Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 11 (2017), pp. 7805–7818

http://www.ripublication.com/gjpam.htm

Lauricella hypergeometric and Pearson's system of partial difference equations

V. Esther Arokkia Rani
Department of mathematics,
Karunya University.

J. Daphy Louis Lovenia
Department of mathematics,
Karunya University,
Coimbatore-641 114, Tamilnadu, India.

Abstract

This paper presents a systematic investigation of the orthogonal polynomial solutions of the third order partial difference equation of lauricella hypergeometric type and the pearson's systems for the orthogonality weight of the solutions are derived.

AMS subject classification: 39A14.
Keywords: Partial difference equation, Lauricella hypergeometric equation, Pearson's system.

1. Introduction

Many model problems of atomic, molecular and nuclear physics can be summed up to the difference equations of hypergeometric type,

$$\sigma(x)\Delta\nabla\Delta y(x)+\tau_1(x)\Delta\nabla y(x)+\tau_2(x)\Delta y(x)+\lambda y(x)=0 \tag{1.1}$$

where $\sigma(x)$, $\tau_1(x)$ and $\tau_2(x)$ are polynomials of third, second and first respectively and λ is a constant [1, 10, 11]. Some solutions of (1.1) are functions used in mathematical physics such as classical orthogonal polynomials and lauricella hypergeometric functions.

Now considered a third order equation,

$$\sum_{i,j,k=1}^{n} a_{ijk}(x)\frac{\partial^3 v}{\partial x_i \partial x_j \partial x_k}+\sum_{i,j=1}^{n} b_{ij}(x)\frac{\partial^2 v}{\partial x_i \partial x_j}+\sum_{i=1}^{n} c_i(x)\frac{\partial v}{\partial x_i}+\lambda v = 0 \tag{1.2}$$

where $a_{ijk}(x) = a_{jki}(x) = a_{kij}(x)$ and the coefficients $a_{ijk}(x), b_{ij}(x)$ and $c_i(x)$ are chosen so that the derivatives of the solutions of equation (1.2).

In addition to this, the functions which are solutions of equation, in many areas of physics and mathematics including major use is made of quantities which are determined on a discrete set of argument values. An extensive study between classical orthogonal polynomials of continuous and discrete argument was noted by Gel'fand [4] in relative with the study of representations of the rotation group playing an important role in theoretical physics. It has proved that all these functions can be explained by means of a combined treatment in terms of polynomials which are solutions of the difference equation of hypergeometric type (1.1) proposed in [9, 11], where $\Delta f(x) = f(x+1) - f(x)$, $\nabla f(x) = f(x) - f(x-1)$. This equation can be obtained by approximating the differential equation (1.1) on a lattice with the constant mesh $\Delta \mathrm{x} = h$ up to the third order in h.

Partial difference equations are difference equations that include functions of three independent variables. In this view, the multivariable generalization of the forward difference operator Δ is defined as

$$\Delta_k f(x_1, \ldots, x_k, \ldots, x_n) = f(x_1, \ldots, x_k + 1, \ldots, x_n) - f(x_1, \ldots, x_k, \ldots, x_n),$$

and the backward difference operator ∇ by

$$\nabla_k f(x_1, \ldots, x_k, \ldots, x_n) = f(x_1, \ldots, x_k, \ldots, x_p) - f(x_1, \ldots, x_k - 1, \ldots, x_n).$$

The theory considered in [10, 11] admits a natural generalization to the case when the differential (1.2) is replaced by a partial difference equation. Using the linear combination of the backward and forward difference quotients and the finite difference schemes we may approximate the first, second and third order partial derivatives, by

$$\frac{\partial v}{\partial x}(x, y, z) \approx \frac{1}{3}\left[\frac{v(x+h, y, z) - v(x, y, z)}{h} + \frac{v(x, y, z) - v(x-h, y, z)}{h} + \frac{v(x+h, y, z) - v(x, y, z)}{h}\right]$$

$$\frac{\partial v}{\partial y}(x, y, z) \approx \frac{1}{3}\left[\frac{v(x, y+k, z) - v(x, y, z)}{k} + \frac{v(x, y, z) - v(x, y-k, z)}{k} + \frac{v(x, y+k, z) - v(x, y, z)}{k}\right]$$

$$\frac{\partial v}{\partial z}(x,y,z) \approx \frac{1}{3}\left[\frac{v(x,y,z+l)-v(x,y,z)}{l}+\frac{v(x,y,z)-v(x,y,z-l)}{l}\right.$$
$$\left.+\frac{v(x,y,z+l)-v(x,y,z)}{l}\right]$$

and

$$\frac{\partial^3 v}{\partial x^3}(x,y,z) \approx \frac{1}{h}\left[\frac{v(x+h,y,z)-v(x,y,z)}{h}+\frac{v(x,y,z)-v(x-h,y,z)}{h}\right.$$
$$\left.+\frac{v(x+h,y,z)-v(x,y,z)}{h}\right]$$

$$\frac{\partial^3 v}{\partial y^3}(x,y,z) \approx \frac{1}{k}\left[\frac{v(x,y+k,z)-v(x,y,z)}{k}+\frac{v(x,y,z)-v(x,y-k,z)}{k}\right.$$
$$\left.+\frac{v(x,y+k,z)-v(x,y,z)}{k}\right]$$

$$\frac{\partial^3 v}{\partial z^3}(x,y,z) \approx \frac{1}{l}\left[\frac{v(x,y,z+l)-v(x,y,z)}{l}+\frac{v(x,y,z)-v(x,y,z-l)}{l}\right.$$
$$\left.+\frac{v(x,y,z+l)-v(x,y,z)}{l}\right]$$

$$\begin{aligned}
&\frac{\partial^3 v}{\partial x\partial y\partial z}(x,y,z)\\
&\quad\approx \frac{1}{3k}\left[\frac{v(x+h,y,z)-v(x,y,z)}{h}+\frac{v(x+h,y-k,z)-v(x,y-k,z)}{h}\right.\\
&\qquad\left.+\frac{v(x+h,y,z)-v(x,y,z)}{h}\right]+\frac{1}{3l}\left[\frac{v(x,y+k,z)-v(x,y,z)}{k}\right.\\
&\qquad\left.+\frac{v(x,y+k,z-l)-v(x,y,z-l)}{k}+\frac{v(x,y+k,z)-v(x,y,z)}{k}\right]\\
&\qquad+\frac{1}{3h}\left[\frac{v(x,y,z+l)-v(x,y,z)}{l}+\frac{v(x-h,y,z+l)-v(x-h,y,z)}{l}\right.\\
&\qquad\left.+\frac{v(x,y,z+l)-v(x,y,z)}{l}\right]
\end{aligned}$$

which [10, 11] yields the error $O(h^3)+O(k^3)+O(l^3)$ for $h\to 0, k\to 0$ and $l\to 0$. The measurement of $h>0, k>0$ and $l>0$ are the discretization parameters or mesh widths. Using the above finite difference approximations to partial derivatives, we can write in terms of the partial forward and backward difference operators, as $h = k = l = 1$

$$\frac{\partial}{\partial x_i} = \frac{\Delta_i + \nabla_i + \Delta_i}{3}, \frac{\partial^3}{\partial {x_i}^3} = \Delta_i \nabla_i \Delta_i, \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} = \frac{\Delta_i \nabla_j \Delta_k + \Delta_j \nabla_k \Delta_i + \Delta_k \nabla_i \Delta_j}{3} \tag{1.3}$$

Since $\nabla_i = \Delta_i - \Delta_i \nabla_i$, equation (1.2) can be summed up to a linear partial difference equation in the form as given below in equation (1.4)

$$\sum_{i,j,k=1}^{n} a_{ijk}(x) \Delta_i \nabla_j \Delta_k v(x) + \sum_{i,j=1}^{n} b_i(x) \Delta_i \nabla_j v(x) + \sum_{i=1}^{n} c_i(x) \Delta_i v(x) + \lambda v(x) = 0, \tag{1.4}$$

where $x = (x_1, \cdots, x_n)$ and λ is the spectral parameter.

