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T_0 Hypergraphs

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Abstract

A hypergraph $H = (V, \mathcal{E})$ is said to be a T_0 *hypergraph* if for any two distinct vertices u and v of V there exists a hyperedge containing one of them bit not the other. In this paper we give examples of T_0 hypergraphs. Sufficient conditions for dual of a hypergraph to be T_0 is derived. The proof of line graph of a T_0 hypergraph is T_0 is also given. Sufficient conditions for join of two hypergraphs, corona, incidence graph, middle graph, and total graph of hypergraph to be T_0 are derived. Sufficient conditions for various products of two hypergraphs to be T_0 are derived.

AMS subject classification: 05C65, 05C76.
Keywords: T_0 hypergraph, Dual of a hypergraph, Join of hypergraphs, Corona, Line graph, Primal graph, Incidence graph, Middle graph, Total graph, Cartesian Product, Minimal rank preserving direct product, Maximal rank preserving direct product, Non-rank preserving direct product, Wreath product.

1. Introduction

All the graphs considered here are finite and simple. In this paper we denote the set of vertices of the graph G by $V(G)$, the set of edges of G by $E(G)$, the maximum degree of G by $\Delta(G)$ and the minimum degree of G by $\delta(G)$.

The *degree* [9] of a vertex v in graph G, denoted by deg(v), is the number of edges incident with v. A simple graph is said to be *complete* [1] if every pair of distinct vertices of G are adjacent in G. A graph H is called a *subgraph*[5] of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. A maximal complete subgraph of graph is a *clique* [4] of the graph. That is if Q is a clique in G, then any supergraph of Q is not complete.

A graph G is said to be T_0[14] if for any two distinct vertices u and v of G, one of the following three conditions hold: (1) At least one of u and v is isolated (2) There exist two nonadjacent edges e_1 and e_2 of G such that e_1 is incident with u and e_2 is incident with v. From the definition of T_0 graph, if G is T_0 graph with no isolated vertices, then any supergraph of G is T_0.

Hypergraphs are generalization of graphs, hence many of the definitions of graphs carry verbatim to hypergraphs. The basic idea of the hypergraph concept is to consider such a generalization of a graph in which any subset of a given set may be an edge rather than two-element subsets [16]. A *hypergraph* [3] H is a pair $(V, \mathcal{E})$, where V is a set of elements called nodes or vertices, and $\mathcal{E}$ is a set of nonempty subsets of V called *hyperedges* or *edges*. Therefore, $\mathcal{E}$ is a subset of $P(X)\setminus\{\emptyset\}$, where $P(X)$ is the power set of X. In drawing hypergraphs, each vertex is a point in the plane and each edge is a closed curve separating the respective subset from the remaining vertices. The cardinality of the finite set V, is denoted by $|V|$, is called the *order* [15] of the hypergraph. The number of edges is usually denoted by m or $m(H)$ [15] and is called the size of the hypergraph.

A *simple hypergraph* [2] is a hypergraph with the property that if e_i and e_j are hyperedges of H with $e_i \subseteq e_j$, then $i = j$. Two vertices in a hypergraph are *adjacent*[16] if there is a hyperedge which contains both vertices. Two hyperedges in a hypergraph are *incident* [16] if their intersection is nonempty.

A *k-uniform hypergraph* [10] or a k-hypergraph is a hypergraph in which every edge consists of k vertices. So a 2-uniform hypergraph is a graph, a 3-uniform hypergraph is a collection of unordered triples, and so on. The *rank* [16] $r(H)$ of a hypergraph is the maximum of the cardinalities of the edges in the hypergraph. The *co-rank* [16] $cr(H)$ of a hypergraph is the minimum of the cardinalities of a hyperedge in the hypergraph. If $r(H) = cr(H) = k$, then H is k-uniform.

The *degree* [13] $d_H(v)$ of a vertex v in a hypergraph H is the number of edges of H that containing the vertex v. A hypergraph H is k-regular if every vertex has degree k. A vertex of a hypergraph which is incident to no edges is called an *isolated vertex*.[16] The degree of an isolated vertex is trivially zero. The *degree* [6] $d(e)$ of a hyperedge, $e \in \mathcal{E}$ is its cardinality.

