, Example 1.2 and Remark 2.3] or [5].

For $w = \exp(-Q) \in \mathcal{F}(C^2+)$, we know that there exists $C > 0$ such that for some $\varepsilon > 0$, and for large enough t,

$$T(a_t) \le Ct^{2-\varepsilon}$$

(see [11, Lemma 3.7]). In [4], $T(a_t)$ is estimated as follows:

Lemma 2.2. [4, Theorem 1.6]

(1) Let $w = \exp(-Q) \in \mathcal{F}(C^2)$, and let $T(x)$ be unbounded. Then for any $\eta > 0$ there exists $C(\eta) > 0$ such that for $t \ge 1$,

$$a_t \le C(\eta)t^{\eta}. \tag{2.1}$$

(2) Let $\lambda := C_1$ be the constant in Definition 2.1 (e), that is,

$$\frac{Q''(x)}{Q'(x)} \le \lambda \frac{Q'(x)}{Q(x)}, \quad a.e. \quad x \in \mathbb{R}\setminus\{0\}.$$

If $1 < \lambda$, then there exists $C(\lambda, \eta)$ such that

$$T(a_t) \le C(\lambda, \eta)t^{\frac{2(\eta+\lambda-1)}{\lambda+1}},$$

and if $0 < \lambda \le 1$, then for any $\eta > 0$ there exists $C(\lambda, \eta)$ such that

$$T(a_t) \le C(\lambda, \eta)t^{\eta}, \quad t \ge 1.$$

Notation 2.3. We use the following notations.

(1) Mhaskar-Rakhmanov-Saff numbers a_x:

$$x = \frac{2}{\pi}\int_0^1 \frac{a_x u Q'(a_x u)}{(1-u^2)^{1/2}} du, \quad x > 0.$$

(2)

$$\varphi_u(x) = \begin{cases} \dfrac{a_u}{u} \dfrac{1 - \frac{|x|}{a_{2u}}}{\sqrt{1 - \frac{|x|}{a_u} + \delta_u}}, & |x| \le a_u; \\ \varphi_u(a_u), & a_u < |x|, \end{cases} \qquad \delta_u = (uT(a_u))^{-2/3}, \quad u > 0.$$

3. Direct Convergence Theorems

We state our theorems. Let

$$X_n(\nu, f; x) := \sum_{k=1}^{n} f(x_{k,n}) l_{k,n}^{\nu}(x) \sum_{i=0}^{\nu-2} e_{0,i}(\ell, \nu, k, n)(x - x_{k,n})^i, \quad \nu \geq 2,$$

$$Y_n(\nu, f; x) := \sum_{k=1}^{n} f(x_{k,n}) l_{k,n}^{\nu}(x) e_{0,\nu-1}(\ell, \nu, k, n)(x - x_{k,n})^{\nu-1}, \quad \nu \geq 1,$$

$$Z_n(\ell, \nu, f; x) := \sum_{k=1}^{n} \sum_{s=1}^{\ell} f^{(s)}(x_{k,n}) l_{k,n}^{\nu}(x) \sum_{i=s}^{\nu-1} e_{s,i}(\ell, \nu, k, n)(x - x_{k,n})^i, \quad \nu \geq 2.$$

Then we have for $\nu \geq 2$

$$L_n(0, \nu, f; x) = X_n(\nu, f; x) + Y_n(\nu, f; x),$$

and

$$L_n(\ell, \nu, f; x) = L_n(0, \nu, f; x) + Z_n(\ell, \nu, f; x).$$

In this section we give the theorems, and write the lemmas which are used to prove the theorems. Through this paper we assume the following:

Assumption 3.1. We consider the weight $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$, and for $\rho \geq 0$ we define $w_\rho(x) = |x|^\rho \exp(-Q(x))$. Then we suppose the following:

(a) If $T(x)$ is bounded, then we suppose that

$$Q(x) \geq C|x|^2,$$

and for some $0 < \delta < 1$,

$$a_n \leq Cn^{1/(1+\nu-\delta)}. \tag{3.1}$$

(b) There exist $0 \leq \gamma < 1$ and $C(\gamma) > 0$ such that

$$T(a_n) \leq C(\gamma)n^\gamma. \tag{3.2}$$

(c) For $0 < c < 1$ small enough and n large enough,

$$a_{\log n} \leq ca_n. \tag{3.3}$$

Remark 3.2. If $T(x)$ is bounded, (3.2) holds with $\gamma = 0$, that is, we see that (3.2) holds with $\gamma = 0$ for the Freud-type weights. We set

$$q_n Q'(q_n) = n, \quad q_{\log n} Q'(q_{\log n}) = \log n,$$

and then we have for some $0 < c_1 \leq c_2 < 1$,

$$n^{c_1} \leq q_n \leq n^{c_2}, \quad (\log n)^{c_1} \leq q_{\log n} \leq (\log n)^{c_2}.$$

Hence,

$$\frac{q_n}{q_{\log n}} = \frac{n}{Q'(q_n)} \frac{Q'(q_{\log n})}{\log n} \sim \frac{n}{\log n} \frac{\log n / q_{\log n}}{n/q_n} = \frac{q_{\log n}}{q_n} \geq \frac{(\log n)^{c_2}}{n^{c_1}} \to \infty$$

as $n \to \infty$. Consequently, we have $\dfrac{q_n}{q_{\log n}} \to \infty$ as $n \to \infty$, that is, from $a_n \sim q_n$, we see that (3.3) holds for the Freud-type weights.

We define $\Phi(x) := \left((1 + Q(x))^{2/3} T(x)\right)^{-1}$. Then we see that $\Phi(x) \sim \dfrac{Q^{1/3}(x)}{xQ'(x)}$ for $0 < d \leq |x|$. Moreover we define

$$\Phi_n(x) := \max\left\{1 - \frac{|x|}{a_n}, \delta_n\right\}, \quad n = 1, 2, 3, ...$$

Then, we have the following:

Lemma 3.3. [6, Lemma 3.4] For $x \in \mathbb{R}$, we have $\Phi(x) \leq C\Phi_n(x)$, $n \geq 1$.

The functions which we consider in this paper are as follows:

Assumption 3.4. Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$ hold Assumption 3.1, and let $\alpha > 0$.

(1) Let $f \in C(\mathbb{R})$ satisfy

$$|f(x)|\,(1 + |x|)^\alpha \left\{\Phi^{-(\frac{3}{4}+(\frac{4-\nu}{2\nu})^+)}(x) w_\rho(x)\right\}^\nu \leq C_f, \quad x \in \mathbb{R}, \tag{3.4}$$

where C_f is a constant depending only on f, and $x^+ = x$, if $x > 0$, $x^+ = 0$, otherwise.

(2) Let $f \in C^\ell(\mathbb{R})$ for a certain $0 \leq \ell \leq \nu - 1$. Then,

$$|f^{(s)}(x)|\,(1 + |x|)^\alpha \left\{\Phi^{-\frac{3}{4}}(x) w_\rho(x)\right\}^\nu \leq C_f, \quad s = 1, 2, \ldots, \ell, \tag{3.5}$$

where C_f is a constant depending only on f.

Remark 3.5. Since f and $f^{(s)}$ are continuous, (3.4) or (3.5) implies the following (3.6) or (3.7) respectively.

(1) Let $f \in C(\mathbb{R})$, then

$$|f(x)|\,(1 + |x|)^\alpha \left\{\Phi^{-(\frac{3}{4}+(\frac{4-\nu}{2\nu})^+)}(x) w(x) \left(|x| + \frac{a_n}{n}\right)^\rho\right\}^\nu \leq C_f, \quad x \in \mathbb{R}. \tag{3.6}$$

(2) Let $f \in C^\ell(\mathbb{R})$ for a certain $0 \leq \ell \leq \nu - 1$. Then,

$$|f^{(s)}(x)|\,(1 + |x|)^\alpha \left\{\Phi_n^{-\frac{3}{4}}(x) w(x) \left(|x| + \frac{a_n}{n}\right)^\rho\right\}^\nu \leq C_f, \quad s = 1, 2, \ldots, \ell, \quad x \in \mathbb{R}. \tag{3.7}$$

(3) Assumption 3.4 (1) implies the following; $f \in C(\mathbb{R})$ satisfy

$$|f(x)|(1+|x|)^{\alpha}\left\{\Phi_n^{-\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu} \le C_f, \qquad x \in \mathbb{R}. \tag{3.8}$$

In what follows, we will use (3.6), (3.7) or (3.8) without notice.

In the rest of this paper we suppose Assumption 3.1, and let $1 < p \le \infty$. Let $C > 0$ mean some absolute constant. We define for $\alpha > 0$

$$\hat{\alpha} = \min\{\alpha, 1\} \quad \text{and} \quad (\log|x|)^* = \begin{cases} \log(1+|x|), & \alpha = 1; \\ 1, & \alpha \ne 1. \end{cases}$$

Let $w_\rho(x) = |x|^\rho w(x)$ and $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$. Then we write $w_\rho(x) \in \mathcal{F}_\rho(C^2+)$.

