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On generalized $*$-n-derivations in $*$-rings

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Abstract

Let R be a $*$-ring. In this paper we introduce the notion of generalized $*$-n-derivation in R. An additive mapping $x \to x^*$ of R into itself is called an involution on R if it satisfies the conditions: (i) $(x^*)^* = x$,(ii) $(xy)^* = y^*x^*$ for all $x, y \in R$. A ring R equipped with an involution $*$ is called a $*$-ring. It is shown that if a prime $*$-ring R admits a nonzero generalized $*$-n-derivation F (resp. a reverse generalized $*$-n-derivation) equipped with a $*$-n-derivation derivation (resp. a reverse $*$-n-derivation) D, then R is commutative. Further, some related properties of generalized $*$-n-derivation in a semiprime $*$-ring have also been investigated.

AMS subject classification:
Keywords: Associative ring, involution, derivation, reverse derivation, $*$-derivation, reverse $*$-derivation, $*$-n-derivation, reverse $*$-n-derivation, generalized $*$-n-derivation, reverse generalized $*$-n-derivation prime $*$-ring, semi prime $*$-ring.

1. Introduction

Throughout the paper, R will represent an assosiative ring with centre $Z(R)$. The ring R is called a prime ring if $xRy = 0$ implies $x = 0$ or $y = 0$.It is called semi prime if $xRx = 0$ implies $x = 0$. Given an integer $n > 1$, the ring R is said to be n-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. An additive mapping $D : R \to R$ is said to be a derivation (resp. a reverse derivation) on R if $D(xy) = D(x)y + xD(y)$(resp. $D(xy) = D(y)x + yD(x)$) holds for all $x, y \in R$. An additive mapping $x \to x^*$ of R into itself is called an involution on R if it satisfies the conditions:

(i) $(x^*)^* = x$,

(ii) $(xy)^* = y^* x^*$ for all $x, y \in R$.

A ring R equipped with an involution $'*'$ is called a $*$-ring. An additive mapping $D : R \rightarrow R$ is called a $*$-derivation (resp. a $*$-reverse derivation) on R if $D(xy) = D(x)y^* + xD(y)$ (resp. $D(xy) = D(y)x^* + yD(x)$) holds for all $x, y \in R$. Let R be a commutative $*$-ring. Then $D : R \rightarrow R$ defined by $D(x) = a(x - x^*)$, where $a \in R$, is a $*$-derivation on R (see [7]). An additive map $T : R \rightarrow R$ is called a left (resp. right) $*$-multiplier if $T(xy) = T(x)y^*$ (resp. $T(xy) = x^* T(y)$) holds for all $x, y \in R$.

There are many works dealing with the commutativity of prime and semi prime rings admitting certain types of derivations (see[3-6,8,12,17]). Ali [2] defined symmetric $*$-bi derivation, a symmetric left (resp. right) $*$-bi multiplier and studied some properties of prime $*$-rings and semi prime $*$-rings, possessing symmetric $*$-bi derivation and a symmetric left (resp. right) $*$-bi multiplier. Motivated by these concepts and the notion of n-derivation given by Park (see [11]). Very recently Ashraf [18] defined the concept of $*$-n-derivation in prime $*$-rings and semi prime $*$ rings and study some of their properties. Thus the notion of $*$-n-derivation gives a generalization of $*$-derivation earlier known to us in $*$-rings.

Let n be a fixed positive integer. An n-additive mapping $D : R \times R \times \cdots \times R \rightarrow R$ is called a $*$-n-derivation of R if the relations

$$D(x_1 x_1^{'}, x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n)(x_1^{'})^* + x_1 D(x_1^{'}, x_2, \ldots, x_n)$$

$$D(x_1, x_2 x_2^{'}, \ldots, x_n) = D(x_1, x_2, \ldots, x_n)(x_2^{'})^* + x_2 D(x_1, x_2^{'}, \ldots, x_n)$$

$$\vdots$$

$$D(x_1, x_2, \ldots, x_n x_n^{'}) = D(x_1, x_2, \ldots, x_n)(x_n^{'})^* + x_n D(x_1, x_2, \ldots, x_n^{'})$$