A hyperedge e of H with $|e| = 1$ is called a *loop*; more specifically a hyperedge $e = \{v\}$ is a loop at the vertex v. A vertex of degree 1 is called a pendant vertex.

A simple hypergraph H with $|e| = 2$ for each $e \in \mathcal{E}$ is a simple graph.

For a hypergraph $H = (V, \mathcal{E})$, any subhypergraph $H' \subseteq H$ such that $H' = (V, \mathcal{E}')$ is called a *partial subhypergraph* [15]. Any spanning subgraph of a graph is a partial subhypergraph.

Theorem 1.1. [12] Let G be a graph. If K_2 is not a component of G then it is T_0.

In this paper we consider only those hypergraphs with no isolated vertices.

2. T_0 Hypergraphs

In this section we introduce the concept of T_0 hypergraphs. We have investigated the T_0 property of various graphs derived from the given hypergraph and different products of hypergraphs.

Definition 2.1. A hypergraph $H = (V, \mathcal{E})$ is said to be a T_0 hypergraph or said to satisfy T_0 axiom if for any two distinct vertices u and v of V there exists a hyperedge containing one of them but not the other.

If $H = (V, \mathcal{E})$ is a T_0 hypergraph, then the topology generated by the elements of $\mathcal{E}$ is a T_0 topology on V.

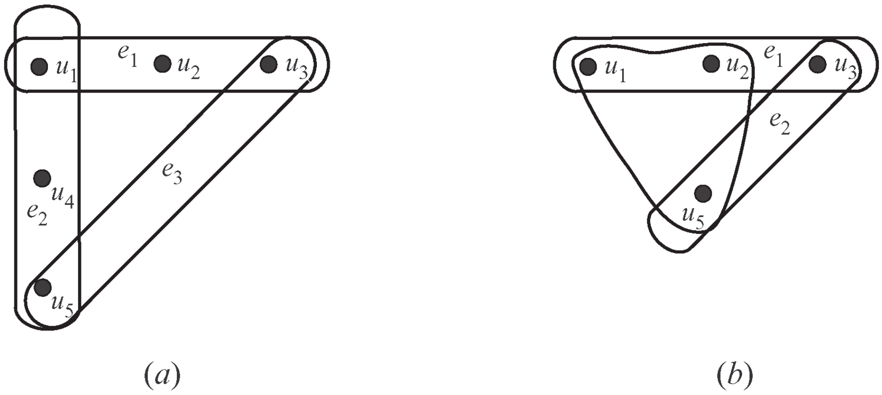

Figure 1: (a) A T_0 hypergraph. (b) A non-T_0 hypergraph.

In the case of graphs, if $\delta(G) \geq 2$, then G is T_0. But this is not true in the case of hypergraphs. For example, graph shown in Figure 1(b) is a non-T_0 hypergraph with $\deg_H(v) \geq 2$, for every $v \in V$.

Next consider the T_0 property of the dual of a give hypergraph.

Theorem 2.2. Let $H = (V, \mathcal{E})$ be a hypergraph. If for any two distinct edges e_i and e_j of H there exists a vertex v_p which belongs to one of them but not the other. Then its dual H^* is T_0.

Proof. Consider two distinct vertices e_i and e_j of H^*. By hypothesis, there exists a vertex v_p which belongs to one of them but not the other. Without loss of generality assume that $v_p \in e_i$. Then, V_p is a hyperedge of H^* containing e_i but not e_j. Hence the theorem. ∎

Hereafter all the hypergraphs considered in this section are simple hypergraphs.

Now, we derive sufficient condition under which line graph of a given hypergraph to be T_0.

Figure 2 shows that line graph of a T_0 hypergraph need not be T_0.

