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T_1 Hypergraphs

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Abstract

A hypergraph $H = (V, \mathcal{E})$ is said to be a T_1 hypergraph or said to satisfy T_1 axiom if for any two distinct vertices u and v of V there exists a hyperedge containing u but not v and another hyperedge containing v but not u. In this paper we give examples of T_1 hypergraphs. Sufficient conditions for dual of a hypergraph to be T_1 is derived. The proof of line graph of a T_1 hypergraph is T_1 is also given. Sufficient conditions for join of two hypergraphs, corona, incidence graph, middle graph, and total graph of hypergraph to be T_1 are derived. Sufficient conditions for various products of two hypergraphs to be T_1 are derived.

AMS subject classification: 05C65, 05C76.
Keywords: T_1 hypergraph, Dual of a hypergraph, Join of hypergraphs, Corona, Line graph, Primal graph, Incidence graph, Middle graph, Total graph, Cartesian Product, Minimal rank preserving direct product, Maximal rank preserving direct product, Non-rank preserving direct product, Wreath product.

1. Introduction

All the graphs considered here are finite and simple. In this paper we denote the set of vertices of the graph G by $V(G)$, the set of edges of G by $E(G)$, the maximum degree of G by $\Delta(G)$ and the minimum degree of G by $\delta(G)$.

The *degree* [10] of a vertex v in graph G, denoted by deg(v), is the number of edges incident with v. A simple graph is said to be *complete* [1] if every pair of distinct vertices of G are adjacent in G. A graph H is called a *subgraph* [5] of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. A maximal complete subgraph of graph is a *clique* [4] of the graph. That is if Q is a clique in G, then any supergraph of Q is not complete.

A graph G is said to be a T_1 graph if for any two distinct vertices u and v of G, one of the following conditions holds:

1. At least one of u and v is isolated.
2. There exist two edges e_1 and e_2 of G such that e_1 is incident with u but not with v and e_2 is incident with v but not with u.

From the definition of T_1 graph, if G is a T_1 graph with no isolated vertices, then any supergraph of G is T_1.

Hypergraphs are generalization of graphs, hence many of the definitions of graphs carry verbatim to hypergraphs. The basic idea of the hypergraph concept is to consider such a generalization of a graph in which any subset of a given set may be an edge rather than two-element subsets [17]. A *hypergraph* [3] H is a pair $(V, \mathcal{E})$, where V is a set of elements called nodes or vertices, and $\mathcal{E}$ is a set of nonempty subsets of V called *hyperedges* or *edges*. Therefore, $\mathcal{E}$ is a subset of $P(X)\backslash\{\emptyset\}$, where $P(X)$ is the power set of X. In drawing hypergraphs, each vertex is a point in the plane and each edge is a closed curve separating the respective subset from the remaining vertices. The cardinality of the finite set V, is denoted by $|V|$, is called the *order* [16] of the hypergraph. The number of edges is usually denoted by m or $m(H)$ [16] and is called the size of the hypergraph.

A *simple hypergraph* [2] is a hypergraph with the property that if e_i and e_j are hyperedges of H with $e_i \subseteq e_j$, then $i = j$. Two vertices in a hypergraph are *adjacent* [17] if there is a hyperedge which contains both vertices. Two hyperedges in a hypergraph are *incident* [17] if their intersection is nonempty.

A *k-uniform hypergraph* [11] or a k-hypergraph is a hypergraph in which every edge consists of k vertices. So a 2-uniform hypergraph is a graph, a 3-uniform hypergraph is a collection of unordered triples, and so on. The *rank* [17] $r(H)$ of a hypergraph is the maximum of the cardinalities of the edges in the hypergraph. The *co-rank* [17] $cr(H)$ of a hypergraph is the minimum of the cardinalities of a hyperedge in the hypergraph. If $r(H) = cr(H) = k$, then H is k-uniform.

The *degree* [15] $d_H(v)$ of a vertex v in a hypergraph H is the number of edges of H that containing the vertex v. A hypergraph H is k-regular if every vertex has degree k. A vertex of a hypergraph which is incident to no edges is called an *isolated vertex.* [17] The degree of an isolated vertex is trivially zero. The *degree* [6] $d(e)$ of a hyperedge, $e \in \mathcal{E}$ is its cardinality.

A hyperedge e of H with $|e| = 1$ is called a *loop*; more specifically a hyperedge $e = \{v\}$ is a loop at the vertex v. A vertex of degree 1 is called a pendant vertex.

A simple hypergraph H with $|e| = 2$ for each $e \in \mathcal{E}$ is a simple graph.

For a hypergraph $H = (V, \mathcal{E})$, any subhypergraph $H' \subseteq H$ such that $H' = (V, \mathcal{E}')$ is called a *partial subhypergraph* [16]. Any spanning subgraph of a graph is a partial subhypergraph.

Theorem 1.1. [14] Let G be a graph with $\delta(G) \geq 2$. Then G is T_1.