holds for all $x_1, x_1^{'}, x_2, x_2^{'}, \ldots, x_n, x_n^{'} \in R$. Similarly, an n-additive mapping $D : R \times R \times \cdots \times R \rightarrow R$ is called the reverse $*$-n-derivation of R if the relations

$$D(x_1 x_1^{'}, x_2, \ldots, x_n) = D(x_1^{'}, x_2, \ldots, x_n)x_1^* + x_1^{'} D(x_1, x_2, \ldots, x_n)$$

$$D(x_1, x_2 x_2^{'}, \ldots, x_n) = D(x_1, x_2^{'}, \ldots, x_n)x_2^* + x_2^{'} D(x_1, x_2^{'}, \ldots, x_n)$$

$$\vdots$$

$$D(x_1, x_2, \ldots, x_n x_n^{'}) = D(x_1, x_2, \ldots, x_n^{'}) x_n^* + x_n^{'} D(x_1, x_2, \ldots, x_n)$$

holds for all $x_1, x_1^{'}, x_2, x_2^{'}, \ldots, x_n, x_n^{'} \in R$. As an example of ∗-n-derivation, consider C be the ring of complex numbers with involution $*$ defined by $z^* = \bar{z}$, where $\bar{z}$ denotes the conjugate of the complex number z. Now define $D : C \times C \times \cdots \times C \to C$ such that $D(z_1, z_2, \ldots, z_n) = \lambda(z_1 - \bar{z_1})(z_2 - \bar{z_2}) \cdots (z_n - \bar{z_n})$ where λ is any fixed complex number. we can easily verify that D is a ∗-n-derivation of C. An n-additive mapping $T_1 : R \times R \times \cdots \times R \to R$ is called the left ∗-n-multiplier of R if

$$T_1(x_1 x_1^{'}, x_2, \ldots, x_n) = T_1(x_1, x_2, \ldots, x_n)(x_1^{'})^*$$

$$T_1(x_1, x_2 x_2^{'}, \ldots, x_n) = T_1(x_1, x_2, \ldots, x_n)(x_2^{'})^*$$

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$$T_1(x_1, x_2, \ldots, x_n x_n^{'}) = T_1(x_1, x_2, \ldots, x_n)(x_n^{'})^*$$

holds for all $x_1, x_1^{'}, x_2, x_2^{'}, \ldots, x_n, x_n^{'} \in R$. An additive mapping $T_2 : R \times R \times \cdots \times R \to R$ is called the right ∗-n-multiplier of R if

$$T_2(x_1 x_1^{'}, x_2, \ldots, x_n) = (x_1^{'})^* T_2(x_1, x_2, \ldots, x_n)$$

$$T_2(x_1, x_2 x_2^{'}, \ldots, x_n) = (x_2^{'})^* T_2(x_1, x_2, \ldots, x_n)$$

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$$T_2(x_1, x_2, \ldots, x_n x_n^{'}) = (x_n^{'})^* T_2(x_1, x_2, \ldots, x_n)$$

holds for all $x_1, x_1^{'}, x_2, x_2^{'}, \ldots, x_n, x_n^{'} \in R$. As examples of a left ∗-n-multiplier and a right ∗-n-multiplier.

Example 1.1. Consider S to be a commutative ring which is not a zero ring and

$$R = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid 0, x, y, z \in S \right\}.$$

$$T_1\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x_1x_2\ldots x_n \\ 0 & 0 & 0 \end{pmatrix}.$$

$$T_2\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & z_1z_2\ldots z_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and

$$\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & z & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}.$$

One can easily verify that $*$ is an involution on R. Also it is straightforward to check that T_1 is a nonzero left$*$-n-multiplier but not a right $*$-n-multiplier of the $*$-ring and T_2 is a nonzero right $*$-n-multiplier but not a left $*$-n-multiplier of the $*$-ring.Finally, an n-additive map $T : R \times R \times \cdots \times R \to R$ is called a $*$-n-multiplier of R if it is both the left$*$-n-multiplier and right $*$-n-multiplier of R. As an example of a $*$-n-multiplier, consider C to be the ring of complex numbers with involution $*$ defined by $z^* = \bar{z}$, where $\bar{z}$ denotes the conjugate of the complex numbers z. Now define $T : C \times C \times \cdots \times C \to C$ such that $T(z_1, z_2, \ldots, z_n) = \mu\bar{z}_1\bar{z}_2\ldots\bar{z}_n$, where μ is any fixed complex number. One can easily verified that T is a $*$-n-multiplier of C.