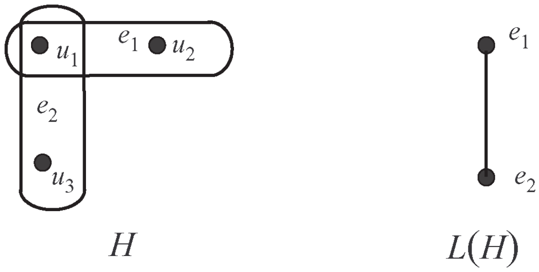

Figure 2: A T_0 hypergraph H and its non-T_0 line graph $L(H)$.

Theorem 2.3. The line graph $L(H)$ of the hypergraph H is T_0 provided H has the property that for any two intersecting hyperedges e_i and e_j of H with cardinality > 1, there exists a hyperedge e_k distinct from e_i and e_j such that either $e_i \cap e_k \neq \emptyset$ or $e_j \cap e_k \neq \emptyset$.

Proof. Let H be the given hypergraph and $L(H)$ be its line graph. Let e_i and e_j be two adjacent vertices of $L(H)$. By the hypothesis there exists an edge e_k of $L(H)$ distinct from e_i and e_j which is incident with either e_i or e_j. This implies K_2 is not a component of $L(H)$. Therefore, by Theorem **??**, $L(H)$ is T_0. ■

Now, we investigate the T_0 property of the primal graph of the given hypergraph.

Figure 3 shows that the primal graph of a non-T_0 hypergraph may be T_0.

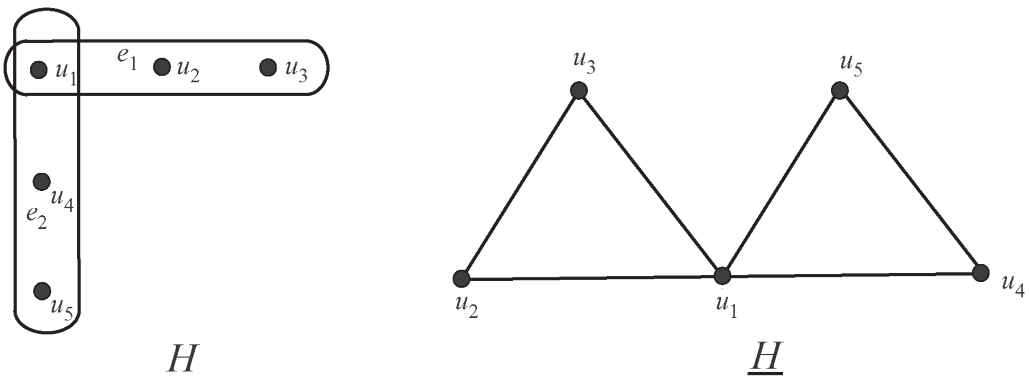

Figure 3: A non- T_0 hypergraph and its T_0 primal graph.

Theorem 2.4. The primal graph $\underline{H}$ of a T_0 hypergraph is T_0.

Proof. Let u and v be two distinct vertices of the primal graph $\underline{H}$ of the hypergraph H. As H is T_0 there exists a hyperedge containing one of them but not the other. Without loss of generality assume that e_1 is a hyperedge of H containing u but not v. If e_1 is a loop then there is nothing to prove. So assume that e_1 is not a loop. Choose a vertex w of e_1 distinct from u. Then uw is an edge of $\underline{H}$ incident with u but not with v. Since u and v are arbitrary, theorem follows. ■

As in the case of T_0 hypergraph, here also we discuss the sufficient condition under which the join and the corona of two hypergraphs to be T_0.

Theorem 2.5. The join $H_1 \vee H_2$ of any two hypergraphs H_1 and H_2 is T_0.

Proof. Let u and v be two distinct vertices of $H_1 \vee H_2$. If $u \in V(H_1)$ and $v \in V(H_2)$ then any hyperedge of H_1 containing u serve the purpose. On the other hand if $u, v \in V(H_1)$ (or $V(H_2)$) then for any vertex w of H_2 (or H_1), $\{u, w\}$ is a hyperedge of $H_1 \vee H_2$ containing u but not v. Therefore, $H_1 \vee H_2$ is T_0. ■

Theorem 2.6. The corona $H_1 \circ H_2$ of any two hypergraphs H_1 and H_2 is T_0.