Proposition 3.6. Let $\nu = 2, 3, \ldots$, and let

$$\Delta > \frac{1}{p} - \hat{\alpha}. \tag{3.9}$$

For $f(x)$ satisfying (3.8), we have

$$\left\|(1+|x|)^{-\Delta}\left\{\Phi^{\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu} X_n(\nu, f; x)\right\|_{L_p(\mathbb{R})} \le CC_f.$$

Proposition 3.7. Let $\nu = 1, 2, 3, \ldots$. Let us suppose (3.9). For $f(x)$ satisfying (3.4), we have

$$\left\|(1+|x|)^{-\Delta}\left\{\Phi^{\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu} Y_n(\nu, f; x)\right\|_{L_p(\mathbb{R})} \le CC_f.$$

Proposition 3.8. Let $\nu = 2, 3, \ldots$. For f satisfying (3.5), we have

$$\begin{aligned} &\left\|(1+|x|)^{-\Delta}\left\{\Phi^{\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu} Z_n(\ell, \nu, f; x)\right\|_{L_p(\mathbb{R})} \\ \le\ & CC_f\frac{a_n\log(1+n)}{n}\begin{cases} a_n^{\frac{1}{p}-\Delta}, & p\Delta < 1; \\ (\log(1+a_n))^{1/p}, & p\Delta = 1; \\ 1, & p\Delta > 1. \end{cases} \end{aligned}$$

Theorem 3.9. Let $\nu = 2, 3, \ldots$, and let (3.9) be satisfied. For $f(x)$ satisfying (3.4), we have

$$\left\|(1+|x|)^{-\Delta}\left\{\Phi^{\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu}(L_n(0, \nu, f; x) - f(x))\right\|_{L_p(\mathbb{R})} \to 0 \quad \text{as} \quad n \to \infty.$$

Theorem 3.10. Let $\nu = 2, 3, \ldots$, and let (3.9) be satisfied. For $f(x)$ satisfying (3.4) and (3.5), we have

$$\left\|(1+|x|)^{-\Delta}\left\{\Phi^{\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu}(L_n(\ell, \nu, f; x) - f(x))\right\|_{L_p(\mathbb{R})} \to 0 \quad \text{as} \quad n \to \infty.$$

4. Proof of Theorems

We have the following lemmas.

Lemma 4.1. Let $w \in \mathcal{F}(C^2+)$ and $0 < p \leq \infty$.

(a) ([7, Lemma 3.2]) For $P \in \mathcal{P}_{\nu n-1}$, $\nu \geq 1$, there exists a certain constant $d > 0$

$$\left\| (1+|x|)^{-\Delta} \left\{ \Phi^{\frac{3}{4}}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right\}^{\nu} P(x) \right\|_{L_p(|x| \geq a_{2n})}$$
$$\leq C n^{-Cn^d} \left\| (1+|x|)^{-\Delta} \left\{ \Phi^{\frac{3}{4}}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right\}^{\nu} P(x) \right\|_{L_p(|x| \leq a_{2n})}.$$

So, if

$$\left\| (1+|x|)^{-\Delta} \left\{ \Phi^{\frac{3}{4}}(x) w(x) \left(1+|x| \right)^{\rho} \right\}^{\nu} P(x) \right\|_{L_p(\mathbb{R})} < \infty,$$

then we have

$$\left\| (1+|x|)^{-\Delta} \left\{ \Phi^{\frac{3}{4}}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right\}^{\nu} P(x) \right\|_{L_p(|x| \geq a_{2n})} \to 0 \quad \text{as} \quad n \to \infty.$$

(b) ([1, Theorem 2.4]) Let $\beta \in \mathbb{R}$. For $P \in \mathcal{P}_n$ and for any $L > 0$, then

$$\left\| w(x) \left(|x| + \frac{a_n}{n} \right)^{\beta} P(x) \right\|_{L_p(\mathbb{R})} \sim \left\| w(x) \left(|x| + \frac{a_n}{n} \right)^{\beta} P(x) \right\|_{L_p(L\frac{a_n}{n} \leq |x| \leq a_n(1-L\eta_n))}.$$

Lemma 4.2. Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$.

(1) ([4, Theorem 2.2 (b)]) For the zeros $x_{j,n}$, $x_{j+1,n}$ there exists $C > 0$ such that for $n \geq 1$ and $1 \leq j \leq n-1$,

$$x_{j,n} - x_{j+1,n} \sim \varphi_n(x_{j,n}).$$

(2) ([4, Theorem 2.2 (a)]) For the minimum positive zero $x_{[n/2],n}$ ($[n/2]$ is the largest integer $\leq n/2$), we have $x_{[n/2],n} \sim a_n n^{-1}$, and for the maximum zero x_{1n} we have $1 - \dfrac{x_{1,n}}{a_n} \sim \delta_n$ for large enough n.

Lemma 4.3. Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$.

(1) ([2, Theorem 2.3, 2.4]) Then, uniformly for $n \geq 1$ we have

$$\sup_{x \in \mathbb{R}} |p_n(x)|\, w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} |x^2 - a_n^2|^{1/4} \sim 1,$$

and

$$\sup_{x \in \mathbb{R}} |p_n(x)|\, w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \sim a_n^{-1/2} (nT(a_n))^{1/6}.$$

(2) ([6, Lemma 4.3]) Uniformly for $n \geq 1$ we have

$$\sup_{x\in\mathbb{R}} \left|\Phi^{\frac{1}{4}}(x)p_n(x)\right| w(x) \left(|x| + \frac{a_n}{n}\right)^{\rho} \sim a_n^{-1/2}.$$

(3) ([2, Theorem 2.5(d)]) For $1 \leq j \leq n-1$ and $x \in [x_{j+1,n}, x_{j,n}]$,

$$|p_n(x)|w(x) \left(|x| + \frac{a_n}{n}\right)^{\rho} \sim \min\left\{|x - x_{j,n}|, |x - x_{j+1,n}|\right\} \varphi_n^{-1}(x_{j,n}) \left(a_n^2 - x_{j,n}^2\right)^{-1/4}.$$

(4) ([2, Theorem 2.5(a), (c)]) For $x \in [x_{1,n}, a_n]$,

$$|p_n(x)|w(x) \left(|x| + \frac{a_n}{n}\right)^{\rho} \leq C(x - x_{1,n})\varphi_n^{-1}(x_{1,n}) \left(a_n^2 - x_{1,n}^2\right)^{-1/4},$$

and for $x \in [-a_n, x_{n,n}]$,

$$|p_n(x)|w(x) \left(|x| + \frac{a_n}{n}\right)^{\rho} \leq C(x_{n,n} - x)\varphi_n^{-1}(x_{n,n}) \left(a_n^2 - x_{n,n}^2\right)^{-1/4}.$$

(5) ([2, Theorem 2.5(a)]) We have

$$\left|p_n' w\right|(x_{j,n}) \left(|x_{j,n}| + \frac{a_n}{n}\right)^{\rho} \sim \varphi_n^{-1}(x_{j,n}) \left(a_n^2 - x_{j,n}^2\right)^{-1/4}$$
$$\sim \frac{n}{a_n} \frac{\sqrt{1 - \frac{|x_{j,n}|}{a_n}}}{1 - \frac{|x_{j,n}|}{a_{2n}}} \left(a_n^2 - x_{j,n}^2\right)^{-1/4} \sim \frac{n}{a_n^{\frac{3}{2}}} \frac{\left|1 - \frac{|x_{j,n}|}{a_n}\right|^{1/4}}{1 - \frac{|x_{j,n}|}{a_{2n}}}.$$

Lemma 4.4. [3, Theorem 2.6] Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$. For each $s = 1, \ldots, \nu - 1$ and $i = s, s+1, \ldots, \nu - 1$, we have $e_{0,0}(\ell, \nu, k, n) = 1$ and

$$|e_{s,i}(\ell, \nu, k, n)| \leq C \left\{ \frac{n}{\left(a_{2n}^2 - x_{k,n}^2\right)^{1/2}} \right\}^{i-s}.$$

Noting Lemma 4.4, we set

$$|X_n|(\nu, f; x) := \sum_{k=1}^{n} \left|f(x_{k,n}) l_{k,n}^{\nu}(x)\right| \sum_{i=0}^{\nu-2} \left\{ \frac{n}{\left(a_{2n}^2 - x_{k,n}^2\right)^{1/2}} \right\}^{i} \left|x - x_{k,n}\right|^{i},$$

$$|Y_n|(\nu, f; x) := \sum_{k=1}^{n} \left|f(x_{k,n}) l_{k,n}^{\nu}(x)\right| \left\{ \frac{n}{\left(a_{2n}^2 - x_{k,n}^2\right)^{1/2}} \right\}^{\nu-1} \left|x - x_{k,n}\right|^{\nu-1}$$

and

$$|Z_n|(\ell, \nu, f; x) := \sum_{k=1}^{n} \sum_{s=1}^{\ell} \left| f^{(s)}(x_{k,n}) l_{k,n}^{\nu}(x) \right| \sum_{i=s}^{\nu-1} \left\{ \frac{n}{\left(a_{2n}^2 - x_{k,n}^2\right)^{1/2}} \right\}^{i-s} \left| x - x_{k,n} \right|^{i}.$$

In what follows, we use the following notation. Let $x_{0,n} := x_{1,n} + \varphi_n(x_{1,n})$ and $x_{n+1,n} = -x_{0,n}$. When $|x| \leq x_{0,n}$, we define

$$x_{m,n} := x_{m(x),n}; \quad |x - x_{m,n}| = \min_{0 \leq j \leq n} |x - x_{j,n}|.$$

If $|x - x_{j+1,n}| = |x - x_{j,n}|$, then we set $x_{m,n} := x_{j,n}$. If $x_{1,n} + \varphi_n(x_{1,n}) < x$, then we put $m = 0$ and if $x < x_{n,n} - \varphi_n(x_{n,n})$, then we put $m = n + 1$.

Lemma 4.5. [cf. [6, Proofs of Proposition 3.7 and 3.9]]

(1) If (3.8) holds, then we have

$$\left\{ \Phi^{\frac{3}{4}}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right\}^{\nu} |X_n|(\nu, f; x) \leq C C_f \sum_{j=0}^{n} (1 + |x_{j,n}|)^{-\alpha}. \quad (4.1)$$

(2) If (3.7) holds, then we have

$$\left\{ \Phi^{\frac{3}{4}}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right\}^{\nu} |Z_n|(\ell, \nu, f; x) \leq C C_f \frac{a_n}{n} \sum_{j=0}^{n} (1 + |x_{j,n}|)^{-\alpha}. \quad (4.2)$$

Proof.