Now we define generalized $*$-derivation (resp. reverse generalized $*$-derivation). An additive map $F : R \to R$ is said to be a generalized derivation if there exist derivation D on R such that

$$F(xy) = F(x)y + xD(y) \text{ for all } x,\ y,\ z \in R.$$

An additive map $F : R \to R$ is said to be a generalized $*$-derivation if there exist a $*$-derivation D on R such that

$$F(xy) = F(x)y^* + xD(y) \text{ for all } x,\ y,\ z \in R.$$

or

$$F(xy) = F(y)x^* + yD(x) \text{ for all } x,\ y,\ z \in R.$$

An additive map $F : R \times R \times \cdots \times R \to R$ is called generalized $*$-n-derivation if there exist a $*$-n-derivation $D : R \times R \times \cdots \times R \to R$ such that

$$F(x_1x_1^{'}, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)(x_1^{'})^* + x_1D(x_1^{'}, x_2, \ldots, x_n)$$

$$F(x_1, x_2x_2^{'}, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)(x_2^{'})^* + x_2D(x_1, x_2^{'}, \ldots, x_n)$$

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$$F(x_1, x_2, \ldots, x_n x_n^{'}) = F(x_1, x_2, \ldots, x_n)(x_n^{'})^* + x_n D(x_1, x_2, \ldots, x_n^{'})$$

and reverse generalized $*$-n-derivation defined as:

$$F(x_1 x_1^{'}, x_2, \ldots, x_n) = F(x_1^{'}, x_2, \ldots, x_n)(x_1)^* + x_1^{'} D(x_1, x_2, \ldots, x_n)$$

$$F(x_1, x_2 x_2^{'}, \ldots, x_n) = F(x_1, x_2^{'}, \ldots, x_n)(x_2)^* + x_2^{'} D(x_1, x_2, \ldots, x_n)$$

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$$F(x_1, x_2, \ldots, x_n x_n^{'}) = F(x_1, x_2, \ldots, x_n^{'})(x_n)^* + x_n^{'} D(x_1, x_2, \ldots, x_n)$$

holds for all $x_i, x_i^{'} \in R$; $i = 1, 2, \ldots, n$. As example of generalized $*$-n-derivation is:

Example 1.2. Let S be a commutative ring which is not a zero ring and

$$R = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid 0, x, y, z \in S \right\}.$$

$F : R \times R \times \cdots \times R \to R$ and $r \to r^*$ such that

$$F\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x_1 x_2 \ldots x_n \\ 0 & 0 & 0 \end{pmatrix}.$$

there exist a $*$-n-derivation D such that

$$D\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & z_1 z_2 \ldots z_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & z & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}.$$

In 1989, Bresar and Vukman (see [7, Proposition 1]) proved that if a prime $*$-ring R admits a nonzero $*$-derivation (resp. $*$-derivation) D, then R is commutative. In this paper, we prove its analogue in the setting of the $*$-n-generalized derivation for prime $*$-rings.We obtain some results related with $*$-n-multipliers in prime $*$-rings and semiprime $*$-rings.In fact our results generalize, extend and complement several results obtained earlier on the $*$-n-derivation,symmetric $*$-biderivation for prime $*$-rings and semiprime $*$-rings.

2. Preliminary Results

We begin with Lemma, most of which have been proved elsewhere.

Lemma 2.1. Let R be a $*$-prime ring having generalized $*$-n-derivations F_1 and F_2 and D_1 and D_2 corresponding $*$-n-derivations. Further assume that $I_1, I_2, \ldots, I_n$ are non-zero ideals of R such that $F_1(i_1, i_2, \ldots, i_n) = F_2(i_1, i_2, \ldots, i_n)$ forall $i_r \in I_r$, $1 \leq r \leq n$, then $D_1 = D_2$.

