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Generalized Polynomials IV

Anwar Habib
Jubail Industrial College, Al Jubail, KSA

Abstract

We have extended the corresponding result of Voronowskaja for Lebesgue integrable function in L_1-norm by our newly defined Generalized Polynomial.

$$A_n^{\alpha}(f,x) = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt\right\} q_{n,k}(x;\alpha)$$

where $$q_{n,k}(x;\alpha) = \binom{n}{k} x(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k}$$

Keywords: Bernstein Polynomials, Convergence, Generalized Polynomials, Integrable function, L_1 norm,

I. INTRODUCTION AND RESULTS

If $f(x)$ is a function defined [0,1], the Bernstein polynomial $B_n^f(x)$ of f is given as

$$B_n^f(x) = \sum_{k=0}^{n} f(k/n)\, P_{n,k}(x) \qquad \ldots\ldots(1.1)$$

where

$$P_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k} \qquad \ldots\ldots\ldots(1.2)$$

One question arises about the rapidity of convergence of B_n^f(x) to $f(x)$. An answer to this question has been given in different directons. One direction is that in which $f(x)$ is supposed to be at least twice differentiable at a point x of [0,1]. . Voronowskaja [6] proved that

$$\lim_{n\to\infty} n \left| f(x) - B_n^f(x) \right| = -\frac{1}{2}x(1-x)f''(x). \quad \ldots(1.3)$$

In particular, if $f''(x) \neq 0$, difference $f(x) - B_n^f(x)$ is exactly of order n^{-1}

A small modification of Bernstein polynomial due to Kantorovitch [4] makes it possible to approximate Lebesgue integrable function in L_1-norm by the modified polynomials

$$A_n^f(x) = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt\right\} p_{n,k}(x) \quad \ldots(1.4)$$

where $P_{n,k}(x)$ is defined by (1.2)

By Abel's formula(see Jensen [3])

$$(x+y+n\alpha)^n = \sum_{k=0}^{n}\binom{n}{k} x(x+k\alpha)^{k-1}(y+(n-k)\alpha)^{n-k} \quad \ldots.(1.5)$$

If we put $= 1 - x - n\alpha$, we obtain

$$1 = \sum_{k=0}^{n}\binom{n}{k} x(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k} \quad \ldots..(1.6)$$

Thus defining

$$p_k(x;\alpha) = \binom{n}{k} x(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k} \quad \ldots..(1.7)$$

we have

$$\sum_{k=0}^{n} p_k(x;\alpha) = 1 \quad \ldots.(1.8)$$

and we now define the Polynomial Operator

$$A_n^\alpha(f,x) = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt\right\} p_k(x;\alpha) \quad \ldots(1.9)$$

where $p_k(x;\alpha)$ is defined in (1.7) and moreover when $\alpha = 0$, (1.7) and (1.7) reduces to (1.2) and (1.4) respectively.

In this paper, we shall prove the corresponding results of approximation due to Voronowskja[6] for Lebesgue integrable function in L_1-norm by the our polynomial (1.11). In fact we state our result is as follows of

Theorem: let $f(x)$ be bounded Lebesgue integrable function with its first derivative in [0,1] and suppose that the second derivative $f''(x)$ exists at a certain point x of [0,1],

then for $\alpha = \alpha_n = o(1/n)$

$$\lim_{n\to\infty} n \left[U_n^\alpha(f,x) - f(x)\right] = \frac{1}{2}[(1-2x)f'(x) - x(1-x)f''(x)]$$

II. LEMMA

we first like to prove the lemma which would be useful for the proof of our theorem

Lemma 1 – For all values of $x \in |0,1|$ and for $\alpha = \alpha_n = o(1/n)$, we have

$$\sum_{k=0}^{n} kp_k(x;\alpha) \leq \frac{nx}{\alpha}$$

Lemma 2 For all values of $x \in |0,1|$ and for $\alpha = \alpha_n = o(1/n)$, we have

$$\sum_{k=0}^{n} kp_k(x;\alpha) \leq n(n-1)x \left\{\frac{x+2\alpha}{4\alpha^2} + \frac{(n-2)}{27\alpha}\right\}$$

Lemma3 - For all values of $x \in |0,1|$ and for $\alpha = \alpha_n = o(1/n)$, we have

$$(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)}(t-x)^2 dt\right\} p_k(x;\alpha) \leq x(1-x)/n$$