This article, additionally analogous case in which the matrix $(a_{ijk}(x))$ is not symmetric. Because of this, we shall use the standard multi-index notation [2, 11]. In the three-dimensional case studied here, the above equation is general. The equation (1.4) is modified as per the convenient of the researcher,

$$\begin{aligned} & a_{111}(\mathrm{x}) \Delta_1 \nabla_1 \Delta_1 v(\mathrm{x}) + a_{112}(\mathrm{x}) \Delta_1 \nabla_1 \Delta_2 v(\mathrm{x}) + a_{113}(\mathrm{x}) \Delta_1 \nabla_1 \Delta_3 v(\mathrm{x}) \\ & + a_{221}(\mathrm{x}) \Delta_2 \nabla_2 \Delta_1 v(\mathrm{x}) + a_{222}(\mathrm{x}) \Delta_2 \nabla_2 \Delta_2 v(\mathrm{x}) + a_{223}(\mathrm{x}) \Delta_2 \nabla_2 \Delta_3 v(\mathrm{x}) \\ & + a_{331}(\mathrm{x}) \Delta_3 \nabla_3 \Delta_1 v(\mathrm{x}) + a_{332}(\mathrm{x}) \Delta_3 \nabla_3 \Delta_2 v(\mathrm{x}) + a_{333}(\mathrm{x}) \Delta_3 \nabla_3 \Delta_3 v(\mathrm{x}) \\ & + b_1(\mathrm{x}) \Delta_1 v(\mathrm{x}) + b_2(\mathrm{x}) \Delta_2 v(\mathrm{x}) + b_3(\mathrm{x}) \Delta_3 v(\mathrm{x}) + c_1(\mathrm{x}) \Delta_1 v(\mathrm{x}) \\ & + c_2(\mathrm{x}) \Delta_2 v(\mathrm{x}) + c_3(\mathrm{x}) \Delta_3 v(\mathrm{x}) + \lambda v(\mathrm{x}) = 0 \end{aligned} \tag{1.5}$$

where $\mathrm{x} = (x, y, z)$ is a point in three-dimensional space. A possibly, the question arises concerning the existence of three variables of total degree n,

$$P_n(\mathrm{x}) = P_n(x, y, z) = \sum_{j=0}^{n} \sum_{k+l+m=j} c_{klm} x^k y^l z^m,$$

which are solution of equation (1.5). And also, we assign the properties of the polynomials as $P_n(x, y, z)$ under some assumptions on equation (1.5).

The outline of the article is as follows: The lauricella hypergeometric class of third order linear partial difference equations presented in section 2. Section 3, gives the partial difference equation which is written in self-adjoint form, giving the Pearson's system.

2. The lauricella hypergeometric class of the linear third order partial difference equation

In this section, we make sure that a number of properties of the solution of equation (1.5) which are analogous to the solution of equation (1.2), for $n = 3$. The polynomial coefficients a_{ijk}, b_i and c_i are selected. Because the difference derivatives of any order of the solutions of equation (1.5) are also the solutions of an equation as of equation (1.5).

Here, we define, $\alpha = (r, s, t)$ a multi-index of order $|\alpha| = r + s + t$ and we let $v_\alpha(\mathrm{x}) = \Delta_1^r \Delta_2^s \Delta_3^t v(\mathrm{x})$ where $v = v(\mathrm{x})$ is a solution of (1.5)

Definition 2.1. The equation (1.5) belongs to the lauricella hypergeometric class if the difference derivatives $v_\alpha(\mathrm{x})$ of the solution $v = v(\mathrm{x})$ of (1.5) are also solutions of an equation of the same type as (1.5).

Lemma 2.2. Equation (1.5) belong to the lauricella hypergeometric class if and only if it has the form

$$\begin{aligned}
&a_{111}(\mathrm{x})\Delta_1\nabla_1\Delta_1 v(\mathrm{x}) + a_{112}(\mathrm{x})\Delta_1\nabla_1\Delta_2 v(\mathrm{x}) + a_{113}(\mathrm{x})\Delta_1\nabla_1\Delta_3 v(\mathrm{x})\\
&+ a_{221}(\mathrm{x})\Delta_2\nabla_2\Delta_1 v(\mathrm{x}) + a_{222}(\mathrm{x})\Delta_2\nabla_2\Delta_2 v(\mathrm{x}) + a_{223}(\mathrm{x})\Delta_2\nabla_2\Delta_3 v(\mathrm{x})\\
&+ a_{331}(\mathrm{x})\Delta_3\nabla_3\Delta_1 v(\mathrm{x}) + a_{332}(\mathrm{x})\Delta_3\nabla_3\Delta_2 v(\mathrm{x}) + a_{333}(\mathrm{x})\Delta_3\nabla_3\Delta_3 v(\mathrm{x})\\
&+ b_1(\mathrm{x})\Delta_1 v(\mathrm{x}) + b_2(\mathrm{x})\Delta_2 v(\mathrm{x}) + b_3(\mathrm{x})\Delta_3 v(\mathrm{x}) + c_1(\mathrm{x})\Delta_1 v(\mathrm{x})\\
&+ c_2(\mathrm{x})\Delta_2 v(\mathrm{x}) + c_3(\mathrm{x})\Delta_3 v(\mathrm{x}) + \lambda v(\mathrm{x}) = 0
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
a_{111}(\mathrm{x}) &= A_1x^3 + B_1x^2 + C_1x + D_1\\
a_{222}(\mathrm{x}) &= A_2y^3 + B_2y^2 + C_2y + D_2\\
a_{333}(\mathrm{x}) &= A_3z^3 + B_3z^2 + C_3z + D_3\\
a_{112}(\mathrm{x}) &= A_4xyz + B_4xy + C_4yz + D_4zx + E_4x + F_4y + G_4z + H_4\\
a_{113}(\mathrm{x}) &= A_5xyz + B_5xy + C_5yz + D_5zx + E_5x + F_5y + G_5z + H_5\\
a_{221}(\mathrm{x}) &= A_6xyz + B_6xy + C_6yz + D_6zx + E_6x + F_6y + G_6z + H_6\\
a_{223}(\mathrm{x}) &= A_7xyz + B_7xy + C_7yz + D_7zx + E_7x + F_7y + G_7z + H_7\\
a_{331}(\mathrm{x}) &= A_8xyz + B_8xy + C_8yz + D_8zx + E_8x + F_8y + G_8z + H_8\\
a_{332}(\mathrm{x}) &= A_9xyz + B_9xy + C_9yz + D_9zx + E_9x + F_9y + G_9z + H_9\\
b_1(\mathrm{x}) &= c_1x + d_1,\ b_2(\mathrm{x}) = c_2y + d_2,\ b_3(\mathrm{x}) = c_3z + d_3\\
c_1(\mathrm{x}) &= e_1x + f_1,\ c_2(\mathrm{x}) = e_2y + f_2,\ c_3(\mathrm{x}) = e_3z + f_3
\end{aligned}$$

Then we have the following theorem.

Theorem 2.3. If $v(\mathrm{x})$ is a solution of equation (2.1), then $v_\alpha(\mathrm{x}) = \Delta_1^r \Delta_2^s \Delta_3^t v(\mathrm{x})$ is a solution of the following equation belonging to the lauricella hypergeometric class

