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Intra-regular Ternary Semigroups

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Abstract

In this paper we study some interesting properties of intra-regular ternary semigroups.

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1. Introduction

The theory of ternary algebraic system was introduced by Lehmer in 1932. Ternary semigroups are universal algebras with one associated ternary operation. In [4] Lehmer investigated certain ternary algebraic systems called triplexes which turn out to be ternary groups. The notion of ternary semigroup was known to S. Banach. He showed by an example that ternary semigroup does not necessarily reduce to an ordinary semigroup.

The notion of regularity was introduced and studied by J. Von Neumann [6] in 1936. Subsequently the notion of regular semiring was also introduced and studied as a generalization of regular ring. In [9], Vasile introduced and studied the notion of regular ternary rings. In [7] M.L. Santiago developed the theory of ternary semigroups. He focused his attention mainly to the study of regular ternary semigroups. In [2], [3] Dutta and Kar introduced and studied the notion of regular ternary semirings. Dutta, Kar and Maity [1] developed the theory of regular, intra-regular and completely regular ternary semigroups.

In this paper we characterize intra-regular ternary semigroups by using ideals of ternary semigroups.

2. Preliminaries

In this section we quote some definitions from [1] and [8] that we need in sequel. Lehmer [4] defined ternary semigroup as follows

Definition 2.1. A non-empty set T together with a ternary operation defined on T is called ternary semigroup if satisfies the associative law.

i.e. $[a\ b[c\ d\ e]] = [a\ [\ b\ c\ d]\ e] = [[\ a\ b\ c\ ]\ d\ e]$, *for all a, b, c, d,* $e \in T$.

By $[A\ B\ C]$ we mean the set $\{[a\ b\ c] : a \in A, b \in B, c \in C\}$ for non-empty subsets *A,B,C* of *T*. If $A = \{a\}$, then $[\{a\}B\ C]$ is simply written as $[a\ B\ C]$.

Example 2.2. Let Z^- be the set of all negative integers. Then together with usual ternary multiplication of negative integers, Z^- forms a ternary semigroup.

Definition 2.3.

(1) A non-empty subset S of T is a ternary sub-semigroup if $[S\ S\ S] \subseteq S$.

(2) A ternary semigroup T is a commutative ternary semigroup if for all $a, b, c \in T$, $[a\ b\ c] = [b\ a\ c] = [c\ b\ a] = [c\ a\ b] = [b\ c\ a] = [a\ c\ b]$.

(3) A ternary semigroup T is a laterally commutative ternary semigroup if for all $a, b, c \in T$, $[a\ b\ c] = [c\ b\ a]$.

Definition 2.4. An element $a \in T$ is an idempotent element if $[aaa] = a$.

Definition 2.5. A non-empty subset I of T is called

(i) a left ideal of T if $[T\ T\ I] \subseteq I$.

(ii) a lateral ideal of T if $[T\ I\ T] \subseteq I$.

(iii) a right ideal of T if $[I\ T\ T] \subseteq I$.

(iv) an ideal if it is a left, a lateral and a right ideal of T.

An ideal I of T is called a proper ideal of T if $I \neq T$.

Definition 2.6. Let T be a ternary semigroup and $a \in T$ Then the principal

(i) left ideal generated by a is given by $< a >_l = [TTa] \cup \{a\}$.

(ii) right ideal generated by a is given by $< a >_r = [aTT] \cup \{a\}$.

(iii) lateral ideal generated by a is given by $< a >_m = [TaT] \cup [TTaTT] \cup \{a\}$.

(iv) ideal generated by a is given by

$$I(a) =< a >= [TTa] \cup [aTT] \cup [TaT] \cup [TTaTT] \cup \{a\}.$$

Definition 2.7. An ideal I of T is called idempotent if $[III] = I^3 = I$.

Definition 2.8. A proper ideal P of T is called a prime ideal of T if $[P_1P_2P_3] \subseteq P$ implies that $P_1 \subseteq P$, $P_2 \subseteq P$, $P_3 \subseteq P$ for any ideals P_1, P_2, P_3 of T.

Definition 2.9. A proper ideal P of T is called a semiprime ideal of T if $P^3 = [PPP] \subseteq P$ implies that $P \subseteq P$ for any ideal P of T.