(1) If we see [6, Proofs of Proposition 3.7 and 3.9] carefully, then we have the results. In fact, in [6, Proof of Proposition 3.7], under our assumption

$$|f(x)| \left\{ \Phi^{-\frac{3}{4}}(x) w_{\rho}(x) \right\}^{\nu} \leq C_f, \quad (4.3)$$

we show

$$\left\{ \Phi^{\frac{3}{4}}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right\}^{\nu} |X_n|(\nu, f; x) \leq C C_f \sum_{j=0}^{n} \sum_{i=0}^{\nu-2} \left(\frac{1}{1 + |m - j|} \right)^{\nu - i}$$

(see [6, (4.18) and (4.25)]). If we suppose (3.8) instead of (4.3), then we easily have (4.1).

(2) In [6, (4.49)], under the assumption

$$|f^{(s)}(x)|\,(1+|x|)^{\alpha}\left\{\Phi_n^{-\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu}\le C_f,\quad s=1,2,\dots,\ell, \tag{4.4}$$

we have $\left\{\Phi^{\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu}|Z_n|(\ell,\nu,f;x)\le CC_f\frac{a_n\log n}{n}$. If we suppose (3.7) instead of (4.4), we easily have (4.2). ■

Since $X_n(\nu,f;x)$, $Y_n(\nu,f;x)$ and $Z_n(\ell,\nu,f;x)$ are polynomials, we can use Lemma 4.1, that is, we may estimate Proposition 3.6, 3.7 and 3.8 for $|x|\le a_{2n}$. In fact, in the rest of this paper we do so.

Proof [Proof of Proposition 3.6]. First we set

$$|X_n|(\nu,f;x)=\sum_{k=1}^{n}\left|f(x_{k,n})l_{k,n}^{\nu}(x)\right|\sum_{i=0}^{\nu-2}\left\{\frac{n}{\left(a_{2n}^2-x_{k,n}^2\right)^{1/2}}\right\}^{i}\left|x-x_{k,n}\right|^{i},$$
$$=\sum_{j:|x_{j,n}|\ge\frac{a_n}{3}}+\sum_{j:|x_{j,n}|\le\frac{a_n}{3}}=:\left|X_{1,n}\right|(\nu,f;x)+\left|X_{2,n}\right|(\nu,f;x).$$

Using Lemma 4.5 (1), we see

$$\begin{aligned}&(1+|x|)^{-\Delta}\left\{\Phi^{\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu}\left|X_{1,n}\right|(\nu,f;x)\\&\le CC_f\,(1+|x|)^{-\Delta}\sum_{|x_{j,n}|\ge\frac{a_n}{3}}(1+|x_{j,n}|)^{-\alpha}\sum_{i=0}^{\nu-2}\left(\frac{1}{1+|m-j|}\right)^{\nu-i}\\&\le CC_fa_n^{-\alpha}(1+|x|)^{-\Delta}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left\|(1+|x|)^{-\Delta}\left\{\Phi^{\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu}\left|X_{1,n}\right|(\nu,f;x)\right\|_{L_p(|x|\le a_{2n})}\\&\le CC_fa_n^{-\alpha}\left\|(1+|x|)^{-\Delta}\right\|_{L_p(|x|\le a_{2n})}\\&\le CC_fa_n^{-\alpha}\begin{cases}a_n^{\frac{1}{p}-\Delta}, & p\Delta<1;\\ \log(1+a_n), & p\Delta=1;\\ 1, & p\Delta>1\end{cases}\le CC_f.\end{aligned}$$

Now, we will estimate $\left|X_{2,n}\right|(\nu,f;x)$. For $|x|\le 1$ we have from (4.1)

$$(1+|x|)^{-\Delta}\left\{\Phi^{\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu}\left|X_{2,n}\right|(\nu,f;x)\le CC_f.$$

So we have

$$\left\|(1+|x|)^{-\Delta}\left\{\Phi^{\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu}\left|X_{2,n}\right|(\nu,f;x)\right\|_{L_p(|x|\le 1)}\le CC_f. \tag{4.5}$$

Let $|x| \geq 1$. We divide $\left|X_{2,n}\right|(\nu, f; x)$ into two sums as follows:

$$\begin{aligned}\left|X_{2,n}\right|(\nu, f; x) &= \sum_{j:|x_{j,n}|<\frac{a_n}{3}, |x-x_{j,n}|\geq\frac{|x|}{2}} + \sum_{j:|x_{j,n}|<\frac{a_n}{3}, |x-x_{j,n}|<\frac{|x|}{2}} \\ &=: \left|X_{2,n}^{[1]}\right|(\nu, f; x) + \left|X_{2,n}^{[2]}\right|(\nu, f; x).\end{aligned}$$

We estimate $\left|X_{2,n}^{[1]}\right|$. Using (3.8), Lemma 4.2 (2) and Lemma 4.3 (2) and (4) with $\varphi_n(x_{j,n}) \sim a_n/n$, we see that

$$\begin{aligned}&\left\{\Phi^{\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu}\left|f(x_{j,n})l_{j,n}^{\nu}(x)\right| \\ \leq\ & CC_f(1+|x_{j,n}|)^{-\alpha}\left|\frac{\Phi^{\frac{3}{4}}(x)p_n(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}}{p_n'(x_{j,n})w(x_{j,n})(|x_{j,n}|+\frac{a_n}{n})^{\rho}}\right|^{\nu} \leq CC_f\left(\frac{a_n}{n}\right)^{\nu},\end{aligned} \tag{4.6}$$

and

$$\sum_{i=0}^{\nu-2}\left\{\frac{n}{\left(a_{2n}^2-x_{k,n}^2\right)^{1/2}}\right\}^{i}\left|x-x_{k,n}\right|^{i-\nu} \leq C\sum_{i=0}^{\nu-2}\left(\frac{n}{a_n}\right)^{i}|x|^{i-\nu} \leq C\left(\frac{n}{a_n}\right)^{\nu-2}|x|^{-2}.$$

Then, we have

$$\begin{aligned}&\left\|(1+|x|)^{-\Delta}\left\{\Phi^{\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu}\left|X_{2,n}^{[1]}\right|(\nu, f; x)\right\|_{L_p(1\leq|x|\leq a_{2n})} \\ &\leq CC_f\frac{a_n^2}{n}\left\|(1+|x|)^{-(\Delta+2)}\right\|_{L_p(1\leq|x|\leq a_{2n})} \leq CC_f\end{aligned} \tag{4.7}$$

(note (3.1) and Lemma 2.2 (2.1)). Since we see $|x_{j,n}| \sim |x|$ for $|x-x_{j,n}| \leq |x|/2$, the inequality (4.1) implies

$$\begin{aligned}&\left|\left\{\Phi^{\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu}\left|X_{2,n}^{[2]}\right|(\nu, f; x)\right| \\ &\leq CC_f\sum_{\substack{j:|x_{j,n}|<a_n/3 \\ |x-x_{j,n}|<|x|/2}}(1+|x_{j,n}|)^{-\alpha}\sum_{i=0}^{\nu-2}\left(\frac{1}{1+|m-j|}\right)^{\nu-i} \leq CC_f(1+|x|)^{-\alpha}.\end{aligned}$$

Therefore, we obtain by (4.6) and (3.9)

$$\begin{aligned}&\left\|(1+|x|)^{-\Delta}\left\{\Phi^{\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu}\left|X_{2,n}^{[2]}\right|(\nu, f; x)\right\|_{L_p(1\leq|x|\leq a_{2n})} \\ &\leq CC_f\left\|(1+|x|)^{-(\alpha+\Delta)}\right\|_{L_p(1\leq|x|\leq a_{2n})} \leq CC_f a_n^{\frac{1}{p}-(\alpha+\Delta)} \leq CC_f.\end{aligned} \tag{4.8}$$

Hence, from (4.5), (4.7), (4.8) and Lemma 4.1 we conclude Proposition 3.6. ■

Next, we show Proposition 3.8.

Proof [Proof of Proposition 3.8]. Using Lemma 4.5 (2) (note (3.1)), we get

$$\begin{aligned}
&\left\|(1+|x|)^{-\Delta}\left\{\Phi^{\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu}|Z_n|(\ell,\nu,f;x)\right\|_{L_p(\mathbb{R})}\\
&\le CC_f\frac{a_n\log(1+n)}{n}\left\|(1+|x|)^{-\Delta}\right\|_{L_p(|x|\le a_{2n})}\\
&\le CC_f\frac{a_n\log(1+n)}{n}\begin{cases} a_n^{\frac{1}{p}-\Delta}, & p\Delta<1;\\ \log(1+a_n), & p\Delta=1;\\ 1, & p\Delta>1.\end{cases}
\end{aligned}$$

■

Now, we show Proposition 3.7.