Definition 1.2. [13] A hypergraph $H = (V, \mathcal{E})$ is said to be a Hausdorff hypergraph (or said to satisfy T_2 axiom) if for any two distinct vertices u and v of V there exist hyperedges $e_1, e_2 \in \mathcal{E}$ such that $u \in e_1$ and $v \in e_2$; and $e_1 \cap e_2 = \emptyset$.

In this paper we consider only those hypergraphs with no isolated vertices.

2. T_1 Hypergraphs

In this section we introduce the concept of T_1 hypergraphs. We have investigated the T_1 property of various graphs derived from the given hypergraph and different products of hypergraphs.

Definition 2.1. A hypergraph $H = (V, \mathcal{E})$ is said to be a T_1 hypergraph or said to satisfy T_1 axiom if for any two distinct vertices u and v of V there exists a hyperedge containing u but not v and another hyperedge containing v but not u.

If $H = (V, \mathcal{E})$ is a T_1 hypergraph then the topology generated by the elements of $\mathcal{E}$ is a T_1 topology on V.

It is immediate from the definition of T_1 hypergraphs that all Hausdorff hypergraphs are T_1.

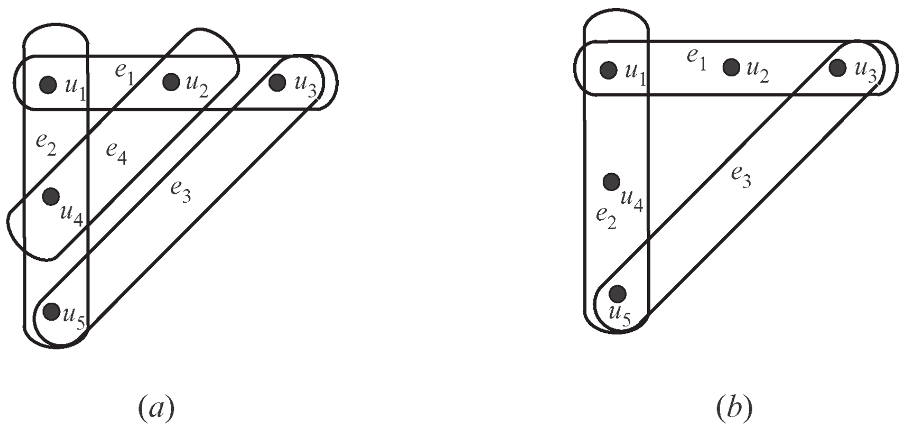

Figure 1: (a) A T_1 hypergraph. (b) A non-T_1 hypergraph.

In the case of graphs it is proved that, if $\delta(G) \geq 2$ then G is T_1. But this is not true in the case of hypergraphs. For example, graph shown in Figure 2 is a non-T_1 hypergraph with $\deg_H(v) \geq 2$, for every $v \in V$.

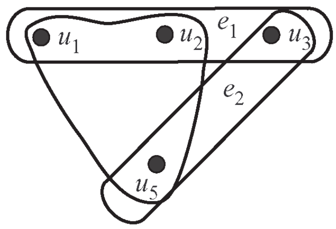

Figure 2: A hypergraph

Now, we discuss the T_1 property of the dual of the given hypergraph.

Theorem 2.2. Let $H = (V, \mathcal{E})$ be a hypergraph. If for any two distinct edges e_i and e_j of H there exist distinct vertices v_p and v_q such that $v_p \in e_i$, $v_p \notin e_j$ and $v_q \in e_j$, $v_q \notin e_i$. Then the dual H^* of H is T_1.

Proof. Consider two distinct vertices e_i and e_j of H^*. By hypothesis, there exist two distinct vertices v_p and v_q such that $v_p \in e_i$, $v_p \notin e_j$ and $v_q \in e_j$ $v_q \notin e_i$. Therefore, the vertices e_i and e_j of H^* are respectively contained in the hyperedges V_p and V_q of H^*. Since $v_p \notin e_j$ and $v_q \notin e_i$, V_p is a hyperedge of H^* containing e_i but not e_j and V_q is a hyperedge of H^* containing e_j but not e_i. ■

Hereafter all the hypergraphs considered in this section are simple hypergraphs.

Next, we investigate the T_1 property of the line graph and the primal graph of the given hypergraph

Theorem 2.3. The line graph $L(H)$ of a T_1 hypergraph H is T_1.

Proof. Let e_1 and e_2 be two distinct vertices of $L(H)$. If one of e_1 and e_2 is isolated then there is nothing to prove. So assume that both e_1 and e_2 are not isolated. Then e_1 and e_2 are two distinct non-loop hyperedges of H. Since $e_1 \neq e_2$, there exist two distinct vertices x and y such that x is contained in e_1 but not in e_2 and y is contained in e_2 but not in e_1. Let u be a vertex of e_1 other than x and v be a vertex of e_2 other than y. Since H is T_1, there exist hyperedges f_1 and f_2 such that f_1 contains x but not u and f_2 contains u but not x. Also there exist hyperedges g_1 and g_2 such that g_1 contains v but not y and g_2 contains y but not v. Then $e_1 f_1$ is an edge of $L(H)$ incident with e_1 but not with e_2 and $e_2 g_2$ is an edge of $L(H)$ incident with e_2 but not with e_1. Therefore, $L(H)$ is T_1. ■

Theorem 2.4. The primal graph $\underline{H}$ of a T_1 hypergraph H is T_1.