Proof. We have

$$F_1(i_1, i_2, \ldots, i_n) = F_2(i_1, i_2, \ldots, i_n) \text{ for all} i_r \in I_r;\ 1 \leq r \leq n \tag{2.1}$$

Now putting $i_1 r_1$, where $r_1 \in R$, $i_1 \in I_1$ for i_1 in relation (2.1), We obtain

$$\begin{aligned} F_1(i_1 r_1, i_2, \ldots, i_n) &= F_2(i_1 r_1, i_2, \ldots, i_n) \\ F_1(i_1, i_2, \ldots, i_n) r_1^* + i_1 D_1(r_1, i_2, \ldots, i_n) &= F_2(i_1, i_2, \ldots, i_n) r_1^* + i_1 D_2(r_1, i_2, \ldots, i_n), \end{aligned}$$

using relation (2.1), we get

$$i_1 D_1(r_1, i_2, \ldots, i_n) = i_1 D_2(r_1, i_2, \ldots, i_n)$$

i.e.

$$i_1 D_1(r_1, i_2, \ldots, i_n) - i_1 D_2(r_1, i_2, \ldots, i_n) = 0.$$

This shows that

$$i_1 R\{D_1(r_1, i_2, \ldots, i_n) - D_2(r_1, i_2, \ldots, i_n)\} = \{0\}.$$

Since $I_1 \neq \{0\}$, the primeness of R implies that

$$D_1(r_1, i_2, \ldots, i_n) = D_2(r_1, i_2, \ldots, i_n) \text{ for all} i_r \in I_r;\ 1 \leq r \leq n. \tag{2.2}$$

Now putting $i_2 r_2$, where $r_2 \in R$, $i_2 \in I_2$ in (2.2), we get

$$D_1(r_1, i_2 r_2, \ldots, i_n) = D_2(r_1, i_2 r_2, \ldots, i_n)$$

i.e.

$$D_1(r_1, i_2, \ldots, i_n) r_2^* + i_2 D_1(r_1, r_2, \ldots, i_n) = D_2(r_1, i_2, \ldots, i_n) r_2^* + i_2 D_2(r_1, r_2, \ldots, i_n).$$

By (2.2), we get

$$\begin{aligned} i_2 D_1(r_1, r_2, i_3, \ldots, i_n) &= i_2 D_2(r_1, r_2, i_3, \ldots, i_n) \\ i_2 D_1(r_1, r_2, i_3, \ldots, i_n) - i_2 D_2(r_1, r_2, i_3, \ldots, i_n) &= 0 \end{aligned}$$

This shows that

$$i_2 R\{D_1(r_1, r_2, i_3, \ldots, i_n) - D_2(r_1, r_2, i_3, \ldots, i_n)\} = \{0\}.$$

Since $I_2 \neq \{0\}$, the primeness of R implies that

$$D_1(r_1, r_2, i_3, \ldots, i_n) - D_2(r_1, r_2, i_3, \ldots, i_n) = 0.$$

Now, continuing this process, we conclude that

$$D_1 = D_2.$$

■

3. Main Result

In 1989, Bresar and Vukman (see [7, Proposition 1]) proved that if a prime $*$-ring R admits a nonzero $*$-derivation (resp reverse $*$-derivation) D, then R is commutative. We obtained the following.

Theorem 3.1. Let R be a prime $*$-ring and if it admits a nonzero generalized $*$-n-derivation (resp. reverse generalized $*$-n-derivation) F associated with a $*$-n-derivation D. Then R is commutative ring.

Proof. By the hypothesis, we have for all $x_1, y, z, x_2, \ldots, x_n \in R$

$$F((x_1 y)z, x_2, \ldots, x_n) = F(x_1 y, x_2, \ldots, x_n)z^* + x_1 y D(z, x_2, \ldots, x_n)$$

$$F((x_1 y)z, x_2, \ldots, x_n) = \{F(x_1, x_2, \ldots, x_n)y^* + x_1 D(y, x_2, \ldots, x_n)\}z^* + x_1 y D(z, x_2, \ldots, x_n)$$

$$F((x_1 y)z, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)y^* z^* + x_1 D(y, x_2, \ldots, x_n)z^* + x_1 y D(z, x_2, \ldots, x_n) \tag{3.1}$$