$$\begin{aligned}
&a_{111}^{(r,s,t)}(\mathrm{x})\Delta_1\nabla_1\Delta_1 v_\alpha(\mathrm{x}) + a_{112}^{(r,s,t)}(\mathrm{x})\Delta_1\nabla_1\Delta_2 v_\alpha(\mathrm{x}) + a_{113}^{(r,s,t)}(\mathrm{x})\Delta_1\nabla_1\Delta_3 v_\alpha(\mathrm{x})\\
&+ a_{221}^{(r,s,t)}(\mathrm{x})\Delta_2\nabla_2\Delta_1 v_\alpha(\mathrm{x}) + a_{222}^{(r,s,t)}(\mathrm{x})\Delta_2\nabla_2\Delta_2 v_\alpha(\mathrm{x}) + a_{223}^{(r,s,t)}(\mathrm{x})\Delta_2\nabla_2\Delta_3 v_\alpha(\mathrm{x})\\
&+ a_{331}^{(r,s,t)}(\mathrm{x})\Delta_3\nabla_3\Delta_1 v_\alpha(\mathrm{x}) + a_{332}^{(r,s,t)}(\mathrm{x})\Delta_3\nabla_3\Delta_2 v_\alpha(\mathrm{x}) + a_{333}^{(r,s,t)}(\mathrm{x})\Delta_3\nabla_3\Delta_3 v_\alpha(\mathrm{x})\\
&+ b_1^{(r,s,t)}(\mathrm{x})\Delta_1 v_\alpha(\mathrm{x}) + b_2^{(r,s,t)}(\mathrm{x})\Delta_2 v_\alpha(\mathrm{x}) + b_3^{(r,s,t)}(\mathrm{x})\Delta_3 v_\alpha(\mathrm{x}) + c_1^{(r,s,t)}(\mathrm{x})\Delta_1 v_\alpha(\mathrm{x})\\
&+ c_2^{(r,s,t)}(\mathrm{x})\Delta_2 v_\alpha(\mathrm{x}) + c_3^{(r,s,t)}(\mathrm{x})\Delta_3 v_\alpha(\mathrm{x}) + \mu^{(r,s,t)} v_\alpha(\mathrm{x}) = 0
\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
a_{111}^{(r,s,t)}(\mathrm{x}) &= a_{111}(\mathrm{x}) - s\Delta_2 a_{221}(\mathrm{x}) - t\Delta_3 a_{331}(\mathrm{x})\\
a_{222}^{(r,s,t)}(\mathrm{x}) &= a_{222}(\mathrm{x}) - t\Delta_3 a_{332}(\mathrm{x}) - r\Delta_2 a_{112}(\mathrm{x})\\
a_{333}^{(r,s,t)}(\mathrm{x}) &= a_{333}(\mathrm{x}) - r\Delta_1 a_{113}(\mathrm{x}) - s\Delta_2 a_{223}(\mathrm{x})\\
a_{112}^{(r,s,t)}(\mathrm{x}) + a_{113}^{(r,s,t)}(\mathrm{x}) &= a_{112}(\mathrm{x}) + a_{113}(\mathrm{x}) + r\Delta_1[a_{112}(\mathrm{x}) + a_{113}(\mathrm{x})]\\
a_{221}^{(r,s,t)}(\mathrm{x}) + a_{223}^{(r,s,t)}(\mathrm{x}) &= a_{221}(\mathrm{x}) + a_{223}(\mathrm{x}) + s\Delta_2[a_{221}(\mathrm{x}) + a_{223}(\mathrm{x})]\\
a_{331}^{(r,s,t)}(\mathrm{x}) + a_{332}^{(r,s,t)}(\mathrm{x}) &= a_{331}(\mathrm{x}) + a_{332}(\mathrm{x}) + t\Delta_3[a_{331}(\mathrm{x}) + a_{332}(\mathrm{x})]
\end{aligned}$$

and

$$\begin{aligned}
b_1^{(r,s,t)} &= b_1 + r\Delta_1(b_1 + a_{111}) + s\Delta_2(a_{123} + a_{231} + a_{312}) + t\Delta_3(a_{123} + a_{231} + a_{312})\\
&\quad + \frac{r(r-1)(r-2)}{3}\Delta_1^3 a_{111} + rst\Delta_1\Delta_2\Delta_3 a_{123}\\
b_2^{(r,s,t)} &= b_2 + s\Delta_2(b_2 + a_{222}) + t\Delta_3(a_{231} + a_{312} + a_{123}) + r\Delta_1(a_{231} + a_{312} + a_{123})\\
&\quad + \frac{s(s-1)(s-2)}{3}\Delta_2^3 a_{222} + rst\Delta_1\Delta_2\Delta_3 a_{231}\\
b_3^{(r,s,t)} &= b_2 + t\Delta_3(b_3 + a_{333}) + r\Delta_1(a_{312} + a_{123} + a_{231}) + s\Delta_2(a_{312} + a_{123} + a_{231})\\
&\quad + \frac{t(t-1)(t-2)}{3}\Delta_3^3 a_{333} + rst\Delta_1\Delta_2\Delta_3 a_{312}\\
c_1^{(r,s,t)} &= c_1 + r\Delta_1(c_1 + a_{111}) + s\Delta_2(a_{123} + a_{231} + a_{312}) + t\Delta_3(a_{123} + a_{231} + a_{312})\\
&\quad + \frac{r(r-1)(r-2)}{3}\Delta_1^3 a_{111} + rst\Delta_1\Delta_2\Delta_3 a_{123}\\
c_2^{(r,s,t)} &= c_2 + s\Delta_2(c_2 + a_{222}) + t\Delta_3(a_{231} + a_{312} + a_{123}) + r\Delta_1(a_{231} + a_{312} + a_{123})\\
&\quad + \frac{s(s-1)(s-2)}{3}\Delta_2^3 a_{222} + rst\Delta_1\Delta_2\Delta_3 a_{231}\\
c_3^{(r,s,t)} &= c_2 + t\Delta_3(c_3 + a_{333}) + r\Delta_1(a_{312} + a_{123} + a_{231}) + s\Delta_2(a_{312} + a_{123} + a_{231})\\
&\quad + \frac{t(t-1)(t-2)}{3}\Delta_3^3 a_{333} + rst\Delta_1\Delta_2\Delta_3 a_{312}
\end{aligned}$$

$$
\begin{aligned}
\mu^{(r,s,t)}(\mathrm{x}) = {} & \lambda + r\Delta_1(b_1 + c_1) + s\Delta_2(b_2 + c_2) + t\Delta_3(b_3 + c_3) \\
& + 2\left[\frac{r(r-1)(r-2)}{3}\Delta_1^3 a_{111} + \frac{s(s-1)(s-2)}{3}\Delta_2^3 a_{222} \right. \\
& \left. + \frac{t(t-1)(t-2)}{3}\Delta_3^3 a_{333}\right] \\
& + rst\Delta_1\Delta_2\Delta_3(a_{123} + a_{231} + a_{312})
\end{aligned}
$$

where

$$
\begin{aligned}
\mu^{(r,s,t)} &= b_1^{(r,s,t)} + b_2^{(r,s,t)} + b_3^{(r,s,t)} + c_1^{(r,s,t)} + c_2^{(r,s,t)} + c_3^{(r,s,t)} \\
\lambda &= b_1 + b_2 + b_3 + c_1 + c_2 + c_3
\end{aligned}
$$

The property of equation (1.5) belonging to the leuricella hypergeometric class allows us to construct a family of particular solutions of (1.5) corresponding to a given λ. In fact, when $\mu^{(r,s,t)} = 0$, equation(2.2) has the particular solution $v_\alpha(\mathrm{x}) = constant$ This means that when

$$
\begin{aligned}
\lambda = \lambda^{(r,s,t)} = {} & -r\Delta_1(b_1 + c_1) - s\Delta_2(b_2 + c_2) - t\Delta_3(b_3 + c_3) \\
& - 2\left[\frac{r(r-1)(r-2)}{3}\Delta_1^3 a_{111} \right. \\
& \left. - \frac{s(s-1)(s-2)}{3}\Delta_2^3 a_{222} - \frac{t(t-1)(t-2)}{3}\Delta_3^3 a_{333}\right] \\
& - rst\Delta_1\Delta_2\Delta_3(a_{123} + a_{231} + a_{312})
\end{aligned}
$$

the equation has a particular solution which is a polynomial of total degree $r + s + t$.