Definition 2.10. A proper ideal P of T is called a completely semiprime ideal of T if $x^3 \in P$ implies that $x \in P$ for an element $x \in T$.

F. M. Sioson [8] defined the notion of regular ternary semigroup as follows:

Definition 2.11. A ternary semigroup T is said to be regular if for each $a \in T$, there exist elements $x, y \in T$ such that $[axaya] = a$.

Subsequently, M.L. Santiago [7] modified the definition of regular ternary semigroup as follows:

Definition 2.12. A ternary semigroup T is said to be regular if for each $a \in T$, there exists an element $x \in T$ such that $[axa] = a$.

A ternary semigroup T is called regular if all of its elements are regular.

Clearly, we see that the above definition of regular ternary semigroup is equivalent to the definition of regular ternary semigroup given by Sioson.

Result 2.13. The following conditions on a ternary semigroup T are equivalent:

(i) T is regular

(ii) $A \cap B = [ATB]$ for every right ideal A and every left ideal B of T

(iii) For $a, b \in T$; $< a >_r \cap < b >_l = [< a >_r T < b >_l]$

(iv) For $a \in T$; $< a >_r \cap < a >_l = [< a >_r T < a >_l]$.

Definition 2.14. A ternary semigroup T is called intra-regular if for each element $a \in T$, there exist elements $x, y \in T$ such that $[xa^3y] = a$.

Result 2.15. If T is an intra-regular ternary semigroup, then for any left ideal L, lateral ideal M and right ideal R of T such that $L \cap M \cap R \subseteq [LMR]$.

Result 2.16. Let T be an intra-regular ternary semigroup. Then a non-empty subset I of T is an ideal of T if and only if I is a lateral ideal of T.

Result 2.17. Every lateral ideal of an intra-regular ternary semigroup T is an intra-regular ternary subsemigroup of T.

Result 2.18. Every ideal of an intra-regular ternary semigroup T is an intra-regular ternary subsemigroup of T.

Result 2.19. Let I be an ideal of an intra-regular ternary semigroup T and J be an ideal of I. Then J is an ideal of an entire ternary semigroup T.

Result 2.20. A ternary semigroup T is intra-regular if and only if every ideal of T is completely semiprime.

Throughout this paper $T = \langle T, [\ ]\rangle$ denotes a ternary semigroup with respect to the ternary operation [ ] unless otherwise stated.

3. Main Results

Theorem 3.1. If I is a two sided ideal of T, then I^3 is also a two sided ideal of T.

Proof. Let I be a two sided ideal of T, then we have $[TTI^3] = [[TTI]II] \subseteq [III] = [I^3]$ (since I is a left ideal of T) and $[I^3TT] = [II[ITT]] \subseteq [III] = [I^3]$ (since I is a right ideal of T). Thus I^3 is a two sided ideal of T. ■

Remark 3.2. If I is an ideal of a ternary semigroup T, then I^3 need not be an ideal of T.

Theorem 3.3. If I is an ideal of a commutative ternary semigroup T, then I^3 is an ideal of T.

Proof. Let I be a two sided ideal a commutative ternary semigroup of T, then we have $[TTI^3] = [[TTI]II] \subseteq [III] = [I^3]$ (since I is a left ideal of T), $[I^3TT] = [II[ITT]] \subseteq [III] = [I^3]$ (since I is a right ideal of T) and $[TI^3T] = [T[III]T]] \subseteq [[TIT]II] = [III] = [I^3]$ (since T is commutative and I is a right ideal of T). Thus I^3 is an ideal of T. ■

Theorem 3.4. If I is an ideal of an intra-regular ternary semigroup T, then I^3 is an ideal of T.

Proof. Let I be a two sided ideal of T, then we have $[TTI^3] = [[TTI]II] \subseteq [III] = [I^3]$ (since I is a left ideal of T) and $[I^3TT] = [II[ITT]] \subseteq [III] = [I^3]$ (since I is a right ideal of T). For $t_1, t_2 \in T$ and $x = [abc] \in I^3$, we have

$$[t_1 x t_2] = [t_1[abc]t_2] = [t_1[x_1 a^3 y_1][x_2 b^3 y_2][x_3 c^3 y_3]t_2]$$

$$= [[t_1 x_1 aaa][y_1 x_2 bbb][[y_2 x_3 c]c[cy_3 t_2]]$$

$$\in [[TTTTI][TTTTI][TTTITTT]] \subseteq [[TTI][TTI][TIT]] \subseteq [III] = I^3.$$

Hence I^3 is an ideal of T. ■

Theorem 3.5. In an intra-regular ternary semigroup every ideal I of T is idempotent, that is $[III] = I^3 = I$.