Proof [Proof of Proposition 3.7]. Noting Lemma 4.4, we set

$$A_j(x):=\left\{\Phi^{\frac{3}{4}}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu}f(x_{j,n})l_{j,n}^{\nu}(x)\left(x-x_{j,n}\right)^{\nu-1}e_{0,\nu-1}(\ell,\nu,j,n).$$

Here, we note that there exists a constant $\delta>0$ such that $|x-x_{m,n}|\le\delta\varphi_n(x_{m,n})$. First, let us consider $j=m$ such that $|x-x_{k,n}|<\delta\varphi_n(x_{m,n})$, where $\delta>0$ is small enough, and then we estimate the term $A_m(x)$. By Lemma 3.3 and Lemma 4.3 (3), (4), and (5), we have

$$\begin{aligned}
|A_m(x)|&\le C\Phi^{\frac{3}{4}\nu}(x)\left|f(x_{m,n})\right|\left\{w(x_{m,n})\left(|x_{m,n}|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu}\\
&\times\left|\frac{p_n(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}}{(x-x_{m,n})p_n'(x_{m,n})w(x_{m,n})(|x_{m,n}|+\frac{a_n}{n})^{\rho}}\right|^{\nu}\left(\frac{n}{a_n}\right)^{\nu-1}\left\{\frac{\delta\varphi_n(x_{m,n})}{\sqrt{1-\frac{|x_{m,n}|}{a_u}+\delta_n}}\right\}^{\nu-1}\\
&\le C\delta^{\nu-1}\Phi^{\frac{3}{4}\nu}(x)\left|f(x_{m,n})\right|\left\{w(x_{m,n})\left(|x_{m,n}|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu}\left(\frac{1-\frac{|x_{m,n}|}{a_{2n}}}{1-\frac{|x_{m,n}|}{a_n}+\delta_n}\right)^{\nu-1}\\
&\le CC_f\delta^{\nu-1}\Phi^{\frac{3}{4}\nu}(x)\left(1+|x_{m,n}|\right)^{-\alpha}\Phi_n^{1-\frac{1}{4}\nu}(x_{m,n})\left(1-\frac{|x_{m,n}|}{a_{2n}}\right)^{\nu-1}\quad\text{(see (3.4))}\\
&\le CC_f\left(1+|x_{m,n}|\right)^{-\alpha}.
\end{aligned}$$

Therefore, we have

$$\left\|(1+|x|)^{-\Delta}A_m(x)\right\|\le C\left\|(1+|x|)^{-(\Delta+\alpha)}\right\|\le C_f, \tag{4.9}$$

where $\|\cdot\|$ means $\|\cdot\|_{L_p(|x-x_{m,n}|<\delta\varphi_n(x_{m,n}))}$. We see that from Lemma 4.3 (2) and (5),

$$\left|\frac{\Phi^{\frac{1}{4}}(x)p_n(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}}{p_n'(x_{x_{j,n}})w(x_{j,n})\left(|x_{j,n}|+\frac{a_n}{n}\right)^{\rho}}\right|^{\nu}\le C\left(\frac{a_n}{n}\right)^{\nu}\left(1-\frac{|x_{j,n}|}{a_{2n}}\right)^{\nu}\left|1-\frac{|x_{j,n}|}{a_n}\right|^{-\frac{\nu}{4}}$$

Therefore, we have for $j \neq m$ by (3.6) and Lemma 4.4,

$$\begin{aligned}\left|A_j(x)\right| &\leq \Phi^{\frac{\nu}{2}}(x)\frac{\left(|fw|(x_{j,n})\left(|x_{j,n}|+\frac{a_n}{n}\right)^{\rho}\right)^{\nu}}{\left|x-x_{j,n}\right|}\left|\frac{\left(\Phi^{\frac{1}{4}}p_n w\right)(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}}{(p_n'w)(x_{j,n})\left(|x_{j,n}|+\frac{a_n}{n}\right)^{\rho}}\right|^{\nu}\left|e_{0,\nu-1}\right| \\ &\leq CC_f\frac{a_n}{n}\Phi^{\frac{\nu}{2}}(x)\frac{\left(1+|x_{j,n}|\right)^{-\alpha}\Phi^{\frac{3}{4}\nu+(\frac{4-\nu}{2})^+}(x_{j,n})}{\left|x-x_{j,n}\right|}\left|1-\frac{|x_{j,n}|}{a_n}\right|^{-\frac{\nu}{4}}\left(1-\frac{|x_{j,n}|}{a_{2n}}\right)^{\frac{\nu+1}{2}}.\end{aligned}$$

Since we see by Lemma 3.3 and Lemma 4.2 (2),

$$\Phi(x_{j,n}) \leq C\Phi_n(x_{j,n}) \sim 1-\frac{|x_{j,n}|}{a_n} \quad \text{and} \quad \left(1-\frac{|x_{j,n}|}{a_{2n}}\right)^{\frac{\nu+1}{2}} \leq 1, \quad j=1,2,\ldots,n,$$

we have

$$\left|\sum_{1\leq j\leq n,\, j\neq m} A_j(x)\right| \leq CC_f\frac{a_n}{n}\Phi^{\frac{\nu}{2}}(x)\sum_{1\leq j\leq n,\, j\neq m}\left(1+|x_{j,n}|\right)^{-\alpha}\frac{\Phi^{\frac{\nu}{2}+(\frac{4-\nu}{2})^+}(x_{j,n})}{\left|x-x_{j,n}\right|}. \tag{4.10}$$

Then, we have the sum of $A_j(x)$ for $j : |x_{j,n}| \geq a_n/3$, $j \neq m$, and $|x-x_{j,n}| \geq \delta\varphi_n(x_{m,n})$ as follows:

$$\left|\sum_j A_j(x)\right| \leq CC_f a_n^{-\alpha}\frac{a_n}{n}\Phi^{\frac{\nu}{2}}(x)\sum_j\frac{\Phi^{\frac{\nu}{2}+(\frac{4-\nu}{2})^+}(x_{j,n})}{\left|x-x_{j,n}\right|}.$$

Now, we see that by Lemma 4.2 (1),

$$\begin{aligned}&\frac{1}{|x-x_{j,n}|} \leq C\frac{1}{\sum_{s=m}^{j}\varphi_n(x_{s,n})} \leq C\frac{1}{\frac{a_n}{n}\sum_{s=m}^{j}\frac{1-\frac{|x_{s,n}|}{a_{2n}}}{\sqrt{1-\frac{|x_{s,n}|}{a_n}+\delta_n}}} \\ \leq\ & C\frac{n}{a_n}\frac{1}{\sum_{s=m}^{j}\sqrt{1-\frac{|x_{s,n}|}{a_n}}} \leq C\frac{n}{a_n}\frac{1}{|j-m+1|\sqrt{1-\frac{\max\{|x|,|x_{j,n}|\}}{a_n}+\delta_n}} \\ \leq\ & C\frac{n}{a_n}\frac{1}{|j-m+1|\sqrt{1-\frac{\max\{|x|,|x_{j,n}|\}}{a_n}}+\delta_n} \leq C\frac{n}{a_n}\frac{\max\left\{\Phi_n^{-1/2}(x_{j,n}),\, \Phi_n^{-1/2}(x)\right\}}{|j-m+1|}.\end{aligned}$$

Then we have the sum of $A_j(x)$ for $j : |x_{j,n}| \geq a_n/3$, $j \neq m$, and $|x-x_{j,n}| \geq \delta\varphi_n(x_{m,n})$

as follows:

$$\left|\sum_j A_j(x)\right| \le CC_f a_n^{-\alpha}\Phi^{\frac{\nu}{2}}(x)\sum_j \Phi^{\frac{\nu}{2}+(\frac{4-\nu}{2})^+}(x_{j,n})\frac{\max\left\{\Phi_n^{-\frac{1}{2}}(x_{j,n}),\, \Phi_n^{-\frac{1}{2}}(x)\right\}}{|j-m+1|}$$
$$\le CC_f a_n^{-\alpha}\Phi^{\frac{\nu-1}{2}}(x)\sum_j \Phi^{\frac{\nu-1}{2}+(\frac{4-\nu}{2})^+}(x_{j,n})\frac{\Phi^{\frac{1}{2}}(x)\Phi^{\frac{1}{2}}(x_{j,n})\max\left\{\Phi_n^{-\frac{1}{2}}(x_{j,n}),\, \Phi_n^{-\frac{1}{2}}(x)\right\}}{|j-m+1|}$$
$$\le CC_f a_n^{-\alpha}\Phi^{\frac{\nu-1}{2}}(x)\sum_j \frac{\Phi^{\frac{\nu-1}{2}+(\frac{4-\nu}{2})^+}(x_{j,n})}{|j-m+1|}$$
$$\le CC_f a_n^{-\alpha}\Phi^{\frac{\nu-1}{2}}(x)\frac{1}{\log n}\sum_j \frac{1}{|j-m+1|} \le CC_f a_n^{-\alpha}.$$

Here, we use the following facts:

$$\Phi^{\frac{1}{2}}(x)\Phi^{\frac{1}{2}}(x_{j,n})\max\left\{\Phi_n^{-\frac{1}{2}}(x_{j,n}),\, \Phi_n^{-\frac{1}{2}}(x)\right\} \le 1$$

and by (3.3) for $|x_{j,n}| \ge a_n/3$,

$$\Phi^{\frac{\nu-1}{2}+(\frac{4-\nu}{2})^+}(x_{j,n}) \le \Phi^{\frac{3}{2}}(a_n/3) = \frac{1}{(1+Q(a_n/3))T^{\frac{3}{2}}(a_n/3)}$$
$$\le \frac{C}{Q(a_{\log n})T^{\frac{3}{2}}(a_{\log n})} \le C\frac{\sqrt{T(a_{\log n})}}{T^{\frac{3}{2}}(a_{\log n})\log n} \le \frac{C}{\log n}$$

(see (3.3) and [11, Lemma 3.4 (3.18)]). Therefore, we have the sum of $A_j(x)$ for $j : |x_{j,n}| \ge a_n/3$, $j \ne m$, and $|x - x_{j,n}| \ge \delta\varphi_n(x_{m,n})$ as follows:

$$\left\|(1+|x|)^{-\Delta}\sum_j A_j(x)\right\|_{L_p(|x|\le a_{2n})} \le C_f a_n^{-\alpha}\left\|(1+|x|)^{-\Delta}\right\|_{L_p(|x|\le a_{2n})}$$
$$\le C_f a_n^{-\alpha}\begin{cases} a_n^{\frac{1}{p}-\Delta}, & p\Delta < 1; \\ \log(1+a_n), & p\Delta = 1; \\ 1, & p\Delta > 1 \end{cases} \le C_f. \tag{4.11}$$

Let $|x| \ge 2a_n/3$. Then we see $|x| \le C|x - x_{j,n}|$ for $|x_{j,n}| \le a_n/3$. So, from Lemma