Proof. Let u and v be two distinct vertices of $H_1 \circ H_2$. We need only consider the cases where both u and v belong to the copy of H_1 or both lie in the same copy of H_2. In the first case fix a vertex w of some copy of H_2. Then $\{u, w\}$ is a hyperedge of $H_1 \circ H_2$ containing u but not v. In the second case we fix a vertex x of copy of H_1. Then $\{x, u\}$ is a hyperedge of $H_1 \circ H_2$ containing u but not v. Therefore, $H_1 \circ H_2$ is T_0. ■

Next, we discuss T_0 property of the incidence graph, the middle graph and the total graph of the given hypergraph.

Theorem 2.7. If H is a hypergraph with no loops, then its incidence graph $I(H)$ is T_0.

Proof. As H is a hypergraph with no loops each hyperedge of H contain at least two vertices. Therefore, each vertex e of $\mathcal{E}$ in the bipartite graph $I(H)$ with bipartition $(V, \mathcal{E})$ is adjacent to at least two vertices of $I(H)$. Therefore, K_2 is not a component of $I(H)$. This implies $I(H)$ is T_0. X ■

Remark 2.8. If $e = \{v\}$ is a loop of the hypergraph H, then ev is an edge of $I(H)$ which is not incident with any other edges of $I(H)$. This implies K_2 is a component of $I(H)$. Hence it is not T_0.

Remark 2.9. As the incidence graph $I(H)$ is the spanning subgraph of its middle graph $M(H)$ and its total graph $T(H)$, if H is a hypergraph with no loops, then $M(H)$ and $T(H)$ are T_0.

Now, we investigate T_0 property of various products of hypergraphs.

3. T_0 Property of Product of Hypergraphs

3.1. Cartesian Product

The Cartesian product of two hypergraphs was introduced by J. Cooper and A. Dutle [7]. It is a natural generalization of the notion of Cartesian products simple graphs. In this section we prove that the Cartesian product of any two hypergraphs is T_0.

Definition 3.1. [7] The *Cartesian product* $H_1 \Box H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is the hypergraph $H = (V, \mathcal{E})$ with vertex set $V = V_1 \times V_2$ and edge set $\mathcal{E} = \big\{\{u\} \times f : u \in V_1, f \in \mathcal{E}_2\big\} \bigcup \big\{e \times \{v\} : e \in \mathcal{E}_1, v \in V_2\big\}$.

Theorem 3.2. Let H_1 and H_2 be two hypergraphs. Then the Cartesian product $H = H_1 \Box H_2$ of H_1 and H_2 is a T_0 hypergraph.

Proof. Let $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_m\}$ be the set of all vertices of H_1 and H_2 respectively. Let the hyperedges of H_1 be $e_1, e_2, \ldots, e_p$ and that of H_2 be $f_1, f_2, \ldots, f_q$. Let us denote the hyperedges $\{u_i\} \times f_j$ and $e_r \times \{v_s\}$ by $f_{i,j}$ and $e_{r,s}$ respectively. Consider two distinct vertices (u_i, v_j) and (u_r, v_s) of H.

Case 1. $v_j = v_s$
Let f_j be a hyperedge of H_2 containing v_j. Then $f_{i,j}$ is a hyperedge of $H_1 \Box H_2$ containing (u_i, v_j) but not (u_r, v_s).

Case 2. $v_j \neq v_s$
Suppose $u_i = u_r$. This case can be dealt as in Case 1.

Now, suppose $u_i \neq u_r$.

Let e_i be a hyperedge of H_1 containing u_i. Then $e_{i,j}$ is a hyperedge of $H_1 \Box H_2$ containing (u_i, v_j) but not (u_r, v_s).