Proposition 2.4. If $v_\alpha(\mathrm{x})$ is a solution of equation (2.2) then $u_1(\mathrm{x}) = \Delta_1 v_\alpha(\mathrm{x})$ is a solution of an equation in the lauricella hypergeometric class of the following from:

$$
\begin{aligned}
& a_{111}^{(r+1,s,t)}(\mathrm{x})\Delta_1\nabla_1\Delta_1 u_1(\mathrm{x}) + a_{112}^{(r+1,s,t)}(\mathrm{x})\Delta_1\nabla_1\Delta_2 u_1(\mathrm{x}) + a_{113}^{(r+1,s,t)}(\mathrm{x})\Delta_1\nabla_1\Delta_3 u_1(\mathrm{x}) \\
& + a_{221}^{(r+1,s,t)}(\mathrm{x})\Delta_2\nabla_2\Delta_1 u_1(\mathrm{x}) + a_{222}^{(r+1,s,t)}(\mathrm{x})\Delta_2\nabla_2\Delta_2 u_1(\mathrm{x}) + a_{223}^{(r+1,s,t)}(\mathrm{x})\Delta_2\nabla_2\Delta_3 u_1(\mathrm{x}) \\
& + a_{331}^{(r+1,s,t)}(\mathrm{x})\Delta_3\nabla_3\Delta_1 u_1(\mathrm{x}) + a_{332}^{(r+1,s,t)}(\mathrm{x})\Delta_3\nabla_3\Delta_2 u_1(\mathrm{x}) + a_{333}^{(r+1,s,t)}(\mathrm{x})\Delta_3\nabla_3\Delta_3 u_1(\mathrm{x}) \\
& + b_1^{(r+1,s,t)}(\mathrm{x})\Delta_1 u_1(\mathrm{x}) + b_2^{(r+1,s,t)}(\mathrm{x})\Delta_2 u_1(\mathrm{x}) + b_3^{(r+1,s,t)}(\mathrm{x})\Delta_3 u_1(\mathrm{x}) + c_1^{(r+1,s,t)}(\mathrm{x})\Delta_1 u_1(\mathrm{x}) \\
& + c_2^{(r+1,s,t)}(\mathrm{x})\Delta_2 u_1(\mathrm{x}) + c_3^{(r+1,s,t)}(\mathrm{x})\Delta_3 u_1(\mathrm{x}) + \mu^{(r+1,s,t)} u_1(\mathrm{x}) = 0
\end{aligned} \tag{2.3}
$$

where

$$
\begin{aligned}
a_{111}^{(r+1,s,t)}(\mathrm{x}) &= a_{111}^{(r,s,t)}(\mathrm{x}) \\
a_{112}^{(r+1,s,t)}(\mathrm{x}) &= a_{112}^{(r,s,t)}(\mathrm{x}) + \Delta_1 a_{112}^{(r,s,t)}(\mathrm{x}) + \Delta_3 a_{332}^{(r,s,t)}(\mathrm{x}) \\
a_{113}^{(r+1,s,t)}(\mathrm{x}) &= a_{113}^{(r,s,t)}(\mathrm{x}) + \Delta_1 a_{113}^{(r,s,t)}(\mathrm{x}) + \Delta_2 a_{223}^{(r,s,t)}(\mathrm{x})
\end{aligned}
$$

$$\begin{aligned}
a_{221}^{(r+1,s,t)}(\mathrm{x}) &= a_{221}^{(r,s,t)}(\mathrm{x})\\
a_{222}^{(r+1,s,t)}(\mathrm{x}) &= a_{222}^{(r,s,t)}(\mathrm{x}) - \Delta_1 a_{112}^{(r,s,t)}(\mathrm{x}) - \Delta_3 a_{332}^{(r,s,t)}(\mathrm{x})\\
a_{223}^{(r+1,s,t)}(\mathrm{x}) &= a_{223}^{(r,s,t)}(\mathrm{x}) + \Delta_2 a_{223}^{(r,s,t)}(\mathrm{x}) + \Delta_1 a_{113}^{(r,s,t)}(\mathrm{x})\\
a_{331}^{(r+1,s,t)}(\mathrm{x}) &= a_{331}^{(r,s,t)}(\mathrm{x})\\
a_{332}^{(r+1,s,t)}(\mathrm{x}) &= a_{332}^{(r,s,t)}(\mathrm{x}) + \Delta_3 a_{332}^{(r,s,t)}(\mathrm{x}) + \Delta_1 a_{112}^{(r,s,t)}(\mathrm{x})\\
a_{333}^{(r+1,s,t)}(\mathrm{x}) &= a_{333}^{(r,s,t)}(\mathrm{x}) - \Delta_1 a_{113}^{(r,s,t)}(\mathrm{x}) - \Delta_2 a_{223}^{(r,s,t)}(\mathrm{x})
\end{aligned}$$

and

$$\begin{aligned}
b_1^{(r+1,s,t)}(\mathrm{x}) &= b_1^{(r,s,t)}(\mathrm{x}) + \Delta_1(b_1^{(r,s,t)}(\mathrm{x}) + a_{111}^{(r,s,t)}(\mathrm{x}))\\
b_2^{(r+1,s,t)}(\mathrm{x}) &= b_2^{(r,s,t)}(\mathrm{x}) + \Delta_1(a_{231}^{(r,s,t)}(\mathrm{x}) + a_{312}^{(r,s,t)}(\mathrm{x}) + a_{123}^{(r,s,t)}(\mathrm{x}))\\
b_3^{(r+1,s,t)}(\mathrm{x}) &= b_3^{(r,s,t)}(\mathrm{x}) + \Delta_1(a_{312}^{(r,s,t)}(\mathrm{x}) + a_{123}^{(r,s,t)}(\mathrm{x}) + a_{231}^{(r,s,t)}(\mathrm{x}))\\
c_1^{(r+1,s,t)}(\mathrm{x}) &= c_1^{(r,s,t)}(\mathrm{x}) + \Delta_1(c_1^{(r,s,t)}(\mathrm{x}) + a_{111}^{(r,s,t)}(\mathrm{x}))\\
c_2^{(r+1,s,t)}(\mathrm{x}) &= c_2^{(r,s,t)}(\mathrm{x}) + \Delta_1(a_{231}^{(r,s,t)}(\mathrm{x}) + a_{312}^{(r,s,t)}(\mathrm{x}) + a_{123}^{(r,s,t)}(\mathrm{x}))\\
c_3^{(r+1,s,t)}(\mathrm{x}) &= c_3^{(r,s,t)}(\mathrm{x}) + \Delta_1(a_{312}^{(r,s,t)}(\mathrm{x}) + a_{123}^{(r,s,t)}(\mathrm{x}) + a_{231}^{(r,s,t)}(\mathrm{x}))\\
\mu^{(r+1,s,t)} &= \mu^{(r,s,t)} + \Delta_1(b_1^{(r,s,t)}(\mathrm{x}) + c_1^{(r,s,t)}(\mathrm{x})).
\end{aligned}$$

Also

Proposition 2.5. If $v_\alpha(\mathrm{x})$ is a solution of equation (2.2) then $u_2(\mathrm{x}) = \Delta_2 v_\alpha(\mathrm{x})$ is a solution of an equation in the lauricella hypergeometric class of the following from:

$$\begin{aligned}
&a_{111}^{(r,s+1,t)}(\mathrm{x})\Delta_1\nabla_1\Delta_1 u_2(\mathrm{x}) + a_{112}^{(r,s+1,t)}(\mathrm{x})\Delta_1\nabla_1\Delta_2 u_2(\mathrm{x}) + a_{113}^{(r,s+1,t)}(\mathrm{x})\Delta_1\nabla_1\Delta_3 u_2(\mathrm{x})\\
&+ a_{221}^{(r,s+1,t)}(\mathrm{x})\Delta_2\nabla_2\Delta_1 u_2(\mathrm{x}) + a_{222}^{(r,s+1,t)}(\mathrm{x})\Delta_2\nabla_2\Delta_2 u_2(\mathrm{x}) + a_{223}^{(r,s+1,t)}(\mathrm{x})\Delta_2\nabla_2\Delta_3 u_2(\mathrm{x})\\
&+ a_{331}^{(r,s+1,t)}(\mathrm{x})\Delta_3\nabla_3\Delta_1 u_2(\mathrm{x}) + a_{332}^{(r,s+1,t)}(\mathrm{x})\Delta_3\nabla_3\Delta_2 u_2(\mathrm{x}) + a_{333}^{(r,s+1,t)}(\mathrm{x})\Delta_3\nabla_3\Delta_3 u_2(\mathrm{x})\\
&+ b_1^{(r,s+1,t)}(\mathrm{x})\Delta_1 u_2(\mathrm{x}) + b_2^{(r,s+1,t)}(\mathrm{x})\Delta_2 u_2(\mathrm{x}) + b_3^{(r,s+1,t)}(\mathrm{x})\Delta_3 u_2(\mathrm{x}) + c_1^{(r,s+1,t)}(\mathrm{x})\Delta_1 u_2(\mathrm{x})\\
&+ c_2^{(r,s+1,t)}(\mathrm{x})\Delta_2 u_2(\mathrm{x}) + c_3^{(r,s+1,t)}(\mathrm{x})\Delta_3 u_2(\mathrm{x}) + \mu^{(r,s+1,t)} u_2(\mathrm{x}) = 0
\end{aligned} \tag{2.4}$$

the result are same as the proposition (2.4).

