Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 10 (2017), pp. 7653–7668

http://www.ripublication.com/gjpam.htm

Oscillation Theorems for Third-Order Half-linear Delay Dynamic Equations with Damping

S. Padmavathi
Associate Professor,
Pachaiyappa's College,
Chennai 600 030, India.

Abstract

This article is concerned with oscillatory behavior of a class of third order half-linear delay dynamic equation with damping

$$\left(r(x^{\Delta^2})^{\gamma}\right)^{\Delta}(t) + p(t)\left(x^{\Delta^2}\right)^{\gamma}(t) + q(t)x^{\gamma}(\tau(t)) = 0,$$

on an arbitrary time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$ and $\lim_{t\to\infty} \tau(t) = \infty$. We establish some new sufficient conditions which ensure that every solution oscillates or converges to zero. Example is given to illustrate our main results.

AMS subject classification: 34K11, 39A10, 34N05.
Keywords: Oscillation, Third order equation, Delay equation, Dynamic equations with damping, Time scale.

1. Introduction

In this article, we consider the third order half-linear delay dynamic equation with damping

$$\left(r(x^{\Delta^2})^{\gamma}\right)^{\Delta}(t) + p(t)\left(x^{\Delta^2}\right)^{\gamma}(t) + q(t)x^{\gamma}(\tau(t)) = 0, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$ unbounded above; here $\gamma > 0$ is the ratio of positive odd integers, r, p and q are real valued rd-continuous positive functions defined on $\mathbb{T}$, $r(t) - \mu(t)p(t) \neq 0$, $\tau \in C_{rd}(\mathbb{T}, \mathbb{T})$, $\tau(t) \leq t$ and $\tau(t) \to \infty$ as $t \to \infty$.

Since we are interested in oscillation, we assume throughout this paper that the given time scale $\mathbb{T}$ unbounded above and is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}} :=$

$[t_0, \infty) \cap \mathbb{T}$ with $t_0 \in \mathbb{T}$. On any time scale we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \;\; \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \tag{1.2}$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ is said to be left dense if $\rho(t) = t$, right dense if $\sigma(t) = t$, left scattered if $\rho(t) < t$, and right scattered if $\sigma(t) > t$. The graininess function μ of the time scale is defined by $\mu(t) := \sigma(t) - t$.

For any function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of$f$ may actually be replaced with any Banach space), the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \tag{1.3}$$

if f is continuous at t and t is right scattered. If t is right dense, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}, \tag{1.4}$$

provided that this limit exists.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense point. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. f is said to be differentiable if its derivative exists. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous function is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{R})$.

The derivative and the shift operator σ are related by the formula

$$f^{\sigma}(t) = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{1.5}$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $g(t)g(\sigma(t)) \neq 0$) of two differentiable functions f and g.

$$\left.\begin{aligned} (fg)^{\Delta}(t) &= f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)), \\ \left(\frac{f}{g}\right)^{\Delta}(t) &= \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}. \end{aligned}\right\} \tag{1.6}$$

For $a, b \in \mathbb{T}$, and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a). \tag{1.7}$$

The integration by parts formula reads

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t, \tag{1.8}$$

and infinite integrals are defined as

$$\int_a^\infty f(s)\Delta s = \lim_{t\to\infty}\int_a^t f(s)\Delta s. \tag{1.9}$$

For more details, see [4,5].

By a solution of (1.1) we mean a nontrivial real function $x \in C^2_{rd}[t_x,\infty)_{\mathbb{T}},\ t_x \in [t_0,\infty)_{\mathbb{T}}$ which has the property that $r(x^{\Delta^2})^\gamma \in C^1_{rd}[t_x,\infty)_{\mathbb{T}}$, and satisfies (1.1) on $[t_x,\infty)_{\mathbb{T}}$. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution x(t) of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory. Our attention is restricted to those solutions of equation (1.1) which exists on some half line $[t_x,\infty)_{\mathbb{T}}$ and satisfy $\sup\{|\ x(t)\ |: t \geqslant T\} > 0$ for all $T \geqslant t_x$.

Recently, Erbe et al.[8] studies a class of second-order delay dynamic equations with a nonlinear damping

$$(r(x^\Delta)^\gamma)^\Delta(t) + p(t)(x^{\Delta^\sigma}(t))^\gamma + q(t)f(x(\tau(t))) = 0. \tag{1.10}$$

Erbe et al. [9–11] studied the oscillatory behavior of third-order dynamic equations

$$(c(t)[a(t)x^\Delta(t)]^\Delta)^\Delta + q(t)f(x(t)) = 0,\ t \in \mathbb{T}, \tag{1.11}$$

$$x^{\Delta^3}(t) + q(t)x(t) = 0,\ t \in \mathbb{T}, \tag{1.12}$$

$$(a(t)[r(t)x^\Delta(t)]^{\Delta^\gamma})^\Delta + f(t,x(t)) = 0,\ t \in \mathbb{T}, \tag{1.13}$$

Saker et al. [22] considered the second order dynamic equation with damping

$$(rx^\Delta)^\Delta(t) + p(t)x^{\Delta^\sigma}(t) + q(t)f(x^\sigma(t)) = 0. \tag{1.14}$$

Hassan [15] and Li et al. [16] considered a third-order nonlinear delay dynamic equation

$$(a((rx^\Delta)^\Delta)^\gamma)^\Delta(t) + f(t,x(\tau(t))) = 0 \tag{1.15}$$

[16] established some new oscillation criteria for (1.15) that can be applied on any time scale$\mathbb{T}$ and result of [16] are different and complement the results established by [15].