4.3 (2), (4), and (4.10) we have the sum of $A_j(x)$ for $j : |x_{j,n}| \le a_n/3$ as follows:

$$\begin{aligned}\left|\sum_{j:|x_{j,n}|\le a_n/3} A_j(x)\right| &\le CC_f \frac{a_n}{n}\frac{\Phi^{\frac{\nu}{2}}(x)}{|x|}\sum_j \left(1+|x_{j,n}|\right)^{-\alpha}\Phi^{\frac{\nu}{2}+(\frac{4-\nu}{2})^+}(x_{j,n}) \\ &\le C_f\frac{1}{|x|}\sum_j \left(1+|x_{j,n}|\right)^{-\alpha}(x_{j,n}-x_{j+1,n}) \\ &\le C_f\frac{1}{|x|}\int_0^{\frac{1}{3}a_n}(1+t)^{-\alpha}dt \le C_f\frac{1}{|x|}a_n^{1-\hat{\alpha}}(\log(1+a_n))^*.\end{aligned}$$

We use the facts: $x_{j,n} - x_{j+1,n} \sim \dfrac{a_n}{n}$ and $\Phi^{\frac{\nu}{2}}(x) < 1$. Therefore, for the sum of $A_j(x)$ for $j : |x_{j,n}| \le a_n/3$ we have

$$\begin{aligned}&\left\|(1+|x|)^{-\Delta}\sum_j A_j(x)\right\|_{L_p(|x|\ge 2a_n/3)} \\ &\le C_f a_n^{1-\hat{\alpha}}(\log(1+a_n))^*\left\|(1+|x|)^{-(\Delta+1)}\right\|_{L_p(|x|\ge 2a_n/3)} \\ &\le C_f a_n^{\frac{1}{p}-(\hat{\alpha}+\Delta)}(\log(1+a_n))^* \le C_f.\end{aligned} \tag{4.12}$$

Finally, for $|x| \le 2a_n/3$ we estimate the sum of $A_j(x)$ for $j : |x_{j,n}| \le a_n/3$, using the method of [10, Theorem 9.2.3]. Let us define the Hilbert transformation by

$$H[f](x) = \lim_{\varepsilon\to+0}\int_{|t-x|\ge\varepsilon}\frac{f(t)}{t-x}dt.$$

■

If $f \in L_1(\mathbb{R})$, then this limit exists a.e.. We need the following lemmas.

Lemma 4.6. [12, Lemma 3.3] Let $1 < p < \infty$, $\quad s < 1 - \dfrac{1}{p}$, $\quad S > -\dfrac{1}{p}$ and $s \le S$. Then for measurable $f : \mathbb{R} \to \mathbb{R}$, for which the right-hand side is finite,

$$\left\|H[f](x)\,(1+|x|)^s\right\|_{L_p(\mathbb{R})} \le C\left\|f(x)\,(1+|x|)^S\right\|_{L_p(\mathbb{R})}.$$

Lemma 4.7. [12, Lemma 2.5] Let $\zeta \in \mathbb{R}$. There exists a polynomial P_γ of degree $\le C\gamma\log\gamma$ for $\gamma \ge 2$ such that uniformly for $t \in [-\gamma, \gamma]$ and $\gamma \ge 2$,

$$P_\gamma(t) \sim (1+t^2)^\zeta.$$

Lemma 4.8. Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$.

(a) ([1, Theorem 2.3]) Given $r > 1$ and $\beta \in \mathbb{R}$, there exist $C_2, n_0, \alpha > 0$ and $L > 0$ such that for $n \ge n_0$ and $P \in \mathcal{P}_n$,

$$\left\|(Pw)(x)|x|^\beta\right\|_{L_p(|x|\ge a_{rn})} \le C_1\exp(-C_2 n^\alpha)\left\|(Pw)(x)|x|^\beta\right\|_{L_p(L\frac{a_n}{n}\le|x|\le a_n(1-L\delta_n))}.$$

(b) ([1, Theorem 2.4]) Let $0 < p \leq \infty$ and $L > 0$. Let $\beta > -1/p$ if $0 < p < \infty$ and $\beta \geq 0$ if $p = \infty$. Then there exist constants $C, L > 0$ such that for $n > 0$, $P \in \mathcal{P}_n$,

$$\left\| (Pw)(x)|x|^\beta \right\|_{L_p(\mathbb{R})} \leq C \left\| (Pw)(x)|x|^\beta \right\|_{L_p(L\frac{a_n}{n} \leq |x| \leq a_n(1-L\delta_n))}.$$

Using the reproducing kernel (say the Christoffel-Darboux formula):

$$K_n(x,t) = \frac{\gamma_{n-1}}{\gamma_n} \frac{\{p_n(x)p_{n-1}(t) - p_n(t)p_{n-1}(x)\}}{x-t}, \tag{4.13}$$

we define the Christoffel function $\lambda_n(w_{rQ}^2; x)$ by

$$\lambda_n^{-1}(x) = \lambda_n^{-1}(w_\rho^2; x) = K_n(x,x) = \frac{\gamma_{n-1}}{\gamma_n} \left\{ p_n'(x)p_{n-1}(x) - p_n(x)p_{n-1}'(x) \right\},$$

where the Cotes number is given by $\lambda_{k,n} = \lambda_n(x_{k,n})$. Then

$$\lambda_{k,n}^{-1} = \frac{\gamma_{n-1}}{\gamma_n} p_n'(x_{k,n})p_{n-1}(x_{k,n})$$

and $\frac{\gamma_{n-1}}{\gamma_n} \sim a_n$ are satisfied. The generalized Christoffel function $\lambda_{n,p}(x)$ is defined by

$$\lambda_{n,p}(w_\rho; x) = \inf_{P \in \mathcal{P}_{n-1}} \int_{-\infty}^{\infty} |P(t)|^p w_\rho^p(t) dt / |P(x)|^p, \quad 0 < p < \infty. \tag{4.14}$$

The function $\lambda_n(x)$ is a special case for $p = 2$.

Lemma 4.9. [1, Theorem 2.7] Let $w \in \mathcal{F}(C^2)$. Let $0 < p < \infty$ and $\rho > -1/2$.

(a) Uniformly for $n \geq 1$ and $|x| \leq a_n$, we have

$$\lambda_{n,p}(w_\rho; x) \sim \varphi_n(x) w^p(x) \left(|x| + \frac{a_n}{n} \right)^{\rho p}.$$

(b) Moreover, uniformly for a constant $C > 0$ and $n \geq 1$,

$$\lambda_{n,p}(w_\rho; x) \geq C\varphi_n(x) w^p(x) \left(|x| + \frac{a_n}{n} \right)^{\rho p}, \quad x \in \mathbb{R}.$$

Lemma 4.10. Let $w \in \mathcal{F}(C^2+)$, $P \in \mathcal{P}_n$ and let $1 \leq p, q \leq \infty$. Then for $q \leq p$,

$$\left\| P w_\rho \right\|_{L_q(\mathbb{R})} \leq C a_n^{\frac{1}{q} - \frac{1}{p}} \left\| P w_\rho \right\|_{L_p(\mathbb{R})} \tag{4.15}$$

and for $p < q$,

$$\left\| \frac{1}{\sqrt{T}} P w_\rho \right\|_{L_q(\mathbb{R})} \leq C \left(\frac{n}{a_n} \right)^{\frac{1}{p} - \frac{1}{q}} \left\| P w_\rho \right\|_{L_p(\mathbb{R})}. \tag{4.16}$$

Proof. By Lemma 4.8 (b),

$$\begin{aligned} \left\| P w_\rho \right\|^q_{L_q(\mathbb{R})} &\leq C \left\| P w_\rho \right\|^q_{L_q(|x| \leq a_n)} \\ &\leq C \int_{-a_n}^{a_n} (P w_\rho)^q(t) dt \leq C \left(\int_{-a_n}^{a_n} dt \right)^{1-q/p} \left(\int_{-a_n}^{a_n} (P w_\rho)^p(t) dt \right)^{q/p} \\ &\leq C \left\{ a_n^{1/q - 1/p} \left\| P w_\rho \right\|_{L_p(\mathbb{R})} \right\}^q. \end{aligned}$$

So, we obtain (4.15). We show (4.16). Let $1 \leq p < q$. Noting $q > \dfrac{q-p}{p}$, we have

$$\begin{aligned} \left\| \frac{1}{\sqrt{T}} P w_\rho \right\|^q_{L_q(\mathbb{R})} &= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{T}} P w_\rho \right)^q (t) dt \\ &\leq \left\| \left(\frac{1}{\sqrt{T}} \right)^q \left(P w_\rho \right)^{q-p} \right\|_{L_\infty(\mathbb{R})} \int_{-\infty}^{\infty} \left(P w_\rho \right)^p (t) dt \\ &\leq \left\| \frac{1}{\sqrt{T}} \left| P w_\rho \right|^p \right\|^{\frac{q-p}{p}}_{L_\infty(\mathbb{R})} \int_{-\infty}^{\infty} \left(P w_\rho \right)^p (t) dt. \end{aligned} \tag{4.17}$$

Here we use L_p type Christoffel function (4.14). By Lemma 4.9 (b)

$$\begin{aligned} \left| \frac{1}{\sqrt{T(t)}} \left| P(t) w_\rho(t) \right|^p \right| &\leq \frac{1}{\sqrt{T(t)}} \lambda^{-1}_{n,p}(t) w_\rho(t) \left\| w_\rho P \right\|^p_{L_p(\mathbb{R})} \\ &\leq C \frac{1}{\sqrt{T(t)}} \varphi_n^{-1}(t) \left\| w_\rho P \right\|^p_{L_p(\mathbb{R})} \leq C \frac{n}{a_n} \left\| w_\rho P \right\|^p_{L_p(\mathbb{R})}. \end{aligned} \tag{4.18}$$

Here, the last inequality follows from [13, Lemma 3.4], that is,

$$\frac{a_n}{n} \frac{1}{\sqrt{T(t)}} \varphi_n^{-1}(t) \leq C.$$

Therefore, from (4.17) and (4.18) we have (4.16). ■

Let $T_k(x)$ be the Chebyshev polynomial of degree k, and then we define

$$K_n^*(x,t) := \begin{cases} \dfrac{1}{2}, & \text{if } n = 1; \\ \dfrac{1}{2} + \displaystyle\sum_{k=1}^{n-1} T_k(x) T_k(t), & n = 2, 3, \ldots. \end{cases}$$

Then we have the following:

Lemma 4.11. [cf. [10, Lemma 9.2.4 and (9.2.20)]] Let $n \geq 1$ be an integer.