Proposition 2.6. If $v_\alpha(\mathrm{x})$ is a solution of equation (2.2) then $u_3(\mathrm{x}) = \Delta_3 v_\alpha(\mathrm{x})$ is a solution of an equation in the lauricella hypergeometric class of the following from:

$$\begin{aligned}
&a_{111}^{(r+1,s,t)}(\mathrm{x})\Delta_1\nabla_1\Delta_1 u_1(\mathrm{x}) + a_{112}^{(r+1,s,t)}(\mathrm{x})\Delta_1\nabla_1\Delta_2 u_1(\mathrm{x}) + a_{113}^{(r+1,s,t)}(\mathrm{x})\Delta_1\nabla_1\Delta_3 u_1(\mathrm{x})\\
&+ a_{221}^{(r+1,s,t)}(\mathrm{x})\Delta_2\nabla_2\Delta_1 u_1(\mathrm{x}) + a_{222}^{(r+1,s,t)}(\mathrm{x})\Delta_2\nabla_2\Delta_2 u_1(\mathrm{x}) + a_{223}^{(r+1,s,t)}(\mathrm{x})\Delta_2\nabla_2\Delta_3 u_1(\mathrm{x})\\
&+ a_{331}^{(r+1,s,t)}(\mathrm{x})\Delta_3\nabla_3\Delta_1 u_1(\mathrm{x}) + a_{332}^{(r+1,s,t)}(\mathrm{x})\Delta_3\nabla_3\Delta_2 u_1(\mathrm{x}) + a_{333}^{(r+1,s,t)}(\mathrm{x})\Delta_3\nabla_3\Delta_3 u_1(\mathrm{x})\\
&+ b_1^{(r+1,s,t)}(\mathrm{x})\Delta_1 u_1(\mathrm{x}) + b_2^{(r+1,s,t)}(\mathrm{x})\Delta_2 u_1(\mathrm{x}) + b_3^{(r+1,s,t)}(\mathrm{x})\Delta_3 u_1(\mathrm{x}) + c_1^{(r+1,s,t)}(\mathrm{x})\Delta_1 u_1(\mathrm{x})\\
&+ c_2^{(r+1,s,t)}(\mathrm{x})\Delta_2 u_1(\mathrm{x}) + c_3^{(r+1,s,t)}(\mathrm{x})\Delta_3 u_1(\mathrm{x}) + \mu^{(r+1,s,t)} u_1(\mathrm{x}) = 0
\end{aligned} \tag{2.5}$$

the result are same as the proposition (2.4).

3. Pearson's system

In this section we study, how to prove equation (2.1) belonging to the lauricella hypergeometric class that can be written in self-adjoint form and establish a number of useful identities.

Introducing the linear partial difference operator of the third order

$$\begin{aligned}\mathcal{D}v(\mathrm{x}) = {} & a_{111}(\mathrm{x})\Delta_1\nabla_1\Delta_1 v(\mathrm{x}) + a_{112}(\mathrm{x})\Delta_1\nabla_1\Delta_2 v(\mathrm{x}) + a_{113}(\mathrm{x})\Delta_1\nabla_1\Delta_3 v(\mathrm{x})\\ & + a_{221}(\mathrm{x})\Delta_2\nabla_2\Delta_1 v(\mathrm{x}) + a_{222}(\mathrm{x})\Delta_2\nabla_2\Delta_2 v(\mathrm{x}) + a_{223}(\mathrm{x})\Delta_2\nabla_2\Delta_3 v(\mathrm{x})\\ & + a_{331}(\mathrm{x})\Delta_3\nabla_3\Delta_1 v(\mathrm{x}) + a_{332}(\mathrm{x})\Delta_3\nabla_3\Delta_2 v(\mathrm{x}) + a_{333}(\mathrm{x})\Delta_3\nabla_3\Delta_3 v(\mathrm{x})\\ & + b_1(\mathrm{x})\Delta_1 v(\mathrm{x}) + b_2(\mathrm{x})\Delta_2 v(\mathrm{x}) + b_3(\mathrm{x})\Delta_3 v(\mathrm{x}) + c_1(\mathrm{x})\Delta_1 v(\mathrm{x})\\ & + c_2(\mathrm{x})\Delta_2 v(\mathrm{x}) + c_3(\mathrm{x})\Delta_3 v(\mathrm{x})\end{aligned} \tag{3.1}$$

we can represent equation (2.1) in the form

$$\mathcal{D}v(\mathrm{x}) + \lambda_n v(\mathrm{x}) = 0 \tag{3.2}$$

where λ_n is a constant.

The formal adjoint operator $\mathcal{D}^\dagger$ of $\mathcal{D}$ is defined by [10, 11], Now we defined the three dimensional operator,

$$\begin{aligned}\mathcal{D}^\dagger v = {} & \Delta_1\nabla_1\Delta_1(a_{111}v) + \Delta_1\nabla_1\Delta_2(a_{112}v) + \Delta_1\nabla_1\Delta_3(a_{113}v)\\ & + \Delta_2\nabla_2\Delta_1(a_{221}v) + \Delta_2\nabla_2\Delta_2(a_{222}v) + \Delta_2\nabla_2\Delta_3(a_{223}v)\\ & + \Delta_3\nabla_3\Delta_1(a_{331}v) + \Delta_3\nabla_3\Delta_2(a_{332}v) + \Delta_3\nabla_3\Delta_3(a_{333}v)\\ & + \Delta_1(b_1v) + \Delta_2(b_2v) + \Delta_3(b_3v) + \Delta_1(c_1v) + \Delta_2(c_2v) + \Delta_3(c_3v)\end{aligned} \tag{3.3}$$

The operator $\mathcal{D}$ is formally self-adjoint if $\mathcal{D}^\dagger v = \mathcal{D}v, \forall v \in \mathcal{C}^2(I)$. The operator $\mathcal{D}$ is symmetrizable if there exists a real function $\varrho(\mathrm{x}) = \varrho(x, y, z)$ such that the operator $\varrho(x, y, z)\mathcal{D}$ is formally self-adjoint. In order that $\mathcal{D}$ be symmetrizable, we multiply equation (2.1) through by a positive function $\varrho(\mathrm{x}) = \varrho(x, y, z)$ in some domain G, generating

$$\begin{aligned}& a_{111}(\mathrm{x})\varrho(\mathrm{x})\Delta_1\nabla_1\Delta_1 v(\mathrm{x}) + a_{112}(\mathrm{x})\varrho(\mathrm{x})\Delta_1\nabla_1\Delta_2 v(\mathrm{x}) + a_{113}(\mathrm{x})\varrho(\mathrm{x})\Delta_1\nabla_1\Delta_3 v(\mathrm{x})\\ & + a_{221}(\mathrm{x})\varrho(\mathrm{x})\Delta_2\nabla_2\Delta_1 v(\mathrm{x}) + a_{222}(\mathrm{x})\varrho(\mathrm{x})\Delta_2\nabla_2\Delta_2 v(\mathrm{x}) + a_{223}(\mathrm{x})\varrho(\mathrm{x})\Delta_2\nabla_2\Delta_3 v(\mathrm{x})\\ & + a_{331}(\mathrm{x})\varrho(\mathrm{x})\Delta_3\nabla_3\Delta_1 v(\mathrm{x}) + a_{332}(\mathrm{x})\varrho(\mathrm{x})\Delta_3\nabla_3\Delta_2 v(\mathrm{x}) + a_{333}(\mathrm{x})\varrho(\mathrm{x})\Delta_3\nabla_3\Delta_3 v(\mathrm{x})\\ & + b_1(\mathrm{x})\varrho(\mathrm{x})\Delta_1 v(\mathrm{x}) + b_2(\mathrm{x})\varrho(\mathrm{x})\Delta_2 v(\mathrm{x}) + b_3(\mathrm{x})\varrho(\mathrm{x})\Delta_3 v(\mathrm{x}) + c_1(\mathrm{x})\varrho(\mathrm{x})\Delta_1 v(\mathrm{x})\\ & + c_2(\mathrm{x})\varrho(\mathrm{x})\Delta_2 v(\mathrm{x}) + c_3(\mathrm{x})\varrho(\mathrm{x})\Delta_3 v(\mathrm{x}) + \lambda_n\varrho(\mathrm{x})v(\mathrm{x}) = 0\end{aligned} \tag{3.4}$$