Before giving the proof of lemma we would like to illustrate some function which are helpful in the proof

The function

$$S(\nu,n,x,y) = \sum_{k=0}^{n}\binom{n}{k}\ (x+k\alpha)^{k+\nu-1}(y+(n-k)\alpha)^{n-k} \tag{1.10}$$

satisfies the reduction formula

$$S(\nu,n,x,y) = xS(\nu-1,n,x,y) + n\alpha S(\nu,n-1,x+\alpha,y) \tag{1.11}$$

by repeated use of reduction formula(1.11) and (1.5) we get

$$S(1,n,x,y) = \sum_{k=0}^{n}\binom{n}{k}\ k!\ \alpha^k(x+y+n\alpha)^{n-k}, \tag{1.12}$$

as $xS(0,n,x,y) = (x+y+n\alpha)^n$, (1.13)

Since $k! = \int_0^\infty e^{-t}t^k dt$ and so using binomial expansion we obtain

$$S(1,n,x,y)=\int_0^\infty e^{-t}(x+y+n\alpha+t\alpha)^n dt\ , \tag{1.14}$$

Similarly

$$S(2,n,x,y) = \sum_{k=0}^{n}\binom{n}{k}\ (x+k\alpha)k!\ \alpha^k S(1,n-k,x+k\alpha,y)$$

reduces to

$S(2,n,x,y) = \int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds[\,x(x+y+n\alpha+t\alpha+s\alpha)^n dt\ \ n\alpha^n s\,(x+y+n\alpha+t\alpha+s\alpha)^{n-1}],$

(1.15)

Proof of lemmas

Proof of lemma 1:

$$\sum_{k=0}^{n} kp_k(x;\alpha) = \sum_{k=0}^{n} k\binom{n}{k} x(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k}$$

$$= nx\sum_{k=1}^{n}\binom{n-1}{k-1}(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k}$$

$$= nxS(1, n-1, x+\alpha, 1-x-\alpha\ )$$

$$= nx\int_0^\infty e^{-t}(1+t\propto)^{n-1}dt$$

$$= nx\int_0^\infty e^{-t}e^{(n-1)t\propto}dt$$

$$= nx\int_0^\infty e^{-t}e^{(n-1)t\propto}dt$$

$$= nx\int_0^\infty e^{-t(1+\propto-n\propto)}dt$$

$$= \frac{nx}{(1+\propto-n\propto)}$$

$$\leq \frac{nx}{\propto} \qquad \text{for } \propto=\propto_n = o\left(\frac{1}{n}\right)$$

Proof of lemma 2:

$$\sum_{k=0}^{n} k(k-1)p_k(x;\alpha) = \sum_{k=0}^{n} k(k-1)\binom{n}{k} x(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k}$$

$$= n(n-1)x\sum_{k=2}^{n}\binom{n-2}{k-2}(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k}$$

$$= n(n-1)xS(2, n-2, x+2\alpha, 1-x-2\alpha\ )$$

$$= n(n-1)x\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds[(x+2\propto)(1+t\propto+s\propto)^{n-2}$$
$$+(n-2)\propto^2 s(1+t\propto+s\propto)^{n-3}]$$

$$= n(n-1)x(x+2\propto)\int_0^\infty e^{-t}dt\{\int_0^\infty e^{-s}(1+t\propto+s\propto)^{n-2}ds\}$$
$$+ n(n-1)(n-2)\propto^2 x\int_0^\infty e^{-t}dt\{\int_0^\infty e^{-s}s(1+t\propto+s\propto)^{n-3}ds\}]$$

$=I_1 + I_2,$

$$\boldsymbol{I_1} = n(n-1)x(x+2\propto)\int_0^\infty e^{-t}dt\{\int_0^\infty e^{-s}(1+t\propto+s\propto)^{n-2}ds\}$$

$$= n(n-1)x(x+2\propto)\int_0^\infty e^{-t}dt\{\int_0^\infty e^{-s}e^{(n-2)(t\propto+s\propto)}ds\}$$

$$= n(n-1)x(x+2\propto)\int_0^\infty e^{-t}e^{(n-2)t\propto}dt\{\int_0^\infty e^{-s}e^{(n-2)s\propto}ds\}$$