Proof. Let u and v be two distinct vertices of the primal graph $\underline{H}$ of the hypergraph H.

As H is T_1 there exist hyperedges e_1 and e_2 of H such that e_1 contains u but not v and e_2 contains v but not u. If e_1 (or e_2) is a loop then u (or v) is an isolated vertex of $\underline{H}$, then there is nothing to prove. So assume that both e_1 and e_2 are not loops. Choose a vertex x of e_1(y of e_2) distinct from u(from v). Then ux is an edge of $\underline{H}$ incident with u but not with v (vy is an edge of $\underline{H}$ incident with v but not with u). Therefore, $\underline{H}$ is T_1. ■

Next consider the T_1 property of the join and the corona of the given two hypergraphs.

Theorem 2.5. The join $H_1 \vee H_2$ of any two hypergraphs H_1 and H_2 is T_1.

Proof. Let u and v be two distinct vertices of $H_1 \vee H_2$. If $u \in V(H_1)$ and $v \in V(H_2)$ then any hyperedge of H_1 containing u and any hyperedge of H_2 containing v serve the purpose. On the other hand if $u, v \in V(H_1)$ (or $V(H_2)$) then for any vertex w of H_2 (or H_1), $\{u, w\}$ is a hyperedge of $H_1 \vee H_2$ containing u but not v and $\{v, w\}$ is a hyperedge of $H_1 \vee H_2$ containing v but not u. Therefore, $H_1 \vee H_2$ is T_1. ■

Theorem 2.6. The corona $H_1 \circ H_2$ of any two hypergraphs H_1 and H_2 is T_1.

Proof. Let u and v be two distinct vertices of $H_1 \circ H_2$. We need only consider the cases where both u and v belong to the copy of H_1 or both lie in the same copy of H_2. In the first case fix a vertex w of some copy of H_2. Then $\{u, w\}$ is a hyperedge of $H_1 \circ H_2$ containing u but not v and $\{v, w\}$ is a hyperedge of $H_1 \circ H_2$ containing v but not u. In the second case we fix a vertex x of copy of H_1. Then $\{x, u\}$ is a hyperedge of $H_1 \circ H_2$ containing u but not v and $\{x, v\}$ is a hyperedge of $H_1 \circ H_2$ containing v but not u. Therefore, $H_1 \circ H_2$ is T_1. ■

3. Incidence Graph, Middle Graph and Total Graph of a Hypergraph

This section investigate the T_1 property of Incidence graph, middle graph, total graph of a given hypergraph. First of all, consider the incidence graph of the given hypergraph, which behaves nicely with the T_1 property, provided $\deg_H(v) \geq 2$, for every vertex v of H.

Definition 3.1. [6] An *incidence graph* of H, $I(H)$, is the bipartite graph with vertices $V \cup \mathcal{E}$ and bipartition $(V, \mathcal{E})$, where $v \in V$ is adjacent to $e \in \mathcal{E}$ if v is contained in the hyperedge e of H.

Theorem 3.2. Let $H = (V, \mathcal{E})$ be a hypergraph. If $\deg_H(v) \geq 2$ for every vertex v of H, then the incidence graph $I(H)$ of H is T_1.

Proof. Let u and v be two distinct vertices of $I(H)$.

Case 1. $u, v \in V$
Since $\deg_H(u)$ *and* $\deg_H(v) \geq 2$, *there exist distinct edges* e_1 *and* e_2 *such that* u *is contained in* e_1 *and* v *is contained in* e_2*. Then* ue_1 *and* ve_2 *are edges of* $I(H)$ *incident*

with u and v respectively.

Case 2. $u, v \in \mathcal{E}$
Let x be a vertex of u and $y \neq x$ be a vertex of v. Then ux and vy are two nonadjacent edges of $I(H)$.

Case 3. $u \in V$ *and* $v \in \mathcal{E}$
Since $\deg_H(u) \geq 2$, *there exists a hyperedge f of H other than v containing u. Since H contains no loops, v contains a vertex x different from u. Then uf and vx are two edges of $I(H)$ incident with u and v respectively.*

Thus given any two distinct vertices u and v of $I(H)$ there exist edges e_1 and e_2 of $I(H)$, such that u is incident with e_1 and v is incident with e_2. Hence $I(H)$ is T_1. ■

Corollary 3.3. If H is a T_1 hypergraph with no loops then the incidence graph $I(H)$ of H is T_1.

If v is a vertex of the hypergraph H contained only in the hyperedge e, then ev is a pendant edge of $I(H)$. Therefore, if H is a hypergraph with $\deg_H(v) = 1$, for some vertex v of H, then the incidence graph $I(H)$ of H cannot be T_1.