The self-adjoint form, provided

$$\begin{aligned}
&\varrho(x-1,y,z)(a_{111}(x-1,y,z)+a_{112}(x-1,y,z)+a_{113}(x-1,y,z)\\
&\qquad +b_1(x-1,y,z)+c_1(x-1,y,z))=\varrho(x,y,z)\varpi_1(x,y,z)\\
&\varrho(x,y-1,z)(a_{221}(x,y-1,z)+a_{222}(x,y-1,z)+a_{223}(x,y-1,z)\\
&\qquad +b_2(x,y-1,z)+c_2(x,y-1,z))=\varrho(x,y,z)\varpi_2(x,y,z)\\
&\varrho(x,y,z-1)(a_{331}(x,y,z-1)+a_{332}(x,y,z-1)+a_{333}(x,y,z-1)\\
&\qquad +b_3(x,y,z-1)+c_3(x,y,z-1))=\varrho(x,y,z)\varpi_3(x,y,z)
\end{aligned}$$

$$\begin{aligned}
&\varrho(x-1,y,z)(a_{123}(x-1,y,z)=\varrho(x,y-1,z)(a_{231}(x,y-1,z)\\
&\qquad =\varrho(x,y,z-1)(a_{312}(x,y,z-1)\\
&\varrho(x-1,y,z)(a_{132}(x-1,y,z)=\varrho(x,y-1,z)(a_{213}(x,y-1,z)\\
&\qquad =\varrho(x,y,z-1)(a_{321}(x,y,z-1)
\end{aligned}$$

Then ϖ_1, ϖ_2 and ϖ_3 are notations,

$$\begin{aligned}
\varpi_1(x,y,z)&=a_{111}(x,y,z)+a_{112}(x,y,z)+a_{113}(x,y,z)\\
\varpi_2(x,y,z)&=a_{221}(x,y,z)+a_{222}(x,y,z)+a_{223}(x,y,z)\\
\varpi_3(x,y,z)&=a_{331}(x,y,z)+a_{332}(x,y,z)+a_{333}(x,y,z)
\end{aligned} \tag{3.5}$$

The function ϱ can be written as,

$$\begin{aligned}
&\Delta_1(\varpi_1(\mathrm{x})\varrho(\mathrm{x}))=\nabla_1((a_{111}(\mathrm{x})+a_{112}(\mathrm{x})+a_{113}(\mathrm{x})+b_1(\mathrm{x})+c_1(\mathrm{x}))\varrho(\mathrm{x}))\\
&\Delta_2(\varpi_2(\mathrm{x})\varrho(\mathrm{x}))=\nabla_2((a_{221}(\mathrm{x})+a_{222}(\mathrm{x})+a_{223}(\mathrm{x})+b_2(\mathrm{x})+c_2(\mathrm{x}))\varrho(\mathrm{x}))\\
&\Delta_3(\varpi_3(\mathrm{x})\varrho(\mathrm{x}))=\nabla_3((a_{331}(\mathrm{x})+a_{332}(\mathrm{x})+a_{333}(\mathrm{x})+b_3(\mathrm{x})+c_3(\mathrm{x}))\varrho(\mathrm{x}))\\
&\varrho(x,y+1,z)(a_{123}(x,y+1,z)=\varrho(x,y,z+1)(a_{231}(x,y,z+1)\\
&\quad =\varrho(x+1,y,z)(a_{312}(x+1,y,z)\\
&\varrho(x,y+1,z)(a_{132}(x,y+1,z)=\varrho(x,y,z+1)(a_{213}(x,y,z+1)\\
&\quad =\varrho(x+1,y,z)(a_{321}(x+1,y,z)
\end{aligned} \tag{3.6}$$

(i.e) A system of equations for determining the unknown function $\varrho(X)$ is obtain,

$$\begin{aligned}
&\Delta_1(\varrho(\mathrm{x})a_{111})+\Delta_2(\varrho(\mathrm{x})a_{112})+\Delta_3(\varrho(\mathrm{x})a_{113})=\varrho(\mathrm{x})(b_1(\mathrm{x})+c_1(\mathrm{x}))\\
&\Delta_1(\varrho(\mathrm{x})a_{221})+\Delta_2(\varrho(\mathrm{x})a_{222})+\Delta_3(\varrho(\mathrm{x})a_{223})=\varrho(\mathrm{x})(b_2(\mathrm{x})+c_2(\mathrm{x}))\\
&\Delta_1(\varrho(\mathrm{x})a_{331})+\Delta_2(\varrho(\mathrm{x})a_{332})+\Delta_3(\varrho(\mathrm{x})a_{333})=\varrho(\mathrm{x})(b_3(\mathrm{x})+c_3(\mathrm{x}))\\
&\Delta_1\nabla_2\Delta_3(a_{312}(\mathrm{x})\varrho(\mathrm{x}))=\Delta_2\nabla_3\Delta_1(a_{123}(\mathrm{x})\varrho(\mathrm{x}))=\Delta_3\nabla_1\Delta_2(a_{231}(\mathrm{x})\varrho(\mathrm{x}))\\
&\Delta_2\nabla_1\Delta_3(a_{321}(\mathrm{x})\varrho(\mathrm{x}))=\Delta_3\nabla_2\Delta_1(a_{132}(\mathrm{x})\varrho(\mathrm{x}))=\Delta_1\nabla_3\Delta_2(a_{213}(\mathrm{x})\varrho(\mathrm{x}))
\end{aligned} \tag{3.7}$$

This equation is called $Pearson's system$ for the three dimensional case, which is a generalization of [10, 11]. The above system of equation gives the conditions for equation (2.1) be self-adjoint.

The equations (3.7) can be considered as a system of partial difference equations,

$$\begin{pmatrix} a_{111}(x,y,z) & a_{112}(x,y,z) & a_{113}(x,y,z) \\ a_{221}(x,y,z) & a_{222}(x,y,z) & a_{223}(x,y,z) \\ a_{331}(x,y,z) & a_{332}(x,y,z) & a_{333}(x,y,z) \\ a_{312}(x+1,y,z) & -a_{123}(x,y+1,z) & -a_{231}(x,y,z+1) \\ a_{321}(x+1,y,z) & -a_{132}(x,y+1,z) & -a_{213}(x,y,z+1) \end{pmatrix} \begin{pmatrix} \dfrac{\Delta_1(\varrho(\mathrm{x}))}{\varrho(\mathrm{x})} \\ \dfrac{\Delta_2(\varrho(\mathrm{x}))}{\varrho(\mathrm{x})} \\ \dfrac{\Delta_3(\varrho(\mathrm{x}))}{\varrho(\mathrm{x})} \end{pmatrix} = \begin{pmatrix} \theta(x,y,z) \\ \phi(x,y,z) \\ \psi(x,y,z) \\ \delta(x,y,z) \\ \xi(x,y,z) \end{pmatrix} \tag{3.8}$$