Han et al. [13] considered the oscillation of third order nonlinear delay dynamic equations on time scales

$$((x^{\Delta\Delta}(t))^\gamma)^\Delta + p(t)x^\gamma(\tau(t)) = 0,\ t \in \mathbb{T}, \tag{1.16}$$

where $\gamma > 0$ is a quotient of odd positive integers, p is positive, real valued and rd-continuous function defined on $\mathbb{T}$, $\tau : \mathbb{T} \to \mathbb{T}$ is a rd-continuous function such that $\tau(t) \leq t$ and $\tau(t) \to \infty$ as $t \to \infty$, and established some new oscillation criteria for (1.16).

Yu and Wang [24] studied asymptotic behavior of solutions of third order nonlinear dynamic equation

$$\left(\frac{1}{a_2(t)}\left[\left(\frac{1}{a_1(t)}(x^\Delta(t))^{\alpha_1}\right)^\Delta\right]^{\alpha_2}\right)^\Delta + q(t)f(x(t)) = 0,\ t \in \mathbb{T}, \tag{1.17}$$

Agarwal et al. [1], studied oscillatory behavior of fourth order half linear delay dynamic equation with damping

$$(r(x^{\Delta^3})^\gamma)^\Delta(t) + p(t)(x^{\Delta^3})^\gamma(t) + q(t)x^\gamma(\tau(t)) = 0,\ t \in \mathbb{T}. \tag{1.18}$$

Clearly, (1.1) is a special case of above equations. The purpose of this paper is to establish some new oscillation criteria for (1.1) which gurantee that every solution x of (1.1) oscillates or converges to zero as $t \to \infty$. This paper is organized as follows:

In Section 2,we present some lemmas which will be used in the proof of our main results. In Section 3, we state and prove main results. In the last section, we give one example to illustrate Theorem 3.1.

2. Some Preliminary Lemmas

In this section, we present some lemmas which are used in the following sections.

Lemma 2.1. ([4, Theorem 2.33]) If $p : \mathbb{T} \to \mathbb{R}$ is rd-continuous such that $1 + \mu(t)p(t) > 0$ and fix $t_0 \in \mathbb{T}$. Then $e_p(., t_0)$ is a solution of the initial value problem

$$y^\Delta = p(t)y,\ y(t_0) = y_0.$$

Lemma 2.2. Assume x is an eventually positive solution of (1.1). If

$$\int_{t_0}^{\infty}\left(\frac{e_{-p/r}(t, t_0)}{r(t)}\right)^{\frac{1}{\gamma}} \Delta t = \infty \tag{2.1}$$

then there are only the following two possible cases for $t \in [t_1, \infty)_\mathbb{T}$, where $t_1 \in [t_0, \infty)_\mathbb{T}$ sufficiently large:

(i) $x(t) > 0,\ x^\Delta(t) > 0,\ x^{\Delta^2}(t) > 0,\ (r(x^{\Delta^2})^\gamma)^\Delta(t) < 0$

(ii) $x(t) > 0,\ x^\Delta(t) < 0,\ x^{\Delta^2}(t) > 0,\ (r(x^{\Delta^2})^\gamma)^\Delta(t) < 0.$

Proof. Let x be an eventually positive solution of (1.1). Then there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$ for $t \in [t_1, \infty)$. From (1.1) we have

$$(r(t)(x^{\Delta^2}(t))^\gamma)^\Delta + p(t)(x^{\Delta^2}(t))^\gamma = -q(t)x^\gamma(\tau(t)) < 0 \tag{2.2}$$

for $t \in [t_1, \infty)_{\mathbb{T}}$. Hence we obtain by (2.2) and Lemma 2.1, that

$$\left(\frac{r(x^{\Delta^2})^\gamma}{e_{-p/r}(.,t_0)}\right)^\Delta = \frac{(r(x^{\Delta^2})^\gamma)^\Delta e_{-p/r}(.,t_0) - r(x^{\Delta^2})^\gamma e^\Delta_{-p/r}(.,t_0)}{e_{-p/r}(.,t_0)e^\sigma_{-p/r}(.,t_0)} = \frac{(r(x^{\Delta^2})^\gamma)^\Delta + p(x^{\Delta^2})^\gamma}{e^\sigma_{-p/r}(.,t_0)} < 0$$

hence $\dfrac{r(x^{\Delta^2})^\gamma}{e^\sigma_{-p/r}(.,t_0)}$ is strictly decreasing on $[t_1, \infty)$. We claim that $x^{\Delta^2} > 0$ on $[t_1, \infty)$. Assume that, then there is a $t_2 \in [t_1, \infty)$ such that $x^{\Delta^2} < 0$ on $t \in [t_2, \infty)$. Then we can choose a negative constant C and $t_2 \in [t_1, \infty)$ such that $\dfrac{r(x^{\Delta^2})^\gamma}{e^\sigma_{-p/r}(.,t_0)} \le C < 0$ for $t \in [t_2, \infty)$. Then

$$x^{\Delta^2} \le \left(\frac{C}{r(t)}\right)^{\frac{1}{\gamma}} (e_{-p/r}(t,t_0))^{\frac{1}{\gamma}}$$