$$= n(n-1)x(x+2\propto)\int_0^\infty e^{-t\{1-n\propto+2\propto\}}dt\{\int_0^\infty e^{-s\{1-n\propto+2\propto\}}ds\}$$

$$= \frac{n(n-1)x(x+2\propto)}{\{1-n\propto+2\propto\}^2}$$

$$\leq \frac{\mathrm{n(n-1)x(x+2\propto)}}{4\propto^2} \qquad \text{for} \quad \propto=\propto_n= o\left(\frac{1}{n}\right)$$

$$I_2 = n(n-1)(n-2)\propto^2 x\int_0^\infty e^{-t}dt\{\int_0^\infty e^{-s}s(1+t\propto+s\propto)^{n-3}ds\}]$$

$$= n(n-1)(n-2)\propto^2 x\int_0^\infty e^{-t}dt\int_0^\infty se^{-s}e^{(n-3)(t\propto+s\propto)}ds$$

$$= n(n-1)(n-2)\propto^2 x\int_0^\infty e^{-t}e^{(n-3)t\propto}dt\int_0^\infty se^{-s}e^{(n-3)s\propto}ds$$

$$= n(n-1)(n-2)\propto^2 x\int_0^\infty e^{-t\{1-n\propto+3\propto\}}dt\int_0^\infty se^{-s\{1-n\propto+3\propto\}}ds$$

$$= \frac{n(n-1)(n-2)\ \propto^2\ x}{\{1-n\propto+3\propto\}^3}$$

$$\leq \frac{n(n-1)(n-2)\ \propto^2\ x}{27\propto^3}$$

$$I_1+I_2 \leq \frac{\mathrm{n(n-1)x(x+2\propto)}}{4\propto^2}+\frac{n(n-1)(n-2)\ \propto^2\ x}{27\propto^3}$$

Therefore

$$\sum_{k=0}^{n} k(k-1)p_k(x;\alpha) \leq \mathrm{n(n-1)x}\left[\frac{\mathrm{x}+2\propto}{4\propto^2}+\frac{(n-2)\ \propto^2}{27\propto^3}\right]$$

Proof of Lemma3:

$$(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)}(t-x)^2dt\right\}q_{n,k}(x;\alpha)$$

$$= \sum_{k=0}^{n}[x^2-\frac{2kx+x}{n+1}+\frac{k^2+k}{(n+1)^2}+\frac{1}{3(n+1)^2}]q_{n,k}(x;\alpha)$$

$$\leq x^2-\frac{1}{(n+1)}[\frac{2nx^2}{1+\alpha}+x]+\frac{1}{(n+1)^2}\ [n(n-1)x\frac{x+2\alpha}{(1+2\alpha)^2}$$

$$+\frac{(n-2)\alpha^2}{(1+3\alpha)^3}+\frac{2nx}{1+\alpha}]+\frac{1}{3(n+1)^2} \qquad (by\ 1.9\ \&\ 1.10)$$

$$\leq \frac{1}{n(1-\alpha)(1+2\alpha)^2(1+3\alpha)^3}\ [x(1-x)+\alpha x(1-x)(2n+9)+x$$

$$+\alpha^2 x(1-x)(17n+23)+9x$$

$$+\alpha^3 x(1-x)(57n-13)+7nx^2+x(5n^2+35)$$

$$+\alpha^4 x(1-x)(96n-144)+86nx^2+x(65n^2+12n)$$

$$+\alpha^5 x(1-x)(54n-216)+x(4n-12n+46)+$$

$162nx^2$

$$+\alpha^6 108x(1-x)+108nx^2]+1/3n^2$$

$$\leq \frac{x(1-x)}{n} \text{ for } \alpha=\alpha_n=o\left(\frac{1}{n}\right) \text{ and for large n}$$

which completes the proof of Lemma.

III. PROOF OF THE THEOREM

Proof of Theorem

We can write (in view of Taylor's Theorem)

$f(t)=f(x)+(t-x)f'(x)+(t-x)^2[\frac{1}{2}f''(x)+\eta(t-x)]$ …..(2.1)

where $\eta(h)$ is bounded $|\eta(h)|\leq H$ for all h and converges to zero with h.