Definition 3.4. [8] The *total graph* $T(H)$ of a hypergraph H is the simple graph with vertices $V \cup \mathcal{E}$, where $u, v \in (V \cup \mathcal{E})$ are adjacent if u is contained in, contains or is adjacent to v in H according as $u \in V$ and $v \in \mathcal{E}$, $u \in \mathcal{E}$ and $v \in V$, and $u, v \in V$ or $u, v \in \mathcal{E}$.

Definition 3.5. [8] The *middle graph* $M(H)$ of a hypergraph H is a spanning subgraph of $T(H)$ formed by deleting all edges connecting the pairs of vertices of V.

The following two corollaries follow from Theorem 3.2 and from the fact that for any hypergraph H, its incidence graph $I(H)$ is a spanning subgraph of its middle graph $M(H)$ and its total graph $T(H)$.

Corollary 3.6. Let $H = (V, \mathcal{E})$ be a hypergraph. If $\deg_H(v) \geq 2$ for every vertex v of H, then the middle graph $M(H)$ and the total graph $T(H)$ of H are T_1.

Figure 3.6 shows that the middle graph of a T_1 hypergraph need not be T_1.

Corollary 3.7. If H is a T_1 hypergraph with no loops, then its middle graph $M(H)$ and total graph $T(H)$ are T_1.

Theorem 3.8. Let H be a hypergraph with $|e| \geq 2$ for every $e \in \mathcal{E}$. Then the total graph $T(H)$ of H is T_1.

Proof. Let u be any vertex of $T(H)$.

Suppose $u \in V$. Let e be a hyperedge of H containing u. Since $|e| \geq 2$, e contains one more vertex, say v of H. Therefore, u is adjacent to at least two vertices v and e of $T(H)$. Hence $\deg_H(u) \geq 2$.

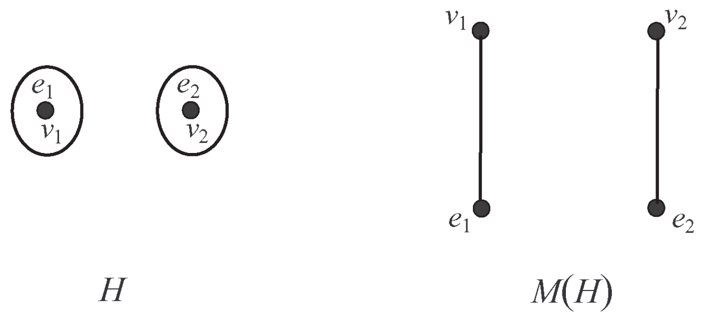

Figure 3: A T_1 hypergraph H and its non-T_1 middle graph $M(H)$.

Suppose $u \in \mathcal{E}$. Then $|u| \geq 2$. Therefore, u contains at least two vertices, say v_1 and v_2 of H. Hence u is adjacent to the vertices v_1 and v_2 of $T(H)$. This implies $\deg_H(u) \geq 2$.

Therefore, $\delta(T(H)) \geq 2$ and hence by Theorem 1.1, $T(H)$ is T_1. ■

4. T_1 Property of Product of Hypergraphs

4.1. Cartesian Product

The Cartesian product of two hypergraphs was introduced by J. Cooper and A. Dutle [7]. It is a natural generalization of the notion of Cartesian products simple graphs. In this section we prove that the Cartesian product of any two hypergraphs is T_1.

Definition 4.1. [7] The *Cartesian product* $H_1 \Box H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is the hypergraph $H = (V, \mathcal{E})$ with vertex set $V = V_1 \times V_2$ and edge set $\mathcal{E} = \big\{\{u\} \times f : u \in V_1, f \in \mathcal{E}_2\big\} \bigcup \big\{e \times \{v\} : e \in \mathcal{E}_1, v \in V_2\big\}$.

Theorem 4.2. Let H_1 and H_2 be two hypergraphs. Then the Cartesian product $H = H_1 \Box H_2$ of H_1 and H_2 is a Hausdorff hypergraph.

Proof. Let $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_m\}$ be the set of all vertices of H_1 and H_2 respectively. Let the hyperedges of H_1 be $e_1, e_2, \ldots, e_p$ and that of H_2 be $f_1, f_2, \ldots, f_q$. Let us denote the hyperedges $\{u_i\} \times f_j$ and $e_r \times \{v_s\}$ by $f_{i,j}$ and $e_{r,s}$ respectively. Consider two distinct vertices (u_i, v_j) and (u_r, v_s) of H.

Case 1. $v_j = v_s$
Let f_j be a hyperedge of H_2 containing v_j. Then $f_{i,j}$ and $f_{r,j}$ are two nonadjacent hyperedges of $H_1 \Box H_2$ such that $(u_i, v_j) \in f_{i,j}$ and $(u_r, v_s) \in f_{r,j}$.