Integrating from t_2 to t, we obtain

$$x^\Delta(t) \le x^\Delta(t_2) + C^{\frac{1}{\gamma}} \int_{t_2}^{t} \frac{(e_{-p/r}(s,t_0))^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)} \Delta s.$$

Letting $t \to \infty$ then $x^\Delta(t) \to -\infty$ by (2.1) and then $x(t) \to -\infty$, which is contradiction. This contradiction proves $x^{\Delta^2} > 0$ and we have only two cases. ■

Lemma 2.3. Assume x is a solution of (1.1) which satisfies case (i) of Lemma 2.2, then

$$x^\Delta(t) \geqslant \left(r^{\frac{1}{\gamma}}(t) \int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}\right) x^{\Delta^2}(t). \tag{2.3}$$

If there exists a function $\phi \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ and a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that

$$\frac{\phi(t)}{r^{\frac{1}{\gamma}}(t) \int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}} - \phi^\Delta(t) \le 0,\ t \in [t_2, \infty)_{\mathbb{T}} \tag{2.4}$$

then x^Δ/ϕ is a nonincreasing function on $t \in [t_2, \infty)_{\mathbb{T}}$ and

$$x(t) \geqslant \left(\frac{1}{\phi(t)} \int_{t_2}^{t} \phi(s)\Delta s\right) x^\Delta(t)\ for\ t \in [t_2, \infty)_{\mathbb{T}}. \tag{2.5}$$

Proof. From $x^{\Delta}(t) > 0,\ x^{\Delta^2}(t) > 0,\ and\ (r(x^{\Delta^2})^{\gamma})^{\Delta}(t) < 0$, we have

$$\begin{aligned} x^{\Delta}(t) &= x^{\Delta}(t_1) + \int_{t_1}^{t} \frac{(r(x^{\Delta^2})^{\gamma})^{\frac{1}{\gamma}}(s)}{r^{\frac{1}{\gamma}}(s)} \Delta s \\ &\geqslant \left(r^{\frac{1}{\gamma}}(t) \int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} \right) x^{\Delta^2}(t). \end{aligned}$$

Thus,

$$\begin{aligned} \left(\frac{x^{\Delta}(t)}{\phi} \right)^{\Delta}(t) &= \frac{x^{\Delta^2}(t)\phi(t) - x^{\Delta}(t)\phi^{\Delta}(t)}{\phi(t)\phi^{\sigma}(t)} \\ &\leq \frac{x^{\Delta}(t)}{\phi(t)\phi^{\sigma}(t)} \left(\frac{\phi(t)}{r^{\frac{1}{\gamma}}(t) \int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}} - \phi^{\Delta}(t) \right) \leq 0 \end{aligned}$$

Therefore, x^{Δ}/ϕ is a nonincreasing function on $t \in [t_2, \infty)_{\mathbb{T}}$. Then, we obtain

$$\begin{aligned} x(t) &= x(t_2) + \int_{t_2}^{t} \frac{x^{\Delta}(s)\phi(s)}{\phi(s)} \Delta s \\ &\geqslant \left(\frac{1}{\phi(t)} \int_{t_2}^{t} \phi(s) \Delta s \right) x^{\Delta}(t). \end{aligned}$$

■

Lemma 2.4. ([14]) If X and Y are nonnegative, then

$$\lambda X Y^{\lambda-1} - X^{\lambda} \leq (\lambda - 1) Y^{\lambda}\ for\ \lambda > 1,$$

where the equality holds if and only if $X = Y$.

3. Main Results

In this section, we state and prove our main results. We establish several new oscillation criteria for (1.1). In the following we use the notation $(D(t))_{+} := \max\{0, D(t)\}$.

Theorem 3.1. Assume that (2.1) hold and $\gamma \geqslant 1$. Furthermore, assume that there exists a positive function $\alpha \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that for all sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$, for some $t_2 \in [t_1, \infty)_{\mathbb{T}}$ and $t_3 \in [t_2, \infty)_{\mathbb{T}}$.

$$\limsup_{t\to\infty} \int_{t_3}^{t} \left[\alpha^{\sigma}(s)q(s)f(s,t_2) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{D^{\gamma+1}(s)}{C^{\gamma}(s)}\right] \Delta s = \infty, \tag{3.1}$$

where ϕ is defined as in Lemma 2.3, and

$$f(s,t_2) := \left(\frac{1}{\phi(\sigma(s))} \int_{t_2}^{\tau(s)} \phi(u)\Delta u \right)^{\gamma},$$

$$D(t) := \left[\frac{\alpha^{\Delta}(t)}{\alpha(t)} - \left(\frac{\phi(t)}{\phi(\sigma(t))}\right)^{\gamma} \frac{\alpha^{\sigma}(t)p(t)}{\alpha(t)r(t)} \right]_{+}$$

$$C(t) := \gamma \left(\frac{\phi(t)}{\phi(\sigma(t))}\right)^{\gamma} \frac{\alpha^{\sigma}(t)}{\alpha^{1+1/\gamma}(t)r^{1/\gamma}(t)}.$$

Then every solution of equation (1.1) is oscillatory.