(a) We have

$$\frac{n}{20} \leq K_n^*(x,x) \leq n, \quad |x| \leq 1.$$

(b) Let $p > 1$, and $\{y_k\}_{k=1}^n \subset [-1,1]$ be a system of points satisfying $|y_k - y_{k+1}| \sim 1/n$ for $k = 1, 2, \ldots, n-1$. Then for $u \in [-1,1]$,

$$\sum_{k=1}^{n} \left|K_n^*(y_k, u)\right|^p \leq Cn^p.$$

Lemma 4.12. [cf. [10, Lemma 9.2.5]] Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$. We fix $\sigma \in (0,1)$, and $\zeta \in \mathbb{R}$. Then, for $P \in \mathcal{P}_n$ and $n \geq 1$,

$$\sum_{|x_{j,n}| \leq \sigma a_n} \lambda_{j,n} T^{-1/2}(x_{j,n}) |P(x_{j,n})| \left\{ w(x_{j,n}) \left(|x_{j,n}| + \frac{a_n}{n} \right)^{\rho} \right\}^{-1} (1 + x_{j,n}^2)^{\zeta}$$
$$\leq C \int_{-a_{8n}}^{a_{8n}} |P(t)| w(t) \left(|t| + \frac{a_n}{n} \right)^{\rho} (1+t^2)^{\zeta} dt,$$

where C is a constant independent of P and n.

Proof. By Lemma 4.10 (4.16) with $q = \infty$ and Lemma 4.8 (b), for every integer $m \geq 1$, $R \in \mathcal{P}_m$ and $x \in \mathbb{R}$,

$$\left|T^{-1/2}(x) w_{\rho}(x) R(x)\right| \leq C \frac{m}{a_m} \int_{|t| \leq a_{2m}} \left|w_{\rho}(t) R(t)\right| dt. \tag{4.19}$$

Now, let $m := (l+2)n$, an integer $l > 1$, and let

$$S(x,t) := K_n^{*2}\left(\frac{x}{a_{2m}}, \frac{t}{a_{2m}} \right).$$

We apply (4.19) with $R(t) = P(t) P_{\gamma}(t) S(x,t) \in \mathcal{P}_m$, where $P_{\gamma}(t)$ is defined in Lemma 4.7, and let $\gamma = a_{4n}$ and $m = 3n + [C a_{4n} \log a_{4n}]$, where $[x]$ is the greatest integer less than or equal x. Then we see

$$2m = 2(3n + [C a_{4n} \log a_{4n}]) \leq 8n.$$

Now we have

$$\left|T^{-1/2}(x) w_{\rho}(x) P(x) P_{\gamma}(x) S(x,x)\right| \leq C \frac{n}{a_n} \int_{|t| \leq a_{8n}} \left|w_{\rho}(t) P(t) P_{\gamma}(t) S(x,t)\right| dt.$$

Here, we know

$$\lambda_{j,n} \left(w(x_{j,n}) \left(|x_{j,n}| + \frac{a_n}{n} \right)^{\rho} \right)^{-2} \sim \varphi_n(x_{j,n}) \leq C \frac{a_n}{n}, \quad |x_{j,n}| \leq \sigma a_n$$

(see the definition of $\varphi_n(x)$). So we have by $S(x,x) \sim n^2$ and Lemma 4.11,

$$\sum_{|x_{j,n}|\le\sigma a_n} \lambda_{j,n}\left(w(x_{j,n})\left(|x_{j,n}|+\frac{a_n}{n}\right)^{\rho}\right)^{-2} T^{-1/2}(x_{j,n})\left|P(x_{j,n})P_\gamma(x_{j,n})\right| w_\rho(x_{j,n})$$
$$\le C\int_{-a_{8n}}^{a_{8n}} \left|w_\rho(t)P(t)P_\gamma(t)\right| \left\{\frac{1}{n^2}\sum_{|x_{j,n}|\le\sigma a_n} K_n^{*2}\left(\frac{x_{j,n}}{a_{8n}},\frac{t}{a_{8n}}\right)\right\} dt. \tag{4.20}$$

Here, we see

$$y_{k,n} := \frac{x_{j,n}}{a_{8n}} \in (-1,1), \quad \left|y_{j,n}-y_{j+1,n}\right| \sim \frac{1}{n},$$

and then we apply Lemma 4.11 (b). So we have

$$\frac{1}{n^2}\sum_{|x_{j,n}|\le\sigma a_n} K_n^{*2}\left(\frac{x_{j,n}}{a_{8n}},\frac{t}{a_{8n}}\right) \le C.$$

Consequently, from (4.20) we conclude

$$\sum_{|x_{j,n}|\le\sigma a_n} \lambda_{j,n}T^{-1/2}(x_{j,n})|P(x_{j,n})|\left\{w(x_{j,n})\left(|x_{j,n}|+\frac{a_n}{n}\right)^{\rho}\right\}^{-1}(1+x_{j,n}^2)^{\zeta}$$
$$\le C\int_{-a_{8n}}^{a_{8n}} |P(t)|w(t)\left(|t|+\frac{a_n}{n}\right)^{\rho}(1+t^2)^{\zeta}dt.$$

■

We continue the proof of Proposition 3.7. Let $|x| \le 2a_n/3$. First, let $\delta > 0$ be small enough. For $|x| \le \delta\frac{a_n}{n}$ and $j \ne m$, we see $|x-x_{j,n}| \ge C\frac{a_n}{n}$. Then for $|x| \le \delta\frac{a_n}{n}$, we have the sum of $A_j(x)$ for $j : |x_{j,n}| \le a_n/3,\ j \ne m$ using (4.10) as follows:

$$\left|\sum_{j;|x_{j,n}|\le\frac{a_n}{3},\,j\ne m} A_j(x)\right| \le CC_f\sum_j \left(1+|x_{j,n}|\right)^{-\alpha}\Phi^{\frac{\nu}{2}+(\frac{4-\nu}{2})^+}(x_{j,n}).$$

Then we have

$$\left\|(1+|x|)^{-\Delta}\sum_j A_j(x)\right\|_{L_p(|x|\le\delta\frac{a_n}{n})}^p \le CC_f\frac{a_n}{n}\sum_j\left\{\left(1+|x_{j,n}|\right)^{-\alpha}\Phi^{\frac{\nu}{2}+(\frac{4-\nu}{2})^+}(x_{j,n})\right\}^p$$
$$\le CC_f\int_{|t|\le\frac{a_n}{3}}(1+|t|)^{-\alpha p}\,\Phi^{\left(\frac{\nu}{2}+(\frac{4-\nu}{2})^+\right)p}(t)dt$$
$$\le CC_f\int_{|t|\le\frac{a_n}{3}}(1+|t|)^{-(\alpha+1)p}\,dt \le CC_f. \tag{4.21}$$

Here, we use $\Phi^{\left(\frac{\nu}{2}+(\frac{4-\nu}{2})^+\right)p}(t) \le C\frac{1}{(1+|x|)^p}$. Next, for $\delta\frac{a_n}{n} \le |x| \le \frac{2}{3}a_n$, we have the sum of $A_j(x)$ for $j : |x_{j,n}| \le a_n/3,\ j \ne m$ as follows:

$$\begin{aligned}\left|\sum_j A_j(x)\right| &\le \Phi^{\frac{\nu}{4}+\frac{1}{2}}(x)\left\{\Phi^{\frac{1}{2}}(x)(p_n w)(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{\nu-1} \\ &\quad\times\left|\sum_{|x_{j,n}|\le\frac{a_n}{3}}\frac{f(x_{j,n})}{\left(p_n'(x_{j,n})\right)^{\nu-1}}l_{j,n}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}e_{0,\nu-1}(\ell,\nu,j,n)\right|.\end{aligned}$$

Now, we define a continuous function ψ_n such that

$$\psi_n(x_{j,n}) := \begin{cases} f(x_{j,n})\left\{p_n'(x_{j,n})\right\}^{-\nu+1}e_{0,\nu-1}(\ell,\nu,j,n), & t = x_{j,n}, \quad |t| \le a_n/3, \\ 0, & a_n/3 + a_n/n \le |t|, \\ \text{linear}, & \text{otherwise}. \end{cases}$$

Then, we see from Lemma 4.3 (2)

$$\begin{aligned}\left|\sum_j A_j(x)\right| &\le a_n^{\frac{1-\nu}{2}}\Phi^{\frac{\nu}{4}+\frac{1}{2}}(x)\left|\sum_{|x_{j,n}|\le\frac{a_n}{3}}\psi_n(x_{j,n})l_{j,n}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right| \\ &\le a_n^{\frac{1-\nu}{2}}\left|L_n(0,1,\psi_n;x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right|.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left\|(1+|x|)^{-\Delta}\sum_j A_j(x)\right\|_{L_p(\delta\frac{a_n}{n}\le|x|\le\frac{2}{3}a_n)} \\ &\le Ca_n^{\frac{1-\nu}{2}}\left\|(1+|x|)^{-\Delta}L_n(0,1,\psi_n;x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\|_{L_p(\delta\frac{a_n}{n}\le|x|\le\frac{2}{3}a_n)}.\end{aligned} \tag{4.22}$$