Case 2. $v_j \neq v_s$
Suppose $u_i = u_r$. This case can be dealt as in Case 1.

Now, suppose $u_i \neq u_r$.

Let e_i and e_r be hyperedges of H_1 containing u_i and u_r respectively. Then $e_{i,j}$ and $e_{r,s}$ are two hyperedges of $H_1 \square H_2$ such that $(u_i, v_j) \in e_{i,j}$ and $(u_r, v_s) \in e_{r,s}$.

Hence the theorem. ■

There are many ways to generalize the direct product of graphs to a product of hypergraphs. As we want such products to coincide with the usual direct product when the factors have rank 2 (and are therefore graphs) it is necessary to impose some rank restricting conditions on the edges. This can be accomplished in different ways and leads to different variants of the direct and strong products of graphs. First of all we consider the minimal rank preserving direct product.

4.2. Minimal Rank Preserving Direct Product

This section begins with the definition of minimal rank preserving direct product of two hypergraphs. The restriction of this product to simple graphs coincides with the direct product of graphs.

Definition 4.3. [12] The *minimal rank preserving direct product* $H_1 \equiv H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$. A subset $e = \{(u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)\}$ of $V_1 \times V_2$ is an edge of $H_1 \equiv H_2$ if

1. $\{u_1, u_2, \ldots, u_r\}$ is an edge of H_1 and $\{v_1, v_2, \ldots, v_r\}$ is a subset of an edge of H_2, or

2. $\{u_1, u_2, \ldots, u_r\}$ is a subset of an edge of H_1 and $\{v_1, v_2, \ldots, v_r\}$ is an edge of H_2.

Now, we derive sufficient condition under which the minimal rank preserving direct product of two hypergraphs becomes T_1.

Theorem 4.4. Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. If the degree of each vertex in any edge of degree 2 of the hypergraph H_1(or H_2) is different from 1, then the minimal rank preserving direct product $H_1 \equiv H_2$ of H_1 and H_2 is T_1.

Proof. Suppose the degree of at least one vertex of every hyperedge of degree 2 of the hypergraph H_1 is different from 1. Let (u_1, v_1) and (u_2, v_2) be two distinct vertices of $H_1 \equiv H_2$.

We consider the following two cases to show that there exists a hyperedge of $H_1 \equiv H_2$ of containing (u_1, v_1) but not (u_2, v_2).

Case 1. *$u_1 = u_2$, $v_1 \neq v_2$*
Let $e = \{u_1, u_3\}$ be a hyperedge of H_1 and $f = \{w_1, w_2\}$, where $w_1 = v_1$ be a hyperedge of H_2. Then $\{(u_1, w_1), (u_3, w_2)\}$ is a hyperedge of $H_1 \equiv H_2$ of containing (u_1, v_1) but not (u_2, v_2).

Case 2. $u_1 \neq u_2$, $v_1 \neq v_2$
Suppose there exist two distinct hyperedges f_1 *and* f_2 *of* H_1 *containing* u_1 *and* u_2 *respectively.*

Let $f_1 = \{w_1, w_2\}$ *and* $f_2 = \{x_1, x_2\}$*, where* $w_1 = u_1$ *and* $x_1 = u_2$*. Let* $g = \{y_1, y_2\}$*, where* $y_1 = v_1$ *be a hyperedge of* H_2 *containing* v_1*. Then* $\{(w_1, y_1), (w_2, y_2)\}$ *is a hyperedge of* $H_1 \equiv H_2$ *of containing* (u_1, v_1) *but not* (u_2, v_2)*.*

Suppose that all the hyperedges of H_1 *containing* u_1 *contains* u_2*. Then* $\{u_1, u_2\}$ *is a hyperedge of* $H_1 \equiv H_2$*. Suppose* $f = \{w_1, w_2\}$*, where* $w_1 = v_1$ *is hyperedge of* H_2 *containing* v_1 *but not* v_2*. Then* $\{(u_1, w_1), (u_2, w_2)\}$ *is a hyperedge of* $H_1 \equiv H_2$ *of containing* (u_1, v_1) *but not* (u_2, v_2)*. If no such hyperedge* f *exist, then* $\{v_1, v_2\}$ *is a hyperedge of* H_2*. By the hypothesis degree of either* u_1 *or* u_2 *is different from 1. Without loss of generality assume that degree of* u_1 *is different from 1. Let* h *be a hyperedge of* H_1 *containing* u_1 *and a vertex* x *different from* u_2*. Then* $\{(u_1, v_1), (x, v_2)\}$ *is a hyperedge of* $H_1 \equiv H_2$ *of containing* (u_1, v_1) *but not* (u_2, v_2)*.*

Hence the theorem. ■

Now, we discuss the T_1 property of the normal product of two hypergraphs. The restriction of the normal product to simple graphs coincides with the usual strong product of graphs.