Proof. Suppose that (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. We may assume without loss of generality that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0,$ *and* $x(\tau(t)) > 0$, for $t \in [t_1, \infty)_{\mathbb{T}}$. Proceeding as in the proof of Lemma 2.2, we get (2.2) and then x satisfies either case (i) or case (ii).
Assume case (i). Define the function w by

$$w(t) = \alpha(t) \frac{r(t)(x^{\Delta^2})^{\gamma}(t)}{(x^{\Delta})^{\gamma}(t)}, \ t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.2}$$

Then $w(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$ and

$$w^{\Delta}(t) = \alpha^{\Delta}(t) \frac{r(t)(x^{\Delta^2})^{\gamma}(t)}{(x^{\Delta})^{\gamma}(t)} + \alpha^{\sigma}(t) \left(\frac{r(t)(x^{\Delta^2})^{\gamma}(t)}{(x^{\Delta})^{\gamma}(t)} \right)^{\Delta},$$

which implies that

$$w^{\Delta}(t) = \alpha^{\Delta}(t) \frac{r(t)(x^{\Delta^2})^{\gamma}(t)}{(x^{\Delta})^{\gamma}(t)} + \alpha^{\sigma}(t) \left[\frac{\left(r(x^{\Delta^2})^{\gamma}\right)^{\Delta}(t)}{(x^{\Delta})^{\gamma}(\sigma(t))} - \frac{r(t)(x^{\Delta^2})^{\gamma}(t)((x^{\Delta})^{\gamma})^{\Delta}(t)}{(x^{\Delta})^{\gamma}(t)(x^{\Delta})^{\gamma}(\sigma(t))} \right]. \tag{3.3}$$

By virtue of Pötzsche chain rule[4, Theorem 1.90], we have

$$((x^{\Delta})^{\gamma})^{\Delta}(t) \geqslant \gamma (x^{\Delta})^{\gamma-1}(t) x^{\Delta^2}(t), \ \gamma \geqslant 1. \tag{3.4}$$

Substituting (3.4) into (3.3), we find

$$w^{\Delta}(t) \leq \alpha^{\Delta}(t)\frac{r(t)(x^{\Delta^2})^{\gamma}(t)}{(x^{\Delta})^{\gamma}(t)} + \alpha^{\sigma}(t)\frac{\left(r(x^{\Delta^2})^{\gamma}\right)^{\Delta}(t)}{(x^{\Delta})^{\gamma}(\sigma(t))} - \frac{\gamma\alpha^{\sigma}(t)r(t)(x^{\Delta^2})^{\gamma}(t)(x^{\Delta})^{\gamma-1}(t)x^{\Delta^2}(t)}{(x^{\Delta})^{\gamma}(t)(x^{\Delta})^{\gamma}(\sigma(t))}.$$

From (2.2), (3.2), and the above inequality, we obtain

$$\begin{aligned} w^{\Delta}(t) &\leq \frac{\alpha^{\Delta}(t)}{\alpha(t)}w(t) - \frac{\alpha^{\sigma}(t)p(t)}{\alpha(t)r(t)}\left(\frac{x^{\Delta}(t)}{x^{\Delta}(\sigma(t))}\right)^{\gamma} w(t) - \alpha^{\sigma}(t)q(t)\left(\frac{x(\tau(t))}{x^{\Delta}(\sigma(t))}\right)^{\gamma} \\ &- \gamma\alpha^{\sigma}(t)\frac{w^{(\gamma+1)/\gamma}(t)}{\alpha^{(\gamma+1)/\gamma}(t)r^{1/\gamma}(t)}\left(\frac{x^{\Delta}(t)}{x^{\Delta}(\sigma(t))}\right)^{\gamma}. \end{aligned} \tag{3.5}$$

In the view of (2.5) and the fact that x^{Δ}/ϕ is a nonincreasing function, we have

$$\begin{aligned} \left(\frac{x(\tau(t))}{x^{\Delta}(\sigma(t))}\right)^{\gamma} &= \left(\frac{x(\tau(t))x^{\Delta}(\tau(t))}{x^{\Delta}(\tau(t))x^{\Delta}(\sigma(t))}\right)^{\gamma} \\ &\geqslant \left(\frac{1}{\phi(\tau(t))}\int_{t_2}^{\tau(t)}\phi(s)\Delta s\right)^{\gamma}\left(\frac{\phi(\tau(t))}{\phi(\sigma(t))}\right)^{\gamma} \\ &= \left(\frac{1}{\phi(\sigma(t))}\int_{t_2}^{\tau(t)}\phi(s)\Delta s\right)^{\gamma} \end{aligned} \tag{3.6}$$

and

$$\left(\frac{x^{\Delta}(t)}{x^{\Delta}(\sigma(t))}\right)^{\gamma} \geqslant \left(\frac{\phi(t)}{\phi(\sigma(t))}\right)^{\gamma}. \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5), we get

$$\begin{aligned} w^{\Delta}(t) &\leq -\alpha^{\sigma}(t)q(t)\left(\frac{1}{\phi(\sigma(t))}\int_{t_2}^{\tau(t)}\phi(s)\Delta s\right)^{\gamma} \\ &+ \left[\frac{\alpha^{\Delta}(t)}{\alpha(t)} - \frac{\alpha^{\sigma}(t)p(t)}{\alpha(t)r(t)}\left(\frac{\phi(t)}{\phi(\sigma(t))}\right)^{\gamma}\right]_{+} w(t) \\ &- \gamma\alpha^{\sigma}(t)\frac{w^{(\gamma+1)/\gamma}(t)}{\alpha^{(\gamma+1)/\gamma}(t)r^{1/\gamma}(t)}\left(\frac{\phi(t)}{\phi(\sigma(t))}\right)^{\gamma}. \end{aligned} \tag{3.8}$$