Also,

$$F(x_1(yz), x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)(yz)^* + x_1 D(yz, x_2, \ldots, x_n)$$

$$F(x_1(yz), x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)z^* y^* + x_1\{D(y, x_2, \ldots, x_n)z^* + y D(z, x_2, \ldots, x_n)\}$$

$$F(x_1(yz), x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)z^* y^* + x_1\{D(y, x_2, \ldots, x_n)z^* + x_1 y D(z, x_2, \ldots, x_n)\}. \tag{3.2}$$

Combining (3.1) and (3.2), we get

$$F(x_1, x_2, \ldots, x_n)z^*y^* = F(x_1, x_2, \ldots, x_n)y^*z^*.$$

Putting y and z instead of y^* and z^*, respectively. We find that,

$$F(x_1x_2, \ldots, x_n)zy = F(x_1, x_2, \ldots, x_n)yz. \tag{3.3}$$

Now replacing y by yr, where $r \in R$ in (3.3), and using it again

$$F(x_1, x_2, \ldots, x_n)yrz = F(x_1, x_2, \ldots, x_n)yzr$$

$$F(x_1, x_2, \ldots, x_n)R[r, z] = \{0\}$$

thus

$$F(x_1, x_2, \ldots, x_n)R[r, z]^* = \{0\}.$$

Since $F \neq \{0\}$, $*$-primeness of R implies that $rz = zr$ for all $z, r \in R$. Therefore R is commutative. ∎

Theorem 3.2. Let R be a 2-torsion free prime $*$-ring possessing generalized $*$-n-derivations F_1 and F_2 associated with $*$-n-derivations D_1 and D_2. If $F_1(x_1, x_2, \ldots, x_n)$ $D_2(y_1, y_2, \ldots, y_n) + F_2(x_1, x_2, \ldots, x_n)D_1(y_1, y_2, \ldots, y_n) = 0$ for all $x_1, x_2, \ldots, x_n$; $y_1, y_2, \ldots, y_n \in R$ then either $F_1 = 0$ or F_2 is a left multiplier and either $F_2 = 0$ or F_1 is a left multiplier.

Proof. Suppose that

$$F_1(x_1, x_2, \ldots, x_n)D_2(y_1, y_2, \ldots, y_n) + F_2(x_1, x_2, \ldots, x_n)D_1(y_1, y_2, \ldots, y_n) = 0.$$

Replacing y_1z instead of y_1, we get

$$F_1(x_1, x_2, \ldots, x_n)D_2(y_1z, y_2, \ldots, y_n) + F_2(x_1, x_2, \ldots, x_n)D_1(y_1z, y_2, \ldots, y_n) = 0$$

$$\begin{aligned} F_1(x_1, x_2, \ldots, x_n)(D_2(y_1, y_2, \ldots, y_n)z^* &+ y_1D_2(z, y_2, \ldots, y_n)) \\ &+ F_2(x_1, x_2, \ldots, x_n)(D_1(y_1, y_2, \ldots, y_n)z^* \\ &+ y_1D_1(z, y_2, \ldots, y_n)) = 0 \\ F_1(x_1, x_2, \ldots, x_n)D_2(y_1, y_2, \ldots, y_n)z^* &+ F_1(x_1, x_2, \ldots, x_n)y_1D_2(z, y_2, \ldots, y_n) \\ &+ F_2(x_1, x_2, \ldots, x_n)D_1(y_1, y_2, \ldots, y_n)z^* \\ &+ F_2(x_1, x_2, \ldots, x_n)y_1D_1(z, y_2, y_3, \ldots, y_n) = 0 \end{aligned}$$

$$\begin{aligned} (F_1(x_1, x_2, \ldots, x_n)D_2(y_1, y_2, \ldots, y_n) &+ F_2(x_1, x_2, \ldots, x_n)D_1(y_1, y_1, \ldots, y_n))z^* \\ &+ F_1(x_1, x_2, \ldots, x_n)y_1D_2(z, y_2, \ldots, y_n) \\ &+ F_2(x_1, x_2, \ldots, x_n)y_1D_1(z, y_2, y_3, \ldots, y_n) = 0. \end{aligned}$$