Let

$$g_n(x) := \chi_{[\frac{\delta a_n}{n},\frac{2}{3}a_n]}(x)\left\{\operatorname{sign}L_n(0,1,\psi_n;x)\right\}|L_n(0,1,\psi_n;x)|^{p-1}w_\rho^{p-2}(x)\,(1+|x|)^{-p\Delta}. \tag{4.23}$$

Then

$$\left\|(1+|x|)^{-\Delta}L_n(0,1,\psi_n;x)w_\rho(x)\right\|^p_{L_p(\frac{\delta a_n}{n}\le|x|\le\frac{2}{3}a_n)} = \int_{-\infty}^{\infty}L_n(0,1,\psi_n;x)g_n(x)w_\rho^2(x)dx. \tag{4.24}$$

Let us denote the nth partial sum of the orthonormal expansion of g by $S_n(g)$. We have the following estimate.

$$\begin{aligned}&\int_{-\infty}^{\infty}L_n(0,1,\psi_n;x)g_n(x)w_\rho^2(x)dx \\ &= \int_{-\infty}^{\infty}L_n(0,1,\psi_n;x)S_n(g_n)(x)w_\rho^2(x)dx \quad (\text{orthogonality of } g - S_n(g) \text{ to } \mathcal{P}_{n-1}) \\ &= \sum_{k=1}^{n}\lambda_{k,n}\psi_n(x_{k,n})S_n(g_n)(x_{k,n}) \quad (\text{Gauss quadrature}).\end{aligned}$$

From (3.6), Lemma 4.3 (5), and $\Phi^{(\frac{3}{4}\nu+(\frac{4-\nu}{2})^+)}(x_{j,n}) \le T^{-1/2}(x_{j,n})$, we see for $x_{j,n} \neq 0$,

$$\left|\psi_n(x_{j,n})\right| \le C_f a_n^{\frac{\nu-1}{2}} \left(1+|x_{j,n}|\right)^{-\alpha} T^{-1/2}(x_{j,n}) \left\{w(x_{j,n})\left(|x_{j,n}|+\frac{a_n}{n}\right)^{\rho}\right\}^{-1}.$$

Therefore, using Lemma 4.12, we have

$$\begin{aligned}\int_{-\infty}^{\infty} L_n(0,1,\psi_n;x)g_n(x)w_\rho^2(x)dx &\le C_f a_n^{\frac{\nu-1}{2}} \sum_{|x_{k,n}|\le\frac{2}{3}a_n} \lambda_{k,n}(1+|x_{k,n}|)^{-\alpha} \\ &\quad \times |S_n(g_n)(x_{k,n})| T^{-1/2}(x_{k,n}) \left\{w(x_{k,n})\left(|x_{k,n}|+\frac{a_n}{n}\right)^{\rho}\right\}^{-1} \\ &\le C_f a_n^{\frac{\nu-1}{2}} \int_{-a_{8n}}^{a_{8n}} |S_n(g_n)(t)| w(t)\left(|t|+\frac{a_n}{n}\right)^{\rho}(1+|t|)^{-\alpha}\,dt \\ &\le C_f a_n^{\frac{\nu-1}{2}} \int_{-\infty}^{\infty} |S_n(g_n)(t)| w(t)\left(|t|+\frac{a_n}{n}\right)^{\rho}(1+|t|)^{-\alpha}\,dt \\ &\le C_f a_n^{\frac{\nu-1}{2}} \int_{\delta a_n/n\le|t|\le a_{\gamma n}} |S_n(g_n)(t)| w_\rho(t)\,|P(\alpha;t)|\,dt,\end{aligned}$$

where $\gamma > 8$. And $P(\alpha;t)$ is a polynomial of degree $\le Ca_{8n}\log a_{8n}$ such that $P(\alpha;t) \sim (1+|t|)^{-\alpha}$ on $[-a_{8n}, a_{8n}]$(see Lemma 4.7). For the last inequality we use Lemma 4.1 (b), Lemma 4.8 (b), and $|t|+a_n/n \sim |t|$ for $\delta a_n/n \le |t| \le a_{\gamma n}$. Let

$$h_n(x) := \operatorname{sign}\{S_n(g_n)(x)\}, \quad G_n(x) := (1+|x|)^{-\alpha}\left\{w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\}^{-1}.$$

Hence, using Hölder inequality with $1/p+1/q=1$, we continue

$$\begin{aligned}&\le C_f a_n^{\frac{\nu-1}{2}} \int_{\delta a_n/n\le|t|\le a_{\gamma n}} S_n(g_n)(t)h_n(t)G_n(t)w_\rho^2(t)dt \\ &\le C_f a_n^{\frac{\nu-1}{2}} \int_{-\infty}^{\infty} S_n(g_n)(t)h_n(t)G_n(t)w_\rho^2(t)dt \\ &\le C_f a_n^{\frac{\nu-1}{2}} \int_{-\infty}^{\infty} S_n(h_nG_n)(t)g_n(t)w_\rho^2(t)dt \\ &\le C_f a_n^{\frac{\nu-1}{2}} \left\|g_n w_\rho(1+|t|)^{\Delta}\right\|_{L_q(\mathbb{R})} \left\|(1+|t|)^{-\Delta} S_n(h_nG_n)(t)w(t)\left(|t|+\frac{a_n}{n}\right)^{\rho}\right\|_{L_p(\mathbb{R})}.\end{aligned}$$

Then from (4.23)

$$\begin{aligned}\left\|g_n w_\rho(1+|x|)^{\Delta}\right\|_{L_q(\mathbb{R})} &= \left\||L_n(0,1,\psi_n;x)|^{p-1} w_\rho^{p-1}(x)(1+|x|)^{-(p-1)\Delta}\right\|_{L_q(\frac{\delta a_n}{n}\le|x|\le\frac{2}{3}a_n)} \\ &= \left\|L_n(0,1,\psi_n;x)w_\rho(x)(1+|x|)^{-\Delta}\right\|_{L_p(\frac{\delta a_n}{n}\le|x|\le\frac{2}{3}a_n)}^{p-1}.\end{aligned}$$

Therefore, we have from (4.24)

$$\begin{aligned}&\left\|(1+|x|)^{-\Delta} L_n(0,1,\psi_n;x)w_\rho(x)\right\|_{L_p(\frac{\delta a_n}{n}\le|x|\le\frac{2}{3}a_n)} \\ &\le C_f a_n^{\frac{\nu-1}{2}} \left\|(1+|x|)^{-\Delta} S_n(h_nG_n)(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\|_{L_p(\mathbb{R})}.\end{aligned}$$

Here, we will show

$$\left\|(1+|x|)^{-\Delta} S_n(h_nG_n)(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\|_{L_p(\mathbb{R})} \leq CC_f. \tag{4.25}$$

We have

$$\left\|(1+|x|)^{-\Delta} L_n(0,1,\psi_n;x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right\|_{L_p(\frac{\delta a_n}{n}\leq|x|\leq\frac{2}{3}a_n)} \leq CC_f a_n^{\frac{\nu-1}{2}}.$$

Consequently, from (4.22)

$$\left\|(1+|x|)^{-\Delta}\sum_j A_j(x)\right\|^p_{L_p(\frac{\delta a_n}{n}\leq|x|\leq\frac{2}{3}a_n)} \leq CC_f, \tag{4.26}$$

and then with (4.9), (4.11), (4.12), (4.21), (4.26) and (4.22), we obtain the result of Proposition 3.7.

Using Lemma 4.6, we will show (4.25). For the proof, we only copy the method of [7, pp.255-257]. First, we consider the Christoffel-Darboux formula (4.13). Using Hilbert transform H we have for $fw_\rho \in L_1(\mathbb{R})$,

$$S_n[f](x) = \frac{\gamma_{n-1}}{\gamma_n}\left\{p_n(x)H[p_{n-1}fw_\rho^2](x) - p_{n-1}(x)H[p_nfw_\rho^2](x)\right\}, \quad \frac{\gamma_{n-1}}{\gamma_n}\sim a_n.$$

Therefore, for $|x|\leq\frac{2}{3}a_n$ we obtain by Lemma 4.3 (1)

$$w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}|S_n[h_nG_n](x)| \leq Ca_n^{1/2}\sum_{j=n-1}^{n}|H[p_jh_nG_nw_\rho^2](x)|. \tag{4.27}$$

For $\sigma\in\left(\frac{2}{3},1\right)$ we consider the characteristic function χ_n of $[-\sigma a_n,\sigma a_n]$, $n\geq 1$. Then, for $|x|\leq\frac{2}{3}a_n$, we have

$$H[p_jh_nG_nw_\rho^2] = H[p_j(1-\chi_n)h_nG_nw_\rho^2] + H[p_j\chi_nh_nG_nw_\rho^2].$$

Here, we obtain for $j=n-1,n$,

$$\begin{aligned}&\left|H[p_j(1-\chi_n)h_nG_nw_\rho^2](x)\right| = \left|\int_{\sigma a_n}^{\infty}\frac{p_j(t)(h_nG_nw_\rho^2)(t)}{x-t}dt\right|\\&\leq C\int_{\sigma a_n}^{\infty}|p_jw_\rho|(t)t^{-1-\alpha}dt \leq C\left(\int_{\sigma a_n}^{\infty}\left(p_jw_\rho\right)^2(t)dt\right)^{1/2}\left(\int_{\sigma a_n}^{\infty}t^{-2-2\alpha}dt\right)^{1/2}\\&\leq Ca_n^{-\frac{1}{2}-\alpha}.\end{aligned} \tag{4.28}$$

Let $\hat{\Delta} := \min\left\{\Delta, \frac{1}{p} - \frac{\hat{\alpha}}{2}\right\}$. Then

$$\frac{1}{p} > \frac{1}{p} - \frac{\hat{\alpha}}{2} \geq \hat{\Delta}. \tag{4.29}$$