Set

$$A := C(t),\ B := D(t),\ y := w(t).$$

Using the inequality

$$By - Ay^{(\gamma+1)/\gamma} \le \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}, \ A > 0,$$

we get

$$w^\Delta(t) \le -\alpha^\sigma(t)q(t)f(t,t_2) + \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{D^{\gamma+1}(t)}{C^\gamma(t)}.$$

Integrating the above inequality from t_3 to t, we obtain

$$\int_{t_3}^{t} \left[\alpha^\sigma(s)q(s)f(s,t_2) - \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{D^{\gamma+1}(s)}{C^\gamma(s)}\right] \Delta s \le w(t_2) - w(t) \le w(t_2)$$

which contradicts (3.1). The proof is complete. ■

Theorem 3.2. Assume that (2.1) hold and $\gamma \geqslant 1$. Furthermore, assume that x is a solution of (1.1) which satisfies the case (ii) in Lemma 2.2. If for all sufficiently large t,

$$\int_{t_0}^{\infty} \int_{v}^{\infty} \left(\frac{1}{r(u)} \int_{u}^{\infty} q(s)\Delta s \right)^{\frac{1}{\gamma}} \Delta u \Delta v = \infty, \tag{3.9}$$

then there is $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume that x is a nonoscillatory solution of (1.1) which satisfies the case (ii) in Lemma 2.2, then $x(t)$ is decreasing and $\lim_{t\to\infty} x(t) = b \geqslant 0$. We claim that $b = 0$. If not, then $x(\tau(t)) \geqslant x(t) \geqslant b > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Integrating (1.1) from sufficiently large t to ∞ and using $x^{\Delta^2} > 0$, we get

$$r(t)(x^{\Delta^2})^\gamma(t) \geqslant \int_{t}^{\infty} q(s)x^\gamma(\tau(s))\Delta s$$

implies that

$$x^{\Delta^2}(t) \geqslant \left(\frac{1}{r(t)} \int_{t}^{\infty} q(s)x^\gamma(\tau(s))\Delta s \right)^{\frac{1}{\gamma}}.$$

Integrating above from t to ∞, we have

$$-x^\Delta(t) \geqslant \int_{t}^{\infty} \left(\frac{1}{r(u)} \int_{u}^{\infty} q(s)x^\gamma(\tau(s))\Delta s \right)^{\frac{1}{\gamma}} \Delta u.$$

Again, integrating above from t_0 to ∞, we can obtain

$$\begin{aligned} x(t_0) &\geqslant \int_{t_0}^{\infty}\int_{v}^{\infty}\left(\frac{1}{r(u)}\int_{u}^{\infty} q(s)x^{\gamma}(\tau(s))\Delta s\right)^{\frac{1}{\gamma}} \Delta u\Delta v \\ &\geqslant b\int_{t_0}^{\infty}\int_{v}^{\infty}\left(\frac{1}{r(u)}\int_{u}^{\infty} q(s)\Delta s\right)^{\frac{1}{\gamma}} \Delta u\Delta v, \end{aligned} \tag{3.10}$$

which is contradiction with the condition (3.9). Therefore, $b = 0$, that is $\lim_{t\to\infty} x(t) = 0$. This completes the proof. ■

Now we establish an oscillation result for (1.1) in the case where $\gamma \leq 1$.

Theorem 3.3. Assume that (2.1) hold and $\gamma \leq 1$. If there exists a positive function $\alpha \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that for all sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$, for some $t_2 \in [t_1, \infty)_{\mathbb{T}}$ and $t_3 \in [t_2, \infty)_{\mathbb{T}}$,

$$\limsup_{t\to\infty}\int_{t_3}^{t}\left[\alpha^{\sigma}(s)q(s)f(s,t_2) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{D^{\gamma+1}(s)}{E^{\gamma}(s)}\right]\Delta s = \infty, \tag{3.11}$$

where ϕ is defined as in Lemma 2.3, f and D are as in Theorem 3.1, and

$$E(t) := \gamma\frac{\phi(t)}{\phi(\sigma(t))}\frac{\alpha^{\sigma}(t)}{\alpha^{1+1/\gamma}(t)r^{1/\gamma}(t)}.$$

Then every solution of equation (1.1) is either oscillatory or converges to zero.