Multiplying eqn. (2.1) by $(n+1)q_{n,k}(x;\alpha)$ and integrating it from *k/(n+1)* to *(k+1)/(n+1)*, then on summing ,we get

$$(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt\right\} q_{n,k}(x;\alpha)$$

$$=(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(x)dt\right\} q_{n,k}(x;\alpha)$$

$$+(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t-x)f'(x)dt\right\} q_{n,k}(x;\alpha)$$

$$+\frac{1}{2}(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 f''(x)dt\right\} q_{n,k}(x;\alpha)$$

$$+(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2\eta(t-x)dt\right\} q_{n,k}(x;\alpha)$$

$= I_1+I_2+I_3+I_4$ (say) ……(2.2)

Now first we evaluate I_1:

$$I_1 = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(x)dt\right\} q_{n,k}(x;\alpha)$$
$$= f(x) \qquad \text{....(2.3)}$$

and then

$$I_2 = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t-x)f'(x)dt\right\} q_{n,k}(x;\alpha)$$
$$= \sum_{k=0}^{n}\left(\frac{2k+1}{2(n+1)} - x\right) f'(x) q_{n,k}(x;\alpha)$$

$\leq \frac{(1-2x)}{2n} f'(x)$ for $\alpha = \alpha_n = o(1/n)$ ......(2.4)

Now we evaluate I_3:

$$I_3 = \frac{1}{2}(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 f''(x)dt\right\} q_{n,k}(x;\alpha)$$
$$\leq x(1-x)f''(x)/2n \text{ (by lemma)} \qquad \text{.....(2.5)}$$

and then in the last we evaluate I_4 :

$$I_4 = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 \eta(t-x)dt\right\} q_{n,k}(x;\alpha)$$

I_4 can be estimated easily. Let $\in > 0$ be arbitrary $\delta > 0$ such that $|\eta(h)| < \epsilon$ for $|h| < \delta$

thus breaking up the sum I_9 into two parts corresponding to those values of t for which

| t - x |< δ, and since in the given range of t, $\left|\frac{k}{n} - x\right| \sim |t-x|$, we have

$$|I_4| \leq \in \sum_{\left|\left(\frac{k}{n}\right)-x\right|<\delta} (n+1)q_{n,k}(x;\alpha)\left|\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^2 dt\right|$$
$$+ H \sum_{\left|\left(\frac{k}{n}\right)-x\right|\geq\delta} (n+1)q_{n,k}(x;\alpha)\left|\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right|$$

$$= I_5 + I_6 \text{ (say)}$$

$|I_5| \leq \frac{\epsilon}{n} \; |\{x(1-x)\}|$, for $\alpha = \alpha_n = o(1/n)$

$$I_6 = (n+1)\; H \sum_{|(\frac{k}{n})-x|\geq\delta} \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt \right\} q_{n,k}(x;\alpha)$$

$$= (n+1) \sum_{\left|(\frac{k}{n})-x\right|\geq\delta} q_{n,k}\,(x;\alpha)\frac{1}{n+1}$$

But if $= n^{-\beta}$, 0 < β < 1/2 (see also Kantorovitch [4]),

then for $\alpha = \alpha_n = o(1/n)$

$$\sum_{\left|(\frac{k}{n})-x\right|\geq n^{-\beta}} q_{n,k}\,(x;\alpha) \leq Cn^{-\nu}$$

For v>0, the constant C= C(β,v).

whence $I_6 < \frac{\epsilon}{n+1} < \epsilon/n$ for all *n* sufficiently large and therefore it follows

$I_4 < \epsilon/n$, for all sufficiently large n …..(2.6)

Hence from (2.2), (2.3), (2.4), (2.5) and (2.6), we have

$$(n+1)\sum_{k=0}^{n}\left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt \right\} q_{n,k}(x;\alpha)$$

$$= f(x) + [\{(1-2x)f'(x) + x(1-x)f''(x)\}/2n] + (\epsilon/n)$$

and therefore, finally we get

$$\lim_{n\to\infty} n\left[U_n^{\alpha}(f,x) - f(x)\right] = \frac{1}{2}[(1-2x)f'(x) - x(1-x)f''(x)]$$

where $\epsilon \to 0 \; as \; n \to \infty$

which completes the proof of the theorem.

IV. CONCLUSION

In this paper we have extended the result of Voronowskaja by taking a Generalized Polynomials $U_n^f(x)$ instead of Bernstein Polynomial $B_n^f(x)$

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