Hence the theorem. ■

There are many ways to generalize the direct product of graphs to a product of hypergraphs. As we want such products to coincide with the usual direct product when the factors have rank 2 (and are therefore graphs) it is necessary to impose some rank restricting conditions on the edges. This can be accomplished in different ways and leads to different variants of the direct and strong products of graphs. First of all we consider the minimal rank preserving direct product.

3.2. Minimal Rank Preserving Direct Product

This section begins with the definition of minimal rank preserving direct product of two hypergraphs. The restriction of this product to simple graphs coincides with the direct product of graphs.

Definition 3.3. [11] The *minimal rank preserving direct product* $H_1 \breve{\times} H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$. A subset $e = \{(u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)\}$ of $V_1 \times V_2$ is an edge of $H_1 \breve{\times} H_2$ if

1. $\{u_1, u_2, \ldots, u_r\}$ is an edge of H_1 and $\{v_1, v_2, \ldots, v_r\}$ is a subset of an edge of H_2, or

2. $\{u_1, u_2, \ldots, u_r\}$ is a subset of an edge of H_1 and $\{v_1, v_2, \ldots, v_r\}$ is an edge of H_2.

Now, we derive sufficient condition under which the minimal rank preserving direct product of two hypergraphs becomes T_0.

Theorem 3.4. Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. If the degree of at least one vertex of every hyperedge of degree 2 of the hypergraph H_1(or H_2) is different from 1, then the minimal rank preserving direct product $H_1 \breve{\times} H_2$ of H_1 and H_2 is T_0.

Proof. Suppose the degree of at least one vertex of every hyperedge of degree 2 of the hypergraph H_1 is different from 1. Let (u_1, v_1) and (u_2, v_2) be two distinct vertices of $H_1 \breve{\times} H_2$.

We consider the following two cases to show that there exists a hyperedge of $H_1 \breve{\times} H_2$ of containing (u_1, v_1) but not (u_2, v_2).

Case 1. $u_1 = u_2$, $v_1 \neq v_2$
Let $e = \{u_1, u_3\}$ be a hyperedge of H_1 and $f = \{w_1, w_2\}$, where $w_1 = v_1$ be a hyperedge of H_2. Then $\{(u_1, w_1), (u_3, w_2)\}$ is a hyperedge of $H_1 \breve{\times} H_2$ of containing (u_1, v_1) but not (u_2, v_2).

Case 2. $u_1 \neq u_2$, $v_1 \neq v_2$
Suppose there exist two distinct hyperedges f_1 and f_2 of H_1 containing u_1 and u_2 respectively.

Let $f_1 = \{w_1, w_2\}$ and $f_2 = \{x_1, x_2\}$, where $w_1 = u_1$ and $x_1 = u_2$. Let $g = \{y_1, y_2\}$, where $y_1 = v_1$ be a hyperedge of H_2 containing v_1. Then $\{(w_1, y_1), (w_2, y_2)\}$ is a hyperedge of $H_1 \breve{\times} H_2$ of containing (u_1, v_1) but not (u_2, v_2).

Suppose that all the hyperedges of H_1 containing u_1 contains u_2. Then $\{u_1, u_2\}$ is a hyperedge of $H_1 \breve{\times} H_2$. Suppose $f = \{w_1, w_2\}$, where $w_1 = v_1$ is hyperedge of H_2 containing v_1 but not v_2. Then $\{(u_1, w_1), (u_2, w_2)\}$ is a hyperedge of $H_1 \breve{\times} H_2$ of containing (u_1, v_1) but not (u_2, v_2). If no such hyperedge f exist, then $\{v_1, v_2\}$ is a hyperedge of H_2. By the hypothesis degree of either u_1 or u_2 is different from 1. Without loss of generality assume that degree of u_1 is different from 1. Let h be a hyperedge of H_1 containing u_1 and a vertex x different from u_2. Then $\{(u_1, v_1), (x, v_2)\}$ is a hyperedge of $H_1 \breve{\times} H_2$ of containing (u_1, v_1) but not (u_2, v_2).