Proof. Suppose that (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. We may assume without loss of generality that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, *and* $x(\tau(t)) > 0$, for $t \in [t_1, \infty)_{\mathbb{T}}$. Proceeding as in the proof of Lemma 2.2, we get (2.2) and then x satisfies either case(i) or case(ii).
Assume case(i). Define the function w by (3.2). Then we obtain (3.3). From Pötzsche chain rule[4, Theorem 1.90], we see that

$$((x^{\Delta})^{\gamma})^{\Delta}(t) \geqslant \gamma(x^{\Delta})^{\gamma-1}(\sigma(t))x^{\Delta^2}(t),\ \gamma \leq 1. \tag{3.12}$$

It follows from (3.3) and (3.12) that

$$\begin{aligned} w^{\Delta}(t) &\leq \alpha^{\Delta}(t)\frac{r(t)(x^{\Delta^2})^{\gamma}(t)}{(x^{\Delta})^{\gamma}(t)} + \alpha^{\sigma}(t)\frac{\left(r(x^{\Delta^2})^{\gamma}\right)^{\Delta}(t)}{(x^{\Delta})^{\gamma}(\sigma(t))} \\ &- \frac{\gamma\alpha^{\sigma}(t)r(t)(x^{\Delta^2})^{\gamma}(t)(x^{\Delta})^{\gamma-1}(\sigma(t))x^{\Delta^2}(t)}{(x^{\Delta})^{\gamma}(t)(x^{\Delta})^{\gamma}(\sigma(t))}. \end{aligned}$$

Proceeding as in the proof of Theorem 3.1, we obtain

$$\begin{aligned} w^{\Delta}(t) &\leq -\alpha^{\sigma}(t)q(t)\left(\frac{1}{\phi(\sigma(t))}\int_{t_2}^{\tau(t)}\phi(s)\Delta s\right)^{\gamma} \\ &+ \left[\frac{\alpha^{\Delta}(t)}{\alpha(t)} - \frac{\alpha^{\sigma}(t)p(t)}{\alpha(t)r(t)}\left(\frac{\phi(t)}{\phi(\sigma(t))}\right)^{\gamma}\right]_{+} w(t) \\ &- \gamma\alpha^{\sigma}(t)\frac{w^{(\gamma+1)/\gamma}(t)}{\alpha^{(\gamma+1)/\gamma}(t)r^{1/\gamma}(t)}\left(\frac{\phi(t)}{\phi(\sigma(t))}\right). \end{aligned} \tag{3.13}$$

Set

$$A := E(t),\ B := D(t),\ y := w(t).$$

Using the inequality

$$By - Ay^{(\gamma+1)/\gamma} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^{\gamma}},\ A > 0,$$

we get

$$w^{\Delta}(t) \leq -\alpha^{\sigma}(t)q(t)f(t,t_2) + \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{D^{\gamma+1}(t)}{E^{\gamma}(t)}.$$

Integrating the above inequality from t_3 to t, we obtain

$$\int_{t_3}^{t}\left[\alpha^{\sigma}(s)q(s)f(s,t_2) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{D^{\gamma+1}(s)}{E^{\gamma}(s)}\right]\Delta s \leq w(t_2) - w(t) \leq w(t_2)$$

which contradicts (3.11).

Assume case (ii). The remainder of the proof is similar to that of Theorem 3.2, so we omit the details. This completes the proof. ■

The following Theorem gives the extension of the Kamenev-type oscillation criterion for equation (1.1).

Theorem 3.4. Assume that (2.1) and $\gamma \geqslant 1$ hold. Furthermore, suppose that there exists functions $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$ where $\mathbb{D} = \left\{(t,s) \in \mathbb{T}^2;\ t \geqslant s \geqslant t_0\right\}$. such that

$$H(t,t) = 0,\ t \geqslant t_0,\ H(t,s) > 0\ t > s \geqslant t_0 \tag{3.14}$$

and H has a nonpositive coninuous Δ -partial derivative $H^{\Delta_s}(t,s)$ with respect to the second variable and satisfies

$$H^{\Delta_s}(t,s) + H(t,s)D(s) = h(t,s) \tag{3.15}$$

and for all sufficiently large t_0,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\alpha^{\sigma}(s)q(s)f(s,t_2)-\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{D^{\gamma+1}(s)}{C^{\gamma}(s)}\frac{h_{+}^{\gamma}(t,s)}{H^{\gamma}(t,s)}\right]\Delta s=\infty, \quad (3.16)$$

where $h_{+}(t,s) := \max\{0, h(t,s)\}$ .Then every solution of (1.1) is either oscillatory or converges to zero.

Proof. Suppose to the contrary that x is a nonoscillatory solution of equation (1.1) and let $t_1 \geqslant t_0$ be such that $x(t) \neq 0$ for all $t \geq t_1$. Without loss of generality we may assume that $x(t) > 0,\ x(\tau(t)) > 0,\ and\ x^{\sigma}(\tau(t)) > 0$ for $t \geqslant T > t_0$ sufficiently large. Assume (1). We proceed as in the proof of Theorem 3.1, to prove that there exists $t_2 \geqslant t_1$ such that (3.8) holds for $t \geqslant t_2$. From (3.8), we have

$$w^{\Delta}(t) \leq -\alpha^{\sigma}(t)q(t)f(t,t_2) + D(t)w(t) - C(t)w^{\frac{\gamma+1}{\gamma}}(t) \quad (3.17)$$

Multiplying (3.17) by $H(t,s)$ and integrating from t_2 to t, we have

$$\begin{aligned}\int_{t_2}^{t} H(t,s)\alpha^{\sigma}(s)q(s)f(s,t_2)\Delta s \ &\leq\ -\int_{t_2}^{t} H(t,s)w^{\Delta}(s)\Delta s+\int_{t_2}^{t} H(t,s)D(s)w(s)\Delta s\\ &-\ \int_{t_2}^{t} H(t,s)C(s)w^{\frac{\gamma+1}{\gamma}}(s)\Delta s. \quad (3.18)\end{aligned}$$