Definition 4.5. [12] The *normal product* $H_1 \equiv H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$ and a subset $e = \{(u_1, v_1), (u_2, v_2), (u_3, v_3), \ldots, (u_n, v_n)\}$ of $V_1 \times V_2$ is an edge of $H_1 \equiv H_2$ if,

1. $\{u_1, u_2, \ldots, u_n\}$ is an edge of H_1 and $v_1 = v_2 = \ldots = v_n \in V_2$, or
2. $\{v_1, v_2, \ldots, v_n\}$ is an edge of H_2 and $u_1 = u_2 = \ldots = u_n \in V_1$, or
3. $\{u_1, u_2, \ldots, u_n\}$ is an edge of H_1 and $\{v_1, v_2, \ldots, v_n\}$ is a subset of an edge of H_2, or
4. $\{v_1, v_2, \ldots, v_n\}$ is an edge of H_2 and $\{u_1, u_2, \ldots, u_n\}$ is a subset of an edge of H_1.

Remark 4.6. Cartesian product $H_1 \square H_2$ of two hypergraphs H_1 and H_2 is a partial subhypergraph of their normal product $H_1 \equiv H_2$.

Theorem 4.7. Let H_1 and H_2 be two hypergraphs. Then the normal product $H_1 \equiv H_2$ of H_1 and H_2 is T_1.

4.3. Maximal Rank Preserving Direct Product

This section discusses the T_1 property of maximal rank preserving direct product of two hypergraphs. As in the case of minimal rank preserving direct product here also the restriction of this product to simple graphs coincides with the direct product of graphs.

Definition 4.8. [12] The *maximal rank preserving direct product* $H_1 := H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$. A subset $e = \{(u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)\}$ of $V_1 \times V_2$ is an edge of $H_1 := H_2$ if

1. $\{u_1, u_2, \ldots, u_r\}$ is an edge of H_1 and there is an edge $f \in \mathcal{E}_2$ of H_2 such that $\{v_1, v_2, \ldots, v_r\}$ is a multiset[1] of elements of f, and $f \subseteq \{v_1, v_2, \ldots, v_r\}$, or
2. $\{v_1, v_2, \ldots, v_r\}$ is an edge of H_2 and there is an edge $e \in \mathcal{E}_1$ of H_1 such that $\{u_1, u_2, \ldots, u_r\}$ is a multiset of elements of e, and $e \subseteq \{u_1, u_2, \ldots, u_r\}$.

If e is the edge of $H_1 := H_2$ determined by the edges e_1 of H_1 and e_2 of H_2, then $|e| = \max\{|e_1|, |e_2|\}$. Therefore, $r(H_1 := H_2) = \max\{r(H_1), r(H_2)\}$ [12].

Theorem 4.9. Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs with no loops. The edges of $H_1 := H_2$ and $H_1 \equiv H_2$ corresponding to the edges of degree 2 in H_1 and H_2 are same. Hence a similar result of Theorem 4.4 also holds in the case of maximal rank preserving direct product.

Theorem 4.10. Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs with no loops. If the degree of at least one vertex of every hyperedge of degree 2 of the hypergraph H_1(or H_2) is different from 1, then the maximal rank preserving direct product $H_1 := H_2$ of H_1 and H_2 is T_1.

Next we consider the T_1 property of the strong product of two hypergraphs. As in the case of normal product of two hypergraphs, the restriction of the strong product to simple graphs coincides with the usual strong product of graphs.

Definition 4.11. [12] The *strong product* $H_1 := H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$ and a subset $e = \{(u_1, v_1), (u_2, v_2),$
$(u_3, v_3), \ldots, (u_n, v_n)\}$ of $V_1 \times V_2$ is an edge of $H_1 := H_2$ if,

1. $\{u_1, u_2, \ldots, u_n\}$ is an edge of H_1 and $v_1 = v_2 = \ldots = v_r \in V_2$, or
2. $\{v_1, v_2, \ldots, v_n\}$ is an edge of H_2 and $u_1 = u_2 = \ldots = u_n \in V_1$, or
3. $\{u_1, u_2, \ldots, u_r\}$ is an edge of H_1 and there is an edge $f \in \mathcal{E}_2$ of H_2 such that $\{v_1, v_2, \ldots, v_r\}$ is a multiset of elements of f, and $f \subseteq \{v_1, v_2, \ldots, v_r\}$, or
4. $\{v_1, v_2, \ldots, v_r\}$ is an edge of H_2 and there is an edge $f \in \mathcal{E}_1$ of H_1 such that $\{u_1, u_2, \ldots, u_r\}$ is a multiset of elements of f, and $f \subseteq \{u_1, u_2, \ldots, u_r\}$.

Remark 4.12. $E(H_1 \overset{\text{def}}{=} H_2) = E(H_1 \Box H_1) \cup E(H_1 := H_2)$. Thus it is immediate that if H_1 and H_2 are two any hypergraphs, then their strong product is T_1.

[1]A multiset is an unordered collection of objects (called the elements) in which, unlike a standard (Cantorian) set, elements are allowed to repeat. In other words a multiset is a set in which elements may belong more than once. $\{1, 1, 1, 2, 3, 3\}$ is a multiset.