By the given hypothesis,

$$F_1(x_1, x_2, \ldots, x_n)y_1 D_2(z, y_2, \ldots, y_n) + F_2(x_1, x_2, \ldots, x_n)y_1 D_1(z, y_2, \ldots, y_n) = 0 \quad (3.4)$$

Now we multiplying (3.4) from the right by $pD_1(r_1, r_2, \ldots, r_n)$; and using (3.4), where $r_i, p \in R$ we get

$$F_1(x_1, x_2, \ldots, x_n)y_1 D_2(z, y_2, \ldots, y_n)pD_1(r_1, r_2, \ldots, r_n) \\ + F_2(x_1, x_2, \ldots, x_n)y_1 D_1(z, y_2, \ldots, y_n)pD_1(r_1, r_2, \ldots, r_n) = 0.$$

Since R is 2-torsion free, we get

$$\begin{aligned} F_2(x_1, x_2, \ldots, x_n)y_1 D_1(z, y_2, \ldots, y_n)pD_1(r_1, r_2, \ldots, r_n) &= 0 \\ F_2(x_1, x_2, \ldots, x_n)y_1 D_1(z, y_2, \ldots, y_n)RD_1(r_1, r_2, \ldots, r_n)^* &= \{0\}. \end{aligned}$$

By $*$-primeness of R, either

$$F_2(x_1, x_2, \ldots, x_n)y_1 D_1(z, y_2, \ldots, y_n) = 0$$

or

$$D_1(r_1, r_2, \ldots, r_n) = 0.$$

But in first,

$$F_2(x_1, x_2, \ldots, x_n)y_1 D_1(z, y_2, \ldots, y_n) = 0$$

$$F_2(x_1, x_2, \ldots, x_n)^* RD_1(z, y_2, \ldots, y_n) = \{0\}.$$

By $*$-primeness of R, either $F_2(x_1, x_2, \ldots, x_n) = 0$ or $D_1(z, y_2, \ldots, y_n) = 0$ this means that $F_2 = 0$ or F_1 is a left multiplier.

Now, multiplying (3.4) from left by $D_2(r_1, r_2, \ldots, r_n)p$, where $r_i, p \in R$, we get

$$D_2(r_1, r_2, \ldots, r_n)pF_1(x_1, x_2, \ldots, x_n)y_1 D_2(z, y_2, \ldots, y_n) \\ +D_2(r_1, r_2, \ldots, r_n)pF_2(x_1, x_2, \ldots, x_n)y_1 D_1(z, y_2, y_3, \ldots, y_n) = 0.$$

Using (3.4)and the fact that R is 2-torsion free, we find

$$D_2(r_1, r_2, \ldots, r_n)pF_1(x_1, x_2, \ldots, x_n)y_1 D_2(z, y_2, \ldots, y_n) = 0$$

i.e.

$$D_2(r_1, r_2, \ldots, r_n)RF_1(x_1, x_2, \ldots, x_n)y_1 D_2(z, y_2, \ldots, y_n) = \{0\}$$

$$D_2(r_1, r_2, \ldots, r_n)^* RF_1(x_1, x_2, \ldots, x_n)y_1 D_2(z, y_2, \ldots, y_n) = \{0\}.$$

Since R is $*$-prime, we get either $D_2(r_1, r_2, \ldots, r_n) = 0$ or $F_1(x_1, x_2, \ldots, x_n)y_1D_2(z, y_2, \ldots, y_n) = 0$ from second relation,

$$F_1(x_1, x_2, \ldots, x_n)y_1D_2(z, y_2, \ldots, y_n) = 0.$$

This gives

$$F_1(x_1, x_2, \ldots, x_n)^* RD_2(z, y_2, \ldots, y_n) = \{0\}$$

either $F_1(x_1, x_2, \ldots, x_n) = 0$ or $D_2(z, y_2, \ldots, y_n) = 0$ i.e. $F_1 = 0$ or $D_2 = 0$. ■

Theorem 3.3. Let R be a semiprime $*$-ring admitting a generalized $*$-n-derivation F. then $F(R, R, \ldots, R) \subseteq Z$.