Hence the theorem. ■

Now, we discuss the T_0 property of the normal product of two hypergraphs. The restriction of the normal product to simple graphs coincides with the usual strong product of graphs.

Definition 3.5. [11] The *normal product* $H_1 \breve{\boxtimes} H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$ and a subset $e = \{(u_1, v_1), (u_2, v_2), (u_3, v_3), \ldots, (u_n, v_n)\}$ of $V_1 \times V_2$ is an edge of $H_1 \breve{\boxtimes} H_2$ if,

1. $\{u_1, u_2, \ldots, u_n\}$ is an edge of H_1 and $v_1 = v_2 = \ldots = v_n \in V_2$, or

2. $\{v_1, v_2, \ldots, v_n\}$ is an edge of H_2 and $u_1 = u_2 = \ldots = u_n \in V_1$, or

3. $\{u_1, u_2, \ldots, u_n\}$ is an edge of H_1 and $\{v_1, v_2, \ldots, v_n\}$ is a subset of an edge of H_2, or

4. $\{v_1, v_2, \ldots, v_n\}$ is an edge of H_2 and $\{u_1, u_2, \ldots, u_n\}$ is a subset of an edge of H_1.

Remark 3.6. Cartesian product $H_1 \Box H_2$ of two hypergraphs H_1 and H_2 is a partial subhypergraph of their normal product $H_1 \breve{\boxtimes} H_2$.

Theorem 3.7. Let H_1 and H_2 be two hypergraphs. Then the normal product $H_1 \breve{\boxtimes} H_2$ of H_1 and H_2 is T_0.

3.3. Maximal Rank Preserving Direct Product

This section discusses the T_0 property of maximal rank preserving direct product of two hypergraphs. As in the case of minimal rank preserving direct product here also the restriction of this product to simple graphs coincides with the direct product of graphs.

Definition 3.8. [11] The *maximal rank preserving direct product* $H_1 \hat{\times} H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$. A subset $e = \{(u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)\}$ of $V_1 \times V_2$ is an edge of $H_1 \hat{\times} H_2$ if

1. $\{u_1, u_2, \ldots, u_r\}$ is an edge of H_1 and there is an edge $f \in \mathcal{E}_2$ of H_2 such that $\{v_1, v_2, \ldots, v_r\}$ is a multiset[1] of elements of f, and $f \subseteq \{v_1, v_2, \ldots, v_r\}$, or

2. $\{v_1, v_2, \ldots, v_r\}$ is an edge of H_2 and there is an edge $e \in \mathcal{E}_1$ of H_1 such that $\{u_1, u_2, \ldots, u_r\}$ is a multiset of elements of e, and $e \subseteq \{u_1, u_2, \ldots, u_r\}$.

If e is the edge of $H_1 \hat{\times} H_2$ determined by the edges e_1 of H_1 and e_2 of H_2, then $|e| = \max\{|e_1|, |e_2|\}$. Therefore, $r(H_1 \hat{\times} H_2) = \max\{r(H_1), r(H_2)\}$ [11].

Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs with no loops. The edges of $H_1 \hat{\times} H_2$ and $H_1 \check{\times} H_2$ corresponding to the edges of degree 2 in H_1 and H_2 are same. Hence a similar result of Theorem 3.4 also holds in the case of maximal rank preserving direct product.

[1] A multiset is an unordered collection of objects (called the elements) in which, unlike a standard (Cantorian) set, elements are allowed to repeat. In other words a multiset is a set in which elements may belong more than once. $\{1, 1, 1, 2, 3, 3\}$ is a multiset.

Theorem 3.9. Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs with no loops. If the degree of at least one vertex of every hyperedge of degree 2 of the hypergraph H_1(or H_2) is different from 1, then the maximal rank preserving direct product $H_1 \hat{\times} H_2$ of H_1 and H_2 is T_0.

Next we consider the T_0 property of the strong product of two hypergraphs. As in the case of normal product of two hypergraphs, the restriction of the strong product to simple graphs coincides with the usual strong product of graphs.

