Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 11 (2017), pp. 7819–7827

http://www.ripublication.com/gjpam.htm

On marker set distance cospectral graphs

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Abstract

Let $G = (V, E)$ be a simple undirected connected graph and let M be a nonempty subset of vertices of G. Using the results we have derived in our previous paper on marker set distance matrix, we define two graphs G_1 with a marker set M_1 and G_2 with a marker set M_2 to be marker set cospectral if they have the same marker distance spectrum. In this paper, we obtain the conditions under which two graphs are marker set cospectral. Algorithms are developed for the same. The polynomials which can be realised as the characteristic polynomial of a M-set distance matrix and real numbers which can be realised as the eigenvalues of a M-set distance are also discussed.

AMS subject classification: Primary 05C12, 05C50, 92E10.
Keywords: Marker set of a graph, M-set distance matrix, M-set distance Laplacian, characteristic polynomial, eigenvalues.

1. Introduction

In [11], we had introduced "Marker set distance eigenvalues" of a graph. Here we extend the notions defined in [11] and consider marker set distance cospectral graphs. For convenience we list some of the definitions here again.

Definition 1.1. [11] Let $G = (V, E)$ be a simple connected graph of order p. Let M be a subset of vertices of G, referred to as a marker set or M-set (in short). We define M-set distance between two vertices v_i and v_j as $d_M(v_i, v_j) = d_{ij} = |d(v_i, M) - d(v_j, M)|$. Here $d(v_i, M) = \min\{d(v_i, w) : w \in M\}$.

Definition 1.2. [11] The M-set eccentricity of a vertex v of G, denoted by $e_M(v)$ is defined as the maximum of all the M-set distances of v.

Definition 1.3. [11] The M-set diameter of a graph G with respect to a marker set M is denoted by $diam_M(G)$ and is defined as the maximum of all the M-set eccentricities of the vertices of G.

The $p \times p$ matrix $D_M(G) = [d_{ij}]$ is called the *M-set distance matrix* of the marker set M in the graph G.

Definition 1.4. [11] The characteristic polynomial of $D_M(G)$ is defined as $\phi(G : M, \mu) = det(\mu I - D_M(G))$ the roots of which are assumed to be in non-increasing order and are called the marker set distance eigen values of M in G.
In case of a graph of order p and its distance matrix of a marker set M being $D_M(G)$, the characteristic polynomial can be written as $\Phi(G : M, \mu) = \Delta(D_M(G) = \mu^p - S_1\mu^{p-1} + S_2\mu^{p-2} - \ldots + (-1)^p S_p$. It is clear from [13] that $(-1)^i S_i = \Sigma M_{D_i}$ where M_{D_i} are the principal minors of $D_M(G)$ with order i. (Minors whose diagonal elements belong to the main diagonal of $D_M(G)$). $S_0 = 1$ and $S_1 = traceD_M(G) = 0$.
The *marker set distance energy* (MSDE in short) of M in G is defined as

$$\varepsilon_M(G) = \sum_{i=1}^{p} |\mu_i|,$$

where $\mu_1, \mu_2, \ldots, \mu_p$ are the M-distance eigen values.

Definition 1.5. [11] Two marker sets M_1 and M_2 of a graph G are said to be equienergetic if the MSDEs of G corresponding to the two sets are equal.

2. Marker set distance cospectrality

We define M-set equienergetic graphs as follows.

Definition 2.1. Two graphs G_1 and G_2 with marker sets M_1 and M_2 are said to be marker set equienergetic if the MSDEs of G_1 corresponding to a marker set M_1 of G_1 and G_2 corresponding to the marker set M_2 of G_2 are equal.

We define cospectrality (depending on the M-set eigenvalues).

Definition 2.2. Two simple connected graphs G_1 and G_2 with marker sets M_1 and M_2 respectively are said to be marker set cospectral if the two graphs have the same set of marker set eigenvalues.

Theorem 2.3. Two simple connected graphs G_1 and G_2 with marker sets M_1 and M_2 respectively are M-set cospectral if the following conditions are satisfied.

1. $|V(G_1)| = |V(G_2)|$,

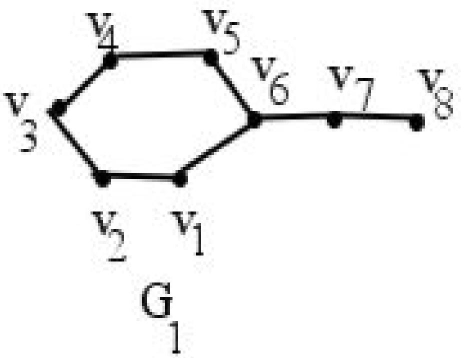

Figure 1:

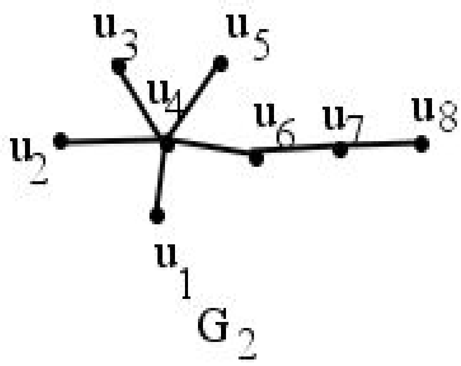

Figure 2:

2. $|M_1| = |M_2|$,

3. $diam_{M_1}(G_1) = diam_{M_2}(G_2) = m$,

4. $DDS_G(M_1) = DDS_G(M_2)$.