with respect to the unknown function $\varrho(\mathrm{x})$, where

$$\begin{aligned} \theta(x,y,z) &= c_1(\mathrm{x}) + b_1(\mathrm{x}) - \mathcal{G}_1(\mathrm{x})\Delta_1(a_{111}(\mathrm{x})) - \mathcal{G}_2(\mathrm{x})\Delta_2(a_{112}(\mathrm{x})) - \mathcal{G}_3(\mathrm{x})\Delta_3(a_{113}(\mathrm{x})) \\ \phi(x,y,z) &= c_2(\mathrm{x}) + b_2(\mathrm{x}) - \mathcal{G}_1(\mathrm{x})\Delta_1(a_{221}(\mathrm{x})) - \mathcal{G}_2(\mathrm{x})\Delta_2(a_{222}(\mathrm{x})) - \mathcal{G}_3(\mathrm{x})\Delta_3(a_{223}(\mathrm{x})) \\ \psi(x,y,z) &= c_3(\mathrm{x}) + b_3(\mathrm{x}) - \mathcal{G}_1(\mathrm{x})\Delta_1(a_{331}(\mathrm{x})) - \mathcal{G}_2(\mathrm{x})\Delta_2(a_{332}(\mathrm{x})) - \mathcal{G}_3(\mathrm{x})\Delta_3(a_{333}(\mathrm{x})) \\ \delta(x,y,z) &= a_{231}(x,y,z+1) + a_{123}(x,y+1,z) - a_{312}(x+1,y,z) \\ \xi(x,y,z) &= a_{213}(x,y,z+1) + a_{132}(x,y+1,z) - a_{321}(x+1,y,z) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \mathcal{G}_1(x,y,z) &= \frac{a_{111}(x,y,z) + a_{112}(x,y,z) + a_{113}(x,y,z) + b_1(x,y,z) + c_1(x,y,z)}{a_{111}(x+1,y,z) + a_{112}(x+1,y,z) + a_{113}(x+1,y,z)} \\ &= \frac{a_{111}(x,y,z) + a_{112}(x,y,z) + a_{113}(x,y,z) + b_1(x,y,z) + c_1(x,y,z)}{\varpi_1(x+1,y,z)} \end{aligned}$$

$$\begin{aligned} \mathcal{G}_2(x,y,z) &= \frac{a_{221}(x,y,z) + a_{222}(x,y,z) + a_{223}(x,y,z) + b_2(x,y,z) + c_2(x,y,z)}{a_{221}(x,y+1,z) + a_{222}(x,y+1,z) + a_{223}(x,y+1,z)} \\ &= \frac{a_{221}(x,y,z) + a_{222}(x,y,z) + a_{223}(x,y,z) + b_2(x,y,z) + c_2(x,y,z)}{\varpi_2(x,y+1,z)} \end{aligned}$$

$$\begin{aligned} \mathcal{G}_3(x,y,z) &= \frac{a_{331}(x,y,z) + a_{332}(x,y,z) + a_{333}(x,y,z) + b_3(x,y,z) + c_3(x,y,z)}{a_{331}(x,y,z+1) + a_{332}(x,y,z+1) + a_{333}(x,y,z+1)} \\ &= \frac{a_{331}(x,y,z) + a_{332}(x,y,z) + a_{333}(x,y,z) + b_3(x,y,z) + c_3(x,y,z)}{\varpi_3(x,y,z+1)} \end{aligned}$$

We assume that the functions θ, ϕ, ψ, δ and ξ do not vanish simultaneously on the domain G and $\varpi_1(x+1,y,z) \neq 0$, $\varpi_2(x,y+1,z) \neq 0$ and $\varpi_3(x,y,z+1) \neq 0$. Hence, there exists a solution ϱ of the Pearson's system and it is non zero in the domain G, if and only if, it satisfies atleast one of the above conditions.

$$\begin{vmatrix} a_{111}(x,y,z) & a_{112}(x,y,z) & a_{113}(x,y,z) & \theta(x,y,z) \\ a_{221}(x,y.z) & a_{222}(x,y,z) & a_{223}(x,y,z) & \phi(x,y,z) \\ a_{331}(x,y,z) & a_{332}(x,y,z) & a_{333}(x,y,z) & \psi(x,y,z) \\ a_{312}(x+1,y,z) & -a_{123}(x,y+1,z) & -a_{231}(x,y,z+1) & \delta(x,y,z) \\ a_{321}(x+1,y,z) & -a_{132}(x,y+1,z) & -a_{213}(x,y,z+1) & \xi(x,y,z) \end{vmatrix} = 0 \quad (3.10)$$

Thus the above 4×5 Matrix can be taken as individual 3×3 Matrix and these can be solved as follows.

$$\begin{aligned} & a_{111}(x,y,z)(a_{222}(x,y,z)a_{333}(x,y,z) - a_{332}(x,y,z)a_{223}(x,y,z)) \\ & - a_{112}(x,y,z)(a_{221}(x,y,z)a_{333}(x,y,z) - a_{223}(x,y,z)a_{331}(x,y,z)) \\ & + a_{113}(x,y,z)(a_{221}(x,y,z)a_{332}(x,y,z) - a_{222}(x,y,z)a_{331}(x,y,z)) \neq 0 \end{aligned}$$

This equation satisfies the conditions mentioned above. And also the remaining 3×3 matrix are satisfied.

As equation (1.1) for $y(x)$, which approximates the differential equation(1.1) up to the third order of accuracy with respect to the mesh h, the Pearson's system (3.7) can be generalized to the multivariable case by means of

$$\begin{aligned} & \sum_{k=1}^{n} \Delta_k(\varrho(x)a_{ijk}(x)) = \varrho(x)(b_i(x) + c_i(x)),\ i,j = 1,2\ldots,n, \\ & \Delta_i \nabla_j \Delta_k(\varrho(x)a_{jki}(x)) = \Delta_j \nabla_k \Delta_i(\varrho(x)a_{kij}(x)) \\ & \quad = \Delta_k \nabla_i \Delta_j(\varrho(x)a_{ijk}(x)),\ i \neq j \neq k \end{aligned} \qquad (3.11)$$

which corresponds to the third order approximation of the system of differential equations

$$\varrho b_i = \sum_{j,k=1}^{n} \frac{\partial^2}{\partial x_j \partial x_k}\varrho(a_{ijk}),\ \varrho c_i = \sum_{j,k=1}^{n} \frac{\partial^2}{\partial x_j \partial x_k}\varrho(a_{ijk}),\ i = 1,2,\ldots,n,$$

given in [7, 11], which appears when we assume that in some domain G there exists a positive and twice continuously differentiable function $\varrho(x)$ such that the differential expression

$$\begin{aligned} & \sum_{i,j,k=1}^{n} \varrho a_{ijk}(x)\frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k} + \sum_{i=1}^{n} \varrho b_i(x)\frac{\partial u}{\partial x_i} + \sum_{i=1}^{n} \varrho c_i(x)\frac{\partial u}{\partial x_i} \\ & = \sum_{i,j,k=1}^{n} \frac{\partial}{\partial x_i}\left(\varrho(x)a_{ijk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k}\right) \end{aligned}$$

is formally self-adjoint in G. Note that the second equation of (3.11) is due to the non symmetry of the matrix $(a_{ijk}(\mathrm{x}))$.