Definition 3.10. [11] The *strong product* $H_1 \widehat{\boxtimes} H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$ and a subset $e = \{(u_1, v_1), (u_2, v_2),$
$(u_3, v_3), \ldots, (u_n, v_n)\}$ of $V_1 \times V_2$ is an edge of $H_1 \widehat{\boxtimes} H_2$ if,

1. $\{u_1, u_2, \ldots, u_n\}$ is an edge of H_1 and $v_1 = v_2 = \ldots = v_r \in V_2$, or

2. $\{v_1, v_2, \ldots, v_n\}$ is an edge of H_2 and $u_1 = u_2 = \ldots = u_n \in V_1$, or

3. $\{u_1, u_2, \ldots, u_r\}$ is an edge of H_1 and there is an edge $f \in \mathcal{E}_2$ of H_2 such that $\{v_1, v_2, \ldots, v_r\}$ is a multiset of elements of f, and $f \subseteq \{v_1, v_2, \ldots, v_r\}$, or

4. $\{v_1, v_2, \ldots, v_r\}$ is an edge of H_2 and there is an edge $f \in \mathcal{E}_1$ of H_1 such that $\{u_1, u_2, \ldots, u_r\}$ is a multiset of elements of f, and $f \subseteq \{u_1, u_2, \ldots, u_r\}$.

Remark 3.11. $E(H_1 \widehat{\boxtimes} H_2) = E(H_1 \Box H_1) \cup E(H_1 \hat{\times} H_2)$. Thus it is immediate that if H_1 and H_2 are two any hypergraphs, then their strong product is T_0.

3.4. Non-rank Preserving Direct Product

This section is devoted to study the T_0 property of non-rank-preserving direct product $H_1 \tilde{\times} H_2$ of two hypergraphs H_1 and H_2. This concept was first introduced by M.Hellmuth *et al.* in their paper entitled "A survey on Hypergraph Products" in the year 2012 [11]. Further he mentioned in the same paper that $r(H_1 \tilde{\times} H_2) = (r(H_1) - 1)(r(H_2) - 1) + 1$ and therefore, the rank of the product will not be the rank of one of its factors. Also if we restrict the definition of this product to simple graphs, then this is exactly the definition of the direct product of graphs.

Definition 3.12. [11] The *non-rank-preserving direct product* $H_1 \tilde{\times} H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$ and edge set $\big\{\{(u, v)\} \cup ((e - \{u\}) \times (f - \{v\})) : u \in e \in \mathcal{E}_1, v \in f \in \mathcal{E}_2\big\}$.

Remark 3.13. If H_1 is a hypergraph with all of its edges are loops then for any hypergraph H_2, the edges of $H_1 \tilde{\times} H_2$ are loops. Hence it is T_0.

Theorem 3.14. Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. Then their non-rank-preserving direct product $H_1 \tilde{\times} H_2$ is T_0 provided one of them is T_0.

Proof. Without loss of generality assume that H_1 is T_0. Let (u_1, v_1) and (u_2, v_2) be two distinct vertices of $H_1 \tilde{\times} H_2$.

Suppose that $u_1 = u_2$.

Let $e = \{u_1, u_3, u_4, \ldots, u_{m+1}\}$ be a hyperedge of H_1 and $f = \{w_1, w_2, \ldots, w_n\}$, where $w_1 = v_1$ be a hyperedge of H_2. Then $\{(u_1, w_1)\} \cup \big(\{u_3, u_4, \ldots, u_{m+1}\} \times \{w_2, w_3, \ldots, w_n\}\big)$ is a hyperedge of $H_1 \tilde{\times} H_2$ containing (u_1, v_1) but not (u_2, v_2).

Suppose that $u_1 \neq u_2$.