Using integration by parts formula, and using (3.14) and (3.15), we have

$$\begin{aligned}\int_{t_2}^{t} H(t,s)\alpha^{\sigma}(s)q(s)f(s,t_2)\Delta s \leq H(t,t_2)w(t_2) &+ \int_{t_2}^{t} h(t,s)w(s)\Delta s\\ &-\int_{t_2}^{t} H(t,s)C(s)w^{\frac{\gamma+1}{\gamma}}(s)\Delta s\end{aligned}$$

Set

$$A := H(t,s)C(s),\ B := h(t,s),\ y := w(t).$$

Using the inequality

$$By - Ay^{(\gamma+1)/\gamma} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^{\gamma}},\ A > 0,$$

we have

$$\int_{t_2}^{t} H(t,s)\alpha^{\sigma}(s)q(s)f(s,t_2)\Delta s \leq H(t,t_2)w(t_2) + \int_{t_2}^{t}\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{h_{+}^{\gamma}(t,s)}{H^{\gamma}(t,s)C^{\gamma}(s)}\Delta s \quad (3.19)$$

Then we have

$$\int_{t_2}^{t}\left[H(t,s)\alpha^{\sigma}(s)q(s)f(s,t_2)-\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{h_+^{\gamma}(t,s)}{H^{\gamma}(t,s)C^{\gamma}(s)}\right]\Delta s\le H(t,t_2)w(t_2)$$

Thus

$$\frac{1}{H(t,t_2)}\int_{t_2}^{t}\left[H(t,s)\alpha^{\sigma}(s)q(s)f(s,t_2)-\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{h_+^{\gamma}(t,s)}{H^{\gamma}(t,s)C^{\gamma}(s)}\right]\Delta s\le w(t_2)<\infty \tag{3.20}$$

which contradicts (3.16).

Assume (2). Again the same arguments as in the proof of Theorem 3.2, we get a contradiction with (3.9). This completes the proof. ∎

Corollary 3.5. Let the assumption (3.16) in Theorem 3.4, be replaced by

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\alpha^{\sigma}(s)q(s)f(s,t_2)\Delta s=\infty,$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{h_+^{\gamma}(t,s)}{H^{\gamma}(t,s)C^{\gamma}(s)}\Delta s<\infty.$$

Then every solution of equation (1.1) is either oscillates or converges to zero on $[t_0,\infty)$.

Corollary 3.6. Assume that (2.1), (3.14) and (3.15) hold. Further assume that $\gamma\le 1$, and for sufficiently large t_0,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\alpha^{\sigma}(s)q(s)f(s,t_2)-\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{D^{\gamma+1}(s)}{E^{\gamma}(s)}\frac{h_+^{\gamma}(t,s)}{H^{\gamma}(t,s)}\right]\Delta s=\infty, \tag{3.21}$$

where $h_+(t,s):=\max\{0,h(t,s)\}$. Then every solution of (1.1) is either oscillatory or converges to zero.

Corollary 3.7. Let the assumption (3.21) in Corolary 3.2, be replaced by

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\alpha^{\sigma}(s)q(s)f(s,t_2)\Delta s=\infty,$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{h_+^{\gamma}(t,s)}{H^{\gamma}(t,s)E^{\gamma}(s)}\Delta s<\infty.$$

Then every solution of equation (1.1) is either oscillates or converges to zero on $[t_0,\infty)$.

4. Examples

In this section we present example to illustrate the main results.

Example 4.1. Consider the following third order delay dynamic equation with damping

$$\left(\frac{1}{t}x^{\Delta^2}(t)\right)^{\Delta}+\frac{1}{t^3}x^{\Delta^2}(t)+\frac{1}{t^2}x(\frac{t}{2})=0 \tag{4.1}$$

where

$$t\in\mathbb{T}:=\bar{2^{\mathbb{Z}}}:=\left\{2^k:k\in\mathbb{Z}\right\}\cup\{0\},t\geq t_0=2.$$

Here $\gamma=1$, $r(t)=\frac{1}{t}$, $p(t)=\frac{1}{t^3}$, $q(t)=\frac{1}{t^2}$, and $\tau(t)=\frac{t}{2}$ then $\mathbb{T}=\bar{2^{\mathbb{Z}}}$ is unbounded above, $\sigma(t)=2t$ and $\mu(t)=\sigma(t)-t=t$. Set $\phi(t)=\int_{t_1}^{t} r^{-\frac{1}{\gamma}}(s)\Delta s=\int_{t_1}^{t} s\Delta s=\frac{t^2-t_1^2}{2}$ and $\alpha(t)=t^2$ By Lemma (2.2), we obtain

$$e_{-p/r}(t,2)\geqslant 1-\int_{2}^{t}\frac{p(s)}{r(s)}\Delta s=\frac{2}{t},\ \forall\, t\geqslant 2$$

so

$$\int_{2}^{t}\left[\frac{1}{r(s)}e_{-p/r}(t,2)\right]^{\frac{1}{\gamma}}\Delta s\geqslant\int_{2}^{t}2\Delta s\to\infty, ast\to\infty$$

then (2.1) holds, $t^2/3\leq\phi(t)\leq t^2/2$ for t large enough,

$$f(s,t_2)=\frac{1}{2(s^2-t_1^2)}\int_{t_2}^{s/2}\frac{s^2-t_1^2}{2}\Delta s\geqslant\frac{s}{96}$$

and $C(t)=\frac{1}{t}$, $D(t)=\frac{2t-1}{t^2}$. Thus equation (3.1) holds. Therefore, by Theorem 3.1, we see that equation (4.1) is oscillatory.