Returning to the three dimensional case, if the positive function $\varrho(\mathrm{x}) = \varrho(x, y, z)$ in some domain G satisfy the above system of difference equations(3.8)(2.1) can be reduced to the self-adjoint form

$$\begin{aligned}
&\nabla_1[\varrho(x+1, y, z)a_{111}(x+1, y, z)\Delta_1 v(\mathrm{x}) + \varrho(x, y+1, z)a_{112}(x, y+1, z)\Delta_2 v(\mathrm{x})\\
&+ \varrho(x, y, z+1)a_{113}(x, y, z+1)\Delta_3 v(\mathrm{x})] + \nabla_2[\varrho(x+1, y, z)a_{221}(x+1, y, z)\Delta_1 v(\mathrm{x})\\
&+ \varrho(x, y+1, z)a_{222}(x, y+1, z)\Delta_2 v(\mathrm{x}) + \varrho(x, y, z+1)a_{223}(x, y, z+1)\Delta_3 v(\mathrm{x})]\\
&+ \nabla_3[\varrho(x+1, y, z)a_{331}(x+1, y, z)\Delta_1 v(\mathrm{x}) + \varrho(x, y+1, z)a_{332}(x, y+1, z)\Delta_2 v(\mathrm{x})\\
&+ \varrho(x, y, z+1)a_{333}(x, y, z+1)\Delta_3 v(\mathrm{x})] + \lambda_n \varrho(\mathrm{x})v(\mathrm{x}) = 0
\end{aligned} \tag{3.12}$$

Similarly equation (2.2) can be reduced to the form,

$$\begin{aligned}
&\nabla_1[\varrho^{(r,s,t)}(x+1, y, z)a_{111}^{(r,s,t)}(x+1, y, z)\Delta_1 v_\alpha(\mathrm{x})\\
&+ \varrho^{(r,s,t)}(x, y+1, z)a_{112}^{(r,s,t)}(x, y+1, z)\Delta_2 v_\alpha(\mathrm{x})\\
&+ \varrho^{(r,s,t)}(x, y, z+1)a_{113}^{(r,s,t)}(x, y, z+1)\Delta_3 v_\alpha(\mathrm{x})]\\
&+ \nabla_2[\varrho^{(r,s,t)}(x+1, y, z)a_{221}^{(r,s,t)}(x+1, y, z)\Delta_1 v_\alpha(\mathrm{x})\\
&+ \varrho^{(r,s,t)}(x, y+1, z)a_{222}^{(r,s,t)}(x, y+1, z)\Delta_2 v_\alpha(\mathrm{x})\\
&+ \varrho^{(r,s,t)}(x, y, z+1)a_{223}^{(r,s,t)}(x, y, z+1)\Delta_3 v_\alpha(\mathrm{x})]\\
&+ \nabla_3[\varrho^{(r,s,t)}(x+1, y, z)a_{331}^{(r,s,t)}(x+1, y, z)\Delta_1 v_\alpha(\mathrm{x})\\
&+ \varrho^{(r,s,t)}(x, y+1, z)a_{332}^{(r,s,t)}(x, y+1, z)\Delta_2 v_\alpha(\mathrm{x})\\
&+ \varrho^{(r,s,t)}(x, y, z+1)a_{333}^{(r,s,t)}(x, y, z+1)\Delta_3 v_\alpha(\mathrm{x})]\\
&+ \mu_n^{(r,s,t)}\varrho^{(r,s,t)}(\mathrm{x})v_\alpha(\mathrm{x}) = 0
\end{aligned} \tag{3.13}$$

for the solution $v_\alpha(\mathrm{x}) = \Delta_1^r \Delta_2^s \Delta_3^t v(\mathrm{x})$, where the function $\varrho^{(r,s,t)}(\mathrm{x})$ satisfies the pearson's system

$$\begin{aligned}
&\Delta_1(\varrho^{(r,s,t)}(\mathrm{x})a_{111}^{(r,s,t)}(\mathrm{x})) + \Delta_2(\varrho^{(r,s,t)}(\mathrm{x})a_{112}^{(r,s,t)}(\mathrm{x})) + \Delta_3(\varrho^{(r,s,t)}(\mathrm{x})a_{113}^{(r,s,t)}(\mathrm{x}))\\
&\qquad = \varrho^{(r,s,t)}(\mathrm{x})[b_1^{(r,s,t)}(\mathrm{x}) + c_1^{(r,s,t)}(\mathrm{x})]\\
&\Delta_1(\varrho^{(r,s,t)}(\mathrm{x})a_{221}^{(r,s,t)}(\mathrm{x})) + \Delta_2(\varrho^{(r,s,t)}(\mathrm{x})a_{222}^{(r,s,t)}(\mathrm{x})) + \Delta_3(\varrho^{(r,s,t)}(\mathrm{x})a_{223}^{(r,s,t)}(\mathrm{x}))\\
&\qquad = \varrho^{(r,s,t)}(\mathrm{x})[b_2^{(r,s,t)}(\mathrm{x}) + c_2^{(r,s,t)}(\mathrm{x})]\\
&\Delta_1(\varrho^{(r,s,t)}(\mathrm{x})a_{331}^{(r,s,t)}(\mathrm{x})) + \Delta_2(\varrho^{(r,s,t)}(\mathrm{x})a_{332}^{(r,s,t)}(\mathrm{x})) + \Delta_3(\varrho^{(r,s,t)}(\mathrm{x})a_{333}^{(r,s,t)}(\mathrm{x}))\\
&\qquad = \varrho^{(r,s,t)}(\mathrm{x})[b_3^{(r,s,t)}(\mathrm{x}) + c_3^{(r,s,t)}(\mathrm{x})]\\
&\Delta_1\nabla_2\Delta_3(a_{312}^{(r,s,t)}(\mathrm{x})\varrho^{(r,s,t)}(\mathrm{x})) = \Delta_2\nabla_3\Delta_1(a_{123}^{(r,s,t)}(\mathrm{x})\varrho^{(r,s,t)}(\mathrm{x}))\\
&\qquad = \Delta_3\nabla_1\Delta_2(a_{231}^{(r,s,t)}(\mathrm{x})\varrho^{(r,s,t)}(\mathrm{x}))\\
&\Delta_2\nabla_1\Delta_3(a_{321}^{(r,s,t)}(\mathrm{x})\varrho^{(r,s,t)}(\mathrm{x})) = \Delta_3\nabla_2\Delta_1(a_{132}^{(r,s,t)}(\mathrm{x})\varrho^{(r,s,t)}(\mathrm{x}))\\
&\qquad = \Delta_1\nabla_3\Delta_2(a_{213}^{(r,s,t)}(\mathrm{x})\varrho^{(r,s,t)}(\mathrm{x}))
\end{aligned}$$

and the coefficients $a_{ijk}^{(r,s,t)}(\mathrm{x})$, $(i, j, k = 1, 2, 3)$ are gives Thm (2.3). The above system generalizes [10, 11].

4. Conclusion

We have given a detailed review of a systematic investigation in order to derive the orthogonal polynomial solution of the third order partial difference equation of Leuricella hypergeometric type. The equation is written in self adjoint form, gave a number of useful identities for the orthogonality weight of the polynomial solutions as well as the difference derivatives of the polynomial solutions.

References

[1] Abramowitz. M and Stegun. I, (1965) "Handbook of Mathematical Functions", Dover, NewYork,

[2] Ch.F. Dunkl, Y. Xu, (2001) "Orthogonal polynomials of several variables", Encyclopedia of Mathematics and its Applications, vol. 81, Cambridge University Press, Cambridge.

[3] Erdlyi. A, (1953) "Higher Transcendental Functions", vol. 1, McGraw-Hill, NewYork.

[4] Gelfand. I.M, Minlos. R.A. and Z.Ya. Sapiro, (1963) "Representations of the Rotation Group and of the Lorentz Group and Their Applications", MacMillan, NewYork.

[5] Jackson. D, (1936) "Formal properties of orthogonal polynomials in two variables", Duke Math. J., 2, 423–434.

[6] Kelley. W.G. and Peterson. A.C., (2001) "Difference Equations. An Introduction with Applications", second ed., Academic Press, San Diego.

[7] Lyskova. A.S., (1991) "Orthogonal polynomials in several variables", Sov. Math., Dokl. 43(1), 264–268.

[8] Lyskova. A.S., (1997) "On some properties of orthogonal polynomials in several variables", Russ. Math. Surv. 52(4), 840–841.

[9] Nikiforov. A.F. and Ouvarov. V.B., (1974) "Moments de la Theorie des Fonctions Spciales", Mir, Moscou.

[10] Nikiforov. A.F., Suslov. S.K. and Uvarov. V.B., (1991) "Classical Orthogonal Polynomials of a Discrete Variable", Springer Series in Computational Physics, Springer, Berlin.

[11] Rodal. J, Area. I and Godoy. E., (2007) "Linear partial difference equations of hypergeometric type: Orthogonal polynomial solutions in two discrete variables, Computational and Applied mathematics", 200, 722–748.

[12] Selvaraj. B. and Daphy Louis Lovenia. J., (2011) "Oscillatory properties of certain frst and second order difference equations", Journal of Computer and Mathematical Sciences, 2(3), 567–571.