From (3.9), we see

$$\Delta > \frac{1}{p} - \hat{\alpha} \geq \frac{1}{p} - 1 \quad \text{and} \quad \frac{1}{p} - \frac{\hat{\alpha}}{2} > \frac{1}{p} - \hat{\alpha} \geq \frac{1}{p} - 1. \tag{4.30}$$

From (4.30), we have

$$\hat{\Delta} = \min\left\{\Delta, \frac{1}{p} - \frac{\hat{\alpha}}{2}\right\} > \frac{1}{p} - 1. \tag{4.31}$$

Then we have by (4.29) and (4.31)

$$-\frac{1}{p} < -\hat{\Delta} < 1 - \frac{1}{p}. \tag{4.32}$$

Then we can apply Lemma 4.6 with $s = S = -\hat{\Delta}$ by (4.32). Then using (4.27), $-\Delta \leq -\hat{\Delta}$ and Lemma 4.6, we have

$$\begin{aligned}
&\left\| S_n[h_n G_n](x) w(x) \left(|x| + \frac{a_n}{n}\right)^{\rho} (1+|x|)^{-\Delta} \right\|_{L_p(|x| \leq \frac{2}{3} a_n)} \\
&\leq C \Bigg\{ a_n^{1/2} \sum_{j=n-1}^{n} \left\| H[p_j \chi_n h_n G_n w_\rho^2](x) \, (1+|x|)^{-\hat{\Delta}} \right\|_{L_p(\mathbb{R})} \\
&\qquad\qquad + a_n^{-\alpha} \left\| (1+|x|)^{-\Delta} \right\|_{L_p(|x| \leq \frac{2}{3} a_n)} \Bigg\} \text{ (see (4.28))} \\
&\leq C a_n^{1/2} \sum_{j=n-1}^{n} \left\| (p_j \chi_n h_n G_n w_\rho^2)(x) \, (1+|x|)^{-\hat{\Delta}} \right\|_{L_p(\mathbb{R})} \\
&\qquad\qquad + C a_n^{-\alpha} \begin{cases} 1, & \Delta p > 1, \\ (\log(1+a_n))^{\frac{1}{p}}, & \Delta p = 1, \\ a_n^{\frac{1}{p} - \Delta}, & \Delta p < 1. \end{cases}
\end{aligned}$$

From (3.9), we see $1 < p(\hat{\alpha} + \Delta) \leq p(\alpha + \Delta)$. Then since $\alpha > 0$, we have

$$a_n^{-\alpha} (\log(1+a_n))^{\frac{1}{p}} \leq C a_n^{-\frac{\alpha}{2}} \leq C, \quad a_n^{-\alpha} \leq C \quad \text{and} \quad a_n^{\frac{1}{p} - (\alpha+\Delta)} \leq C.$$

Let $j = n-1, n$. From Lemma 4.3 (1) we have

$$\begin{aligned}
&a_n^{1/2} \left\| (p_j \chi_n h_n G_n w_\rho^2)(x) \, (1+|x|)^{-\hat{\Delta}} \right\|_{L_p(\mathbb{R})} \\
&\leq C \left\| (1+|x|)^{-(\alpha+\hat{\Delta})} \right\|_{L_p(|x| \leq \sigma a_n)} \leq C \left(a_n^{\frac{1}{p} - (\alpha+\hat{\Delta})} + 1 \right) \leq C.
\end{aligned}$$

So, we have

$$\left\| S_n[h_n G_n](x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} (1+|x|)^{-\Delta} \right\|_{L_p(|x| \le \frac{2}{3} a_n)} \le C.$$

Consequently, Proposition 3.7 is complete. ■

Proof [Proof of Theorem 3.9]. By (3.2) and [14, Theorem 1.4] we see that for given ε there exists a polynomial P such that

$$\left\| (1+|x|)^{\alpha} w_{\rho}^{\nu}(x)(f(x) - P(x)) \right\|_{L_p(\mathbb{R})} < \varepsilon =: C_{f-P}.$$

Therefore, from Proposition 3.6 and 3.7 we have

$$\begin{aligned}
& \left\| (1+|x|)^{-\Delta} \left(\Phi^{\frac{3}{4}}(x) w_{\rho}(x) \right)^{\nu} (L_n(0, \nu, f; x) - f(x)) \right\|_{L_p(\mathbb{R})} \\
\le & \left\| (1+|x|)^{-\Delta} \left(\Phi^{\frac{3}{4}}(x) w_{\rho}(x) \right)^{\nu} L_n(0, \nu, f - P; x) \right\|_{L_p(\mathbb{R})} \\
& + \left\| (1+|x|)^{-\Delta} \left(\Phi^{\frac{3}{4}}(x) w_{\rho}(x) \right)^{\nu} (L_n(0, \nu, P; x) - P(x)) \right\|_{L_p(\mathbb{R})} \\
& + \left\| (1+|x|)^{-\Delta} \left(\Phi^{\frac{3}{4}}(x) w_{\rho}(x) \right)^{\nu} (P(x) - f(x)) \right\|_{L_p(\mathbb{R})} \\
\le & \; C \left\| (1+|x|)^{-\Delta} \left(\Phi^{\frac{3}{4}}(x) w_{\rho}(x) \right)^{\nu} (L_n(0, \nu, P; x) - P(x)) \right\|_{L_p(\mathbb{R})} + \varepsilon.
\end{aligned}$$

Here, we see from Proposition 3.8,

$$\begin{aligned}
& \left\| (1+|x|)^{-\Delta} \left(\Phi^{\frac{3}{4}}(x) w_{\rho}(x) \right)^{\nu} (L_n(0, \nu, P; x) - P(x)) \right\|_{L_p(\mathbb{R})} \\
= & \left\| (1+|x|)^{-\Delta} \left(\Phi^{\frac{3}{4}}(x) w_{\rho}(x) \right)^{\nu} (L_n(0, \nu, P; x) - L_n(\nu - 1, \nu, P; x)) \right\|_{L_p(\mathbb{R})} \\
= & \left\| (1+|x|)^{-\Delta} \left(\Phi^{\frac{3}{4}}(x) w_{\rho}(x) \right)^{\nu} Z_n(\nu, P; x) \right\|_{L_p(\mathbb{R})} \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Therefore, we have the result. ■

Proof [Proof of Theorem 3.10]. From Theorem 3.9 we have

$$\begin{aligned}
& \left\| (1+|x|)^{-\Delta} \left(\Phi^{\frac{3}{4}}(x) w_{\rho}(x) \right)^{\nu} (L_n(\ell, \nu, f; x) - f(x)) \right\|_{L_p(\mathbb{R})} \\
\le & \left\| (1+|x|)^{-\Delta} \left(\Phi^{\frac{3}{4}}(x) w_{\rho}(x) \right)^{\nu} (L_n(0, \nu, f; x) - f(x)) \right\|_{L_p(\mathbb{R})} \\
& + \left\| (1+|x|)^{-\Delta} \left(\Phi^{\frac{3}{4}}(x) w_{\rho}(x) \right)^{\nu} (Z_n(\ell, \nu, f; x)) \right\|_{L_p(\mathbb{R})} \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Therefore, we have the result. ■

References

[1] H. S. Jung and R. Sakai, Inequalities with exponential weights, JCAM, 212(2008), 359–373.

[2] H. S. Jung and R. Sakai, Orthonormal polynomials with exponential-type weights, J. Approx. Theory, 152(2008), 215–238.

[3] H. S. Jung and R. Sakai, Markov-Bernstein inequality and Hermite-Fejér interpolation for exponential-type weights, J. Approx. Theory, 162(2010), 1381–1397.

[4] H. S. Jung and R. Sakai, Derivatives of orthonormal polynomials and coefficients of Hermite-Fejér interpolation polynomial with exponential-type weights, J. Inequalities and Applications, Vol. 2010, Article ID 816363, 29 pages.

[5] H. S. Jung and R. Sakai, Specific examples of exponential weights, Commun. Korean Math. Soc., 24 (2009), No. 2, 303–319.

[6] H. S. Jung, G. Nakamura, R. Sakai and N. Suzuki, Convergence or divergence of higher order Hermite or Hermite-Fejér interpolation polynomials with exponential-type weights, ISRN Mathematical Analysis. Vol. 2012, Article ID 904169, 31 pages, doi: 10.5402/2012/904169.

[7] H. S. Jung and R. Sakai, L_p-Convergence of higher order Hermite or Hermite-Fejér interpolation polynomials with exponential-type weights, Advanced Studies in Contemporary Mathematics, 25 (2015), No. 3, 317–332.

[8] T. Kasuga and R. Sakai, Uniform or Mean Convergence of Hermite-Fejér Interpolation of Higher Order for Freud Weights, J. Approx. Theory, 101, 330–358 (1999).

[9] T. Kasuga and R. Sakai, Conditions for Uniform or Mean Convergence Hermite-Fejér Interpolation of Higher Order for Generalized Freud Weights, FJMS, Vol. 19, 145–199 (2005).

[10] H. N. Mhaskar, Introduction to the Theory of Weighted Polynomial Approximation, World Scientific, Singapore, 1996.

[11] A. L. Levin and D. S. Lubinsky, Orthogonal Polynomials for Exponential Weights, Springer, New York, 2001.

[12] D. S. Lubinsky and D. M. Matjila, Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Freud weights, SIAM J. Math Anal., 26(1995), 238–262.

[13] R. Sakai and N. Suzuki, Mollification of exponential weights and its application to the Markov-Bernstein inequality, Pioneer J. of Math., Vol. 7, no. 1, pp. 83–101, 2013.

[14] D. S. Lubinsky, A Survey of Weighted Polynomial Approximation with Exponential Weights, Surveys In Approximation Theory, 3, 1–105 (2007).