Acknowledgement

The authors thank the referees for his/her suggestions and corrections which improved the content of the paper.

References

[1] Agarwal, R.P., Bohner, M., Li, T., and Zhang, C., 2014, "Oscillation theorems for fourth order half-linear delay dynamic equations with damping," Mediterr. J. Math., 11, pp. 463–475.

[2] Agarwal, R.P., O'Regan, D., and Saker, S.H., 2004, "Oscillation criteria for second order nonlinear neutral delay dynamic equations," J. Math. Anal. Appl., 300, pp. 203–217.

[3] Baculikova, B., and Dzurina, J., 2010, "Oscillation of third order neutral differential equations," Math. Comp. Modelling, 52, pp. 215–226.

[4] Bohner, M., and Peterson, A., 2001, "Dynamic Equations on Time Scales: An Introduction with Applications," Birkhäuser: Boston.

[5] Bohner, M., and Peterson, A., 2003, "Advances in Dynamic Equations on Time Scales," Birkhäuser: Boston.

[6] Erbe, L., Hassan, T.S., and Peterson, A., 2008, "Oscillation criteria for nonlinear damped dynamic equations on time scales," Appl. Math. Comput., 203, pp. 343–357.

[7] Erbe, L., and Peterson, A., 2003, "An oscillation result for a nonlinear dynamic equation on a time scale," Canad. Appl. Math. Quart., 11, pp. 143–157.

[8] Erbe, L., Peterson, A., and Saker, S.H., 2003, "Oscillation criteria for second order nonlinear dynamic equations on time scales," J. London Math. Soc., 76(2), pp. 701–714.

[9] Erbe, L., Peterson, A., and Saker, S.H., 2005, "Asymptotic behavior of solutions of a third order non linear dynamic equations on time scales," J. Comp. Appl. Math., 181, pp. 92–102.

[10] Erbe, L., Peterson, A., and Saker, S.H., 2006, "Oscillation and asymptotic behavior of a third order nonlinear dynamic equations," Can. Appl. Math. Q., 14, pp. 129–147.

[11] Erbe, L., Peterson, A., and Saker, S.H., 2007, "Hilla and Nehari type criteria for third order dynamic equations," J. Math. Anal. Appl., 329, pp. 112–131.

[12] Graef, J.R., and Thandapani, E., 1999, "Oscillatory and asymptotic behavior of solutions of third order delay difference equations," Funkcial. Ekvac., 42, pp. 355–369.

[13] Han, Z., Li, T., Sun, S., and Cao, F., 2010, "Oscillation criteria for third order nonlinear delay dynamic equations on time scales," Annales Polonici Mathematici, 99, pp. 143–156.

[14] Hardy, G.H., Littlewood, J.E., and Pólya, G., 1988, "Inequalities," second ed., Cambridge: Cambridge University Press.

[15] Hassan, T.S., 2009, "Oscillation of third order non-linear delay dynamic equations on time scales," Math. Comput. Modelling, 49, pp. 1573–1586.

[16] Li, T., Han, Z., Sun, S., and Zhao, Y., 2011, “Oscillation results for third order non-linear delay dynamic equations on time scales,” Bull. Malays. Math. Sci. Soc., 34, pp. 639–648.

[17] Li, T., Han, Z., Zhang, C., and Sun, Y., 2011, “Oscillation criteria for third order nonlinear delay dynamic equations on time scales,” Bull. Math. Anal. Appl., 3, pp. 52–60.

[18] Li, T., Thandapani., and Tang, S., 2011, “Oscillation theorems for fourth order delay dynamic equations on time scales,” Bull. Math. Anal. Appl., 3, pp. 190–199.

[19] Saker, S.H., 2006, “Oscillation criteria of third order non-linear delay differential equations,” Mathematica Slovaca, 56, pp. 433–450.

[20] Saker, S,H., 2010, “Oscillation Theory of Dynamic Equations on Time Scales, Second and Third orders,” Lambert Academic Publisher.

[21] Saker, S.H., 2011, “On oscillation of a certain class of third order nonlinear functional dynamic equations on time scales,” Bull. Math. Soc. Sci. Math. Roumanie Tome, 54(102), pp. 365–389.

[22] Saker, S,H., Agarwal, R.P., and O’Regan, D., 2007, “Oscillation of second order damped dynamic equations on time scales,” J. Math. Anal. Appl., 330, pp. 1317–1337.

[23] Sun, Yi., Han, Z., Sun, Y., and Pan, Y., 2011, “Oscillation theorems for certain third order nonlinear delay dynamic equations on time scales,” Electron. J. Qual. Theory Differ. Equ., 75, pp. 1–14.

[24] Yu, Z.H., and Wang, Q.R., 2009, “Asymptotic behavior of solutions of third order nonlinear dynamic equations on time scales,” J. Comput. Appl. Math., 225, pp. 531–540.