Proof. Let T be an intra-regular ternary semigroup and I be an ideal of T. We prove that $I \subseteq I^3$. Let $a \in I$, then $a^3 \in I^3$. This implies that $a \in I^3$, since I^3 is an ideal of T and every ideal in an intra regular ternay semigroup is completely semiprime (by Theorem 3.4 and Result 2.20). Since $I^3 \subseteq I$. Therefore $I^3 = I$. Thus every ideal I of T is idempotent. ■

Notation 3.6. We denote by $< b >$ the ternary subsemigroup of T defined by $< b >= \{b^n : n$ is an odd natural number $\}$. (i.e. the cyclic ternary subsemigroup of T generated by b.)

Theorem 3.7. Let T be a ternary semigroup, P a completely semiprime ideal of T and $a \in T$. If $a^n \in P$ for some odd natural number n, then $a \in P$.

Proof. For $n = 1, 3$, it is obviously true. Assume that $a^m \in P$ for some odd natural number $m \geq 5$, let $a^{m+2} \in P$. Then prove that $a \in P$.

Since $m \geq 5$, we have $a^{m-2} \in P$. Then

$$(a^m)^3 = a^{3m} = a^{3m+2-2} = [a^{m-2}a^{m+2}a^m] \in [TPT] \subseteq P$$

(Since P is an ideal of T). As P is completely semiprime and $a^m \in T$ such that $(a^m)^3 \in P$, we have $a^m \in P$. Then by assumption $a \in P$. ■

Theorem 3.8. Let T be a ternary semigroup, P a completely semiprime ideal of T and $b \in T$ such that $b \notin P$. Then $P\cap < b >= \emptyset$.

Proof. Suppose that $P\cap < b >\neq \emptyset$. Let $x \in P\cap < b >$. Then $T \ni x = b^n$ for some odd natural number n. Since P a completely semiprime ideal of T and $b \in T$ and $b^n \in P$ for some odd natural number n. Hence by Theorem 3.7, we have $b \in P$, which is not possible. Therefore $P\cap < b >= \emptyset$. ■

Remark 3.9. In a ternary semigroup T, for $a \in T$ $[TaT]$ need not be an ideal of T.

Theorem 3.10. If T is an intra-regular ternary semigroup, then $[TaT]$ is an ideal of T, for any $a \in T$.

Proof. Let $a \in T$, then $[TaT]$ is a non-empty subset of T. Now we have $[TT[TaT]] = [[TTT]aT] \subseteq [TaT]$. Therefore $[TaT]$ is a left ideal of T. Since $[[TaT]TT] = [Ta[TTT]] \subseteq [TaT]$. Hence $[TaT]$ is a right ideal of T.

Now let $x, y \in T$ and $u \in [TaT] \subseteq T$.Then $u = [t_1at_2]$ and $u = [x_1u^3y_1]$. We have

$$\begin{aligned}[xuy] &= [x[x_1u^3y_1]y] = [xx_1uuuy_1y] \\ &= [xx_1uu[t_1at_2]y_1]y] \\ &= [[xx_1uut_1]a[t_2y_1y]] \in [TaT].\end{aligned}$$

Therefore $[TaT]$ is a lateral ideal of T. $[TaT]$ is an ideal of T. ■

Theorem 3.11. If T is an intra-regular ternary semigroup, then $I(a) = [TaT]$ for any $a \in T$.

Proof. Let $a \in T$, since T is an intra-regular ternary semigroup. We have $a \in [Ta^3T] = [[Taa]aT] \subseteq [TaT]$ and by Theorem 3.10, $[TaT]$ is an ideal of T containing a. Hence $I(a) \subseteq [TaT]$.

On the other hand, $[TaT] \subseteq \{a\} \cup [TTa] \cup [TaT] \cup [TTaTT] \cup [aTT]$. Hence $[TaT] \subseteq I(a)$. Thus$I(a) = [TaT]$ for any $a \in T$. ■

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