4.4. Non-rank Preserving Direct Product

This section is devoted to study the T_1 property of non-rank-preserving direct product $H_1 \stackrel{\text{def}}{=} H_2$ of two hypergraphs H_1 and H_2. This concept was first introduced by M.Hellmuth *et al.* in their paper entitled "A survey on Hypergraph Products" in the year 2012 [12]. Further he mentioned in the same paper that $r(H_1 \stackrel{\text{def}}{=} H_2) = (r(H_1) - 1)(r(H_2) - 1) + 1$ and therefore, the rank of the product will not be the rank of one of its factors. Also if we restrict the definition of this product to simple graphs, then this is exactly the definition of the direct product of graphs.

Definition 4.13. [12] The *non-rank-preserving direct product* $H_1 \stackrel{\text{def}}{=} H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$ and edge set $\big\{\{(u, v)\} \cup ((e - \{u\}) \times (f - \{v\})) : u \in e \in \mathcal{E}_1, v \in f \in \mathcal{E}_2\big\}$.

Now, we discuss T_1 property of non-rank-preserving direct product of two hypergraphs.

Theorem 4.14. Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. Then their non-rank-preserving direct product $H_1 \stackrel{\text{def}}{=} H_2$ is T_1 provided one of them is T_1.

Proof. Without loss of generality assume that H_1 is T_1. Let (u_1, v_1) and (u_2, v_2) be two distinct vertices of $H_1 \stackrel{\text{def}}{=} H_2$.

Suppose that $u_1 = u_2$.

Let $e = \{u_1, u_3, u_4, \ldots, u_{m+1}\}$ be a hyperedge of H_1. Let $f = \{v_1, v_2, \ldots, v_n\}$ be a hyperedge of H_2 containing both v_1 and v_2. Then the hyperedge $\{(u_1, v_1)\} \cup \big(\{u_3, u_4, \ldots, u_{m+1}\} \times \{v_2, v_3, \ldots, v_n\}\big)$ of $H_1 \stackrel{\text{def}}{=} H_2$ contains (u_1, v_1) but not (u_2, v_2) and the hyperedge $\{(u_1, v_2)\} \cup \big(\{u_3, u_4, \ldots, u_{m+1}\} \times \{v_1, v_3, \ldots, v_n\}\big)$ of $H_1 \stackrel{\text{def}}{=} H_2$ contains (u_2, v_2) but not (u_1, v_1). If no such hyperedge f exist, then let $g_1 = \{w_1, w_2, \ldots, w_p\}$, where $w_1 = v_1$ and $g_2 = \{y_1, y_2, \ldots, y_q\}$, where $y_1 = v_2$ be the hyperedges of H_2 containing v_1 and v_2 respectively. Then $\{(u_1, w_1)\} \cup \big(\{u_3, u_4, \ldots, u_{m+1}\} \times \{w_2, w_3, \ldots, w_p\}\big)$ is a hyperedge of $H_1 \stackrel{\text{def}}{=} H_2$ containing (u_1, v_1) but not (u_2, v_2) and $\{(u_1, y_1)\} \cup \big(\{u_3, u_4, \ldots, u_{m+1}\} \times \{y_2, y_3, \ldots, y_q\}\big)$ is a hyperedge of $H_1 \stackrel{\text{def}}{=} H_2$ containing (u_2, v_2) but not (u_1, v_1).

Suppose that $u_1 \neq u_2$.

As H_1 is T_1, there exists a hyperedge e_1 of H_1 containing u_1 but not u_2 and another hyperedge e_2 of H_1 containing u_2 but not u_1. Let $e_1 = \{u_1, u_3, \ldots, u_{p+1}\}$ and $e_2 = \{x_1, x_2, \ldots, x_q\}$, where $x_1 = u_2$.

Let $f = \{v_1, v_2, \ldots, v_n\}$ be a hyperedge of H_2 containing both v_1 and v_2. Then $\{(u_1, v_1)\} \cup \big(\{u_3, u_4, \ldots, u_{p+1}\} \times \{v_2, v_3, \ldots, v_n\}\big)$ is a hyperedge of $H_1 \stackrel{\text{def}}{=} H_2$ containing (u_1, v_1) but not (u_2, v_2) and $\{(x_1, v_2)\} \cup \big(\{x_2, x_3, \ldots, x_q\} \times \{v_1, v_3, v_4, \ldots, v_n\}\big)$ is a hyperedge of $H_1 \stackrel{\text{def}}{=} H_2$ containing (u_1, v_1) but not (u_2, v_2).