As H_1 is T_0, there exists a hyperedge of H_1 containing u_1 but not u_2. Let $g = \{x_1, x_2, \ldots, x_p\}$, where $x_1 = u_1$ be a hyperedge of H_1 containing u_1 but not u_2. Let $h = \{y_1, y_2, \ldots, y_m\}$, where $y_1 = v_1$ be hyperedge of H_2 containing v_1. Then $\{(x_1, y_1)\} \cup \big(\{x_2, x_3, \ldots, x_p\} \times \{y_2, y_3, \ldots, y_n\}\big)$ is a hyperedge of $H_1 \tilde{\times} H_2$ containing (u_1, v_1) but not (u_2, v_2). ■

3.5. Wreath Product

In [8], G. Hahn introduced the definition of wreath product of two hypergraphs as a generalisation of that of graphs. He used the following notations while defining the same.

Let A and B be two sets. For each $e \in P(A \times B)$, we denote the set of first coordinates of elements of e by e^1 and the set of second coordinates of elements of e by e^2.

Definition 3.15. [8] Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs with disjoint vertex sets. The *wreath product* $H = (V, \mathcal{E}) = H_1[H_2]$ of H_1 and H_2 is a hypergraph with vertex set $V = V_1 \times V_2$ and a subset e of $V \times V$ belongs to $\mathcal{E}$ if at least one of the following conditions holds.

1. $e^1 \in \mathcal{E}_1$ and $|e \cap (\{u\} \times V_2)| \leq 1$ for each $u \in V_1$, or
2. $|e^1| = 1$ and $e^2 \in \mathcal{E}_2$.

Theorem 3.16. Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. Then their wreath product $H_1[H_2]$ is T_0.

Proof. Consider two distinct vertices (u_1, v_1) and (u_2, v_2) of $H_1[H_2]$.

Case 1. $u_1 = u_2$ *and* $v_1 \neq v_2$
Let $e = \{u_1, u_3, u_4, \ldots, u_{n+1}\}$ *be an edge of* H_1. *Then* $\{(u_1, v_1), (u_3, v_1), (u_4, v_1), \ldots, (u_{n+1}, v_1)\}$ *is a hyperedge of* $H_1[H_2]$ *containing* (u_1, v_2) *but not* (u_2, v_2).

Case 2. $u_1 \neq u_2$ *and* $v_1 = v_2$
Let $f = \{v_1, v_3, v_4, \ldots, v_{p+1}\}$ *be an edge of* H_2. *Then the hyperedge* $\{(u_1, v_1), (u_1, v_3), (u_1, v_4), \ldots, (u_1, v_{p+1})\}$ *is a hyperedge of* $H_1[H_2]$ *containing* (u_1, v_2) *but not* (u_2, v_2).

Case 3. $u_1 \neq u_2$ *and* $v_1 \neq v_2$

Subcase 1. *There exists an edge of* H_2 *containing both* v_1 *and* v_2.
Let $f = \{v_1, v_2, \ldots, v_p\}$ *be an edge of* H_2. *Then the hyperedge* $\{(u_1, v_1), (u_1, v_2), (u_1, v_3), \ldots, (u_1, v_p)\}$ *is a hyperedge of* $H_1[H_2]$ *containing* (u_1, v_2) *but not* (u_2, v_2).

Subcase 2. *There exist no edge of* H_2 *containing both* v_1 *and* v_2.
Let $f_1 = \{v_1, w_2, w_3, \ldots, w_p\}$ *and* $f_2 = \{v_2, z_2, z_3, \ldots, z_q\}$ *be edges* H_2 *containing* v_1 *and* v_2 *respectively. Then the hyperedge* $\{(u_1, v_1), (u_1, w_2), (u_1, w_3), \ldots, (u_1, w_p)\}$ *is a hyperedge of* $H_1[H_2]$ *containing* (u_1, v_2) *but not* (u_2, v_2).

■

4. Conclusion

In this paper T_0 hypergraphs have been discussed with examples. Sufficient conditions for dual, incidence graph, middle graph, and total graph of hypergraph to be T_0 is derived. Further more, sufficient condition for join and corona of two hypergraphs to be T_0 is derived. We also discussed conditions under which minimal rank, maximal rank , non-rank, preserving direct product of two hypergraphs to be T_0. It is proved that Cartesian, normal, strong and wreath product of any two hypergraphs is always T_0.

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