Proof. Since R is a $*$-ring and having a generalized $*$-n-derivation F, we have relation (3.3) in theorem (3.1).

Now putting $yF(x_1, x_2, \ldots, x_n)$ instead of y, we get

$F(x_1, x_2, \ldots, x_n)y[F(x_1, x_2, \ldots, x_n), z] = 0$ for all$x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z \in R$.

This relation gives

$$zF(x_1, x_2, \ldots, x_n)y[F(x_1, x_2, \ldots, x_n), z] = 0 \tag{3.5}$$

Now replacing y by zy in the relation $F(x_1, x_2, \ldots, x_n)y[F(x_1, x_2, \ldots, x_n), z] = 0$, we obtain

$$F(x_1, x_2, \ldots, x_n)zy[F(x_1, x_2, \ldots, x_n), z] = 0 \tag{3.6}$$

comparing (3.5) and (3.6), we get

$$F(x_1, x_2, \ldots, x_n)zy[F(x_1, x_2, \ldots, x_n), z] = zF(x_1, x_2, \ldots, x_n)y[F(x_1, x_2, \ldots, x_n), z].$$

This means that

$$[F(x_1, x_2, \ldots, x_n), z]y[F(x_1, x_2, \ldots, x_n), z] = 0$$

i.e.

$$[F(x_1, x_2, \ldots, x_n), z]R[F(x_1, x_2, \ldots, x_n), z] = \{0\}.$$

This gives

$$[F(x_1, x_2, \ldots, x_n), z]^* R[F(x_1, x_2, \ldots, x_n), z]^* = \{0\}.$$

Now by the $*$- semiprimeness of R, yields that

$$[F(x_1, x_2, \ldots, x_n), z] = 0$$

i.e.

$$F(R, R, \ldots, R) \subseteq Z.$$

■

Theorem 3.4. Let R be a semiprime ring with involution $*$. If F is a generalized $*$-n-derivation of R associated with a $*$-n-derivation D such that

$F(x_1, x_2, \ldots, x_n)y_1 = x_1F(y_1, y_2, \ldots, y_n)$ for all$x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n \in R$,

then F is a left multiplier.

Proof. By the given hypothesis, we have

$F(x_1, x_2, \ldots, x_n)y_1 = x_1F(y_1, y_2, \ldots, y_n)$ for all$x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n \in R$.

Using y_1z instead of y_1 where $z \in R$, we obtain

$$F(x_1, x_2, \ldots, x_n)y_1z = x_1F(y_1z, y_2, \ldots, y_n)$$

$$F(x_1, x_2, \ldots, x_n)y_1z = x_1(F(y_1, y_2, \ldots, y_n)z^* + y_1D(z, y_2, y_3, \ldots, y_n))$$

$$F(x_1, x_2, \ldots, x_n)y_1z = x_1F(y_1, y_2, \ldots, y_n)z^* + x_1y_1D(z, y_2, y_3, \ldots, y_n).$$

Using hypothesis

$$x_1F(x_1, x_2, \ldots, x_n)z = x_1F(y_1, y_2, \ldots, y_n)z^* + x_1y_1D(z, y_2, y_3, \ldots, y_n).$$

We replace z^* in place of z, we get

$$x_1F(x_1, x_2, \ldots, x_n)z = x_1F(y_1, y_2, \ldots, y_n)z + x_1y_1D(z, y_2, y_3, \ldots, y_n)$$

$$x_1y_1D(z, y_2, y_3, \ldots, y_n) = 0.$$

Replacing $D(z, y_2, y_3, \ldots, y_n)$ instead of x_1, we obtain

$$D(z, y_2, y_3, \ldots, y_n)y_1D(z, y_2, y_3, \ldots, y_n) = 0.$$

This implies that

$$D(z, y_2, y_3, \ldots, y_n)RD(z, y_2, y_3, \ldots, y_n) = \{0\}.$$

This gives

$$D(z, y_2, y_3, \ldots, y_n)^*RD(z, y_2, y_3, \ldots, y_n)^* = \{0\}.$$

Using $*$-semiprimeness F is a left multiplier. ■

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