If no such hyperedge f exist, then let $g_1 = \{w_1, w_2, \ldots, w_l\}$, where $w_1 = v_1$ and $g_2 = \{y_1, y_2, \ldots, y_m\}$, where $y_1 = v_2$ be the hyperedges of H_2 containing v_1 and v_2 respectively. Then $\{(u_1, w_1)\} \cup \big(\{u_3, u_4, \ldots, u_{p+1}\} \times \{w_2, w_3, \ldots, w_l\}\big)$ is a hyperedge of $H_1 \overset{\text{def}}{=} H_2$ containing (u_1, v_1) but not (u_2, v_2) and $\{(x_1, y_1)\} \cup \big(\{x_2, x_3, \ldots, x_q\} \times \{y_2, y_3, \ldots, y_q\}\big)$ is a hyperedge of $H_1 \overset{\text{def}}{=} H_2$ containing (u_2, v_2) but not (u_1, v_1). ∎

5. Wreath Product

In [9], G. Hahn introduced the definition of wreath product of two hypergraphs as a generalisation of that of graphs. He used the following notations while defining the same.

Let A and B be two sets. For each $e \in P(A \times B)$, we denote the set of first coordinates of elements of e by e^1 and the set of second coordinates of elements of e by e^2.

Definition 5.1. [9] Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs with disjoint vertex sets. The *wreath product* $H = (V, \mathcal{E}) = H_1[H_2]$ of H_1 and H_2 is a hypergraph with vertex set $V = V_1 \times V_2$ and a subset e of $V \times V$ belongs to $\mathcal{E}$ if at least one of the following conditions holds.

1. $e^1 \in \mathcal{E}_1$ and $|e \cap (\{u\} \times V_2)| \leq 1$ for each $u \in V_1$, or
2. $|e^1| = 1$ and $e^2 \in \mathcal{E}_2$.

Theorem 5.2. Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. Then their wreath product $H_1[H_2]$ is Hausdorff.

Proof. Consider two distinct vertices (u_1, v_1) and (u_2, v_2) of $H_1[H_2]$.

Case 1. $u_1 = u_2$ *and* $v_1 \neq v_2$
Let $e = \{u_1, u_3, u_4, \ldots, u_{n+1}\}$ *be an edge of* H_1. *Then the edges* $e_1 = \{(u_1, v_1), (u_3, v_1), (u_4, v_1), \ldots, (u_{n+1}, v_1)\}$ *and* $e_2 = \{(u_1, v_2), (u_3, v_2), (u_4, v_2), \ldots, (u_{n+1}, v_2)\}$ *of* $H_1[H_2]$ *are such that* $(u_1, v_1) \in e_1$ *and* $(u_1, v_2) \in e_2$.

Case 2. $u_1 \neq u_2$ *and* $v_1 = v_2$
Let $f = \{v_1, v_3, v_4, \ldots, v_{p+1}\}$ *be an edge of* H_2. *Now, the edges* $e_1 = \{(u_1, v_1), (u_1, v_3), (u_1, v_4), \ldots, (u_1, v_{p+1})\}$ *and* $e_2 = \{(u_2, v_1), (u_2, v_3), (u_2, v_4), \ldots, (u_2, v_{p+1})\}$ *of* $H_1[H_2]$ *are such that* $(u_1, v_1) \in e_1$ *and* $(u_2, v_1) \in e_2$.

Case 3. $u_1 \neq u_2$ *and* $v_1 \neq v_2$

Subcase 1. *There exists an edge of* H_2 *containing both* v_1 *and* v_2.
Let $f = \{v_1, v_2, \ldots, v_p\}$ *be an edge of* H_2. *Now, the edges* $e_1 = \{(u_1, v_1), (u_1, v_2), (u_1, v_3), \ldots, (u_1, v_p)\}$ *and* $e_2 = \{(u_2, v_1), (u_2, v_2), (u_2, v_3), \ldots, (u_2, v_p)\}$ *of* $H_1[H_2]$ *are edges such that* $(u_1, v_1) \in e_1$ *and* $(u_2, v_2) \in e_2$.

Subcase 2. *There exist no edge of* H_2 *containing both* v_1 *and* v_2.
Let $f_1 = \{v_1, w_2, w_3, \ldots, w_p\}$ *and* $f_2 = \{v_2, z_2, z_3, \ldots, z_q\}$ *be edges* H_2 *containing* v_1 *and* v_2 *respectively. Then the edges* $e_1 = \{(u_1, v_1), (u_1, w_2), (u_1, w_3), \ldots, (u_1, w_p)\}$ *and* $e_2 = \{(u_2, v_2), (u_2, z_2), (u_2, z_3), \ldots, (u_2, z_q)\}$ *of* $H_1[H_2]$ *are edges such that* $(u_1, v_1) \in e_1$ *and* $(u_2, v_2) \in e_2$.

∎

6. Conclusion

In this paper T_1 hypergraphs have been discussed with examples. Sufficient conditions for dual, incidence graph, middle graph, and total graph of hypergraph to be T_1 is derived. Further more, sufficient condition for join and corona of two hypergraphs to be T_1 is derived. We also discussed conditions under which minimal rank, maximal rank , non-rank, preserving direct product of two hypergraphs to be T_1. It is proved that Cartesian, normal, strong and wreath product of any two hypergraphs is always T_1.

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