Proof. Let the graphs G_1 and G_2 with marker sets M_1 and M_2 respectively satisfy the conditions given above. Then the marker set distance matrices of graphs G_1 and G_2 are similar to the matrix $D_M(G)$ given above. That is, the graphs G_1 and G_2 have the same set of marker set eigenvalues. Hence the graphs G_1 and G_2 are marker set cospectral. ■

Corollary 2.4. Two marker sets M_1 and M_2 of a simple connected graph G are equienergetic if the following conditions are satisfied.

1. $|M_1| = |M_2|$,

2. $diam_{M_1}(G) = diam_{M_2}(G) = m$,

3. $DDS_G(M_1) = DDS_G(M_2)$.

Remark 2.5. Two M-set cospectral graphs satisfying the conditions of the above theorem have their MSDEs equal. but need not be isomorphic.

The following example illustrates the above remark.

Example 2.6. The graphs G_1 (Figure 1) and G_2 (Figure 2) are nonisomophic graphs. But with M-sets $M_1 = \{v_1, v_2, v_3, v_4, v_5\}$ for G_1 and $M_2 = \{u_1, u_2, u_3, u_4, u_5\}$ for

G_2, the marker set distance matrices $D_{M_1}(G_1) \simeq D_{M_2}(G_2) \simeq$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the two graphs are M-set cospectral and are equienergetic.

Next, we answer the prominent question of which real numbers can be eigenvalues of some M-set in a graph, and call it a realizable eigenvalue.

Theorem 2.7. A square root of a positive integer is a realizable eigenvalue of an M-set distance matrix.

Proof. Consider $\sqrt{n}$ where n is a positive integer. Let $n = km$ where k and m are positive integers ($k \geq 1, m \geq 1$). Put $p = m + k$. Then the M-distance matrix can be taken as a block matrix

$$D_M(G) = \begin{bmatrix} 0_{m\times m} & 1_{m\times k} \\ 1_{k\times m} & 0_{k\times k} \end{bmatrix}$$

where $j_{t_1\times t_2}$ is a matrix of order $t_1 \times t_2$ with all entries equal to j where$j \in \{0, 1\}$. The row reduced echelon form of the matrix has two nonzero rows and so the rank of the matrix is 2. It can be seen that $D_M(G)$ has $2km$ entries as 1. Every subset of the principal diagonal gives a principal minor. It follows that there are km minors of order 2. Hence $S_i = 0$ for $3 \leq i \leq p$, as each principal minor of order at least three has at least two equal rows (columns) making its value zero. Hence the M-distance characteristic polynomial for G can be written as $\mu^p - km\mu^{(p-2)} = 0$, and the M-set distance eigen values are $0, \sqrt{km}, -\sqrt{km}$ with multiplicities $(p-2)$, 1, 1 respectively. In other words, the eigenvalues are $\sqrt{n}$,$-\sqrt{n}$ and zero of multiplicity $(p-2)$.

The graphical interpretation of this matrix is a graph G on p vertices is a sequential join of $\overline{K_{(p-m)}}$ and $\langle M\rangle$, where $\langle M\rangle$ is the induced graph on m vertices. Note that in this case $\langle M\rangle$ becomes a dominating set. That is, $G = \langle M\rangle + \overline{K_{(p-m)}}$.

For any other decomposition of n, $n = k_1m_1, n, k_1, m_1 \in Z$, let $p_1 = k_1 + m_1$. Then the M-set distance matrix can be taken as

$$\begin{bmatrix} 0_{m_1\times m_1} & 1_{m_1\times k_1} \\ 1_{k_1\times m_1} & 0_{k_1\times k_1} \end{bmatrix}$$

where $j_{t_1\times t_2}$ is a matrix of order $t_1 \times t_2$ with all entries equal to j where $j \in \{0, 1\}$ and $G = \langle M\rangle + \overline{K_{(p_1-m_1)}}$ is the realizable graph again.

If n is a prime number then $n = n.1$ is the only 2-integer decomposition. Put $p = n + 1$. Then

$$D_M(G) = \begin{bmatrix} 0_{n\times n} & 1_{n\times 1} \\ 1_{1\times n} & 0_{1\times 1}. \end{bmatrix}$$

Then, it can be seen that the graph G of order p whose eigenvalues are $\sqrt{n}$, $-\sqrt{n}$ and zero of multiplicity $(p-2)$ is $G = \langle M\rangle + \overline{K_{(p-1)}}$ where $|M| = 1$ or $G = \langle M\rangle + \overline{K_1}$ where $|M| = p - 1$.

Hence, we can find a graph with M-set distance eigenvalues $\sqrt{n}$, $-\sqrt{n}$ and zero of multiplicity $(p - 2)$ for each distinct 2 integer factor decomposition of n. ■

Corollary 2.8. Every even positive integer can be realised as the M-set distance energy of a simple connected graph.

Proof. The proof follows from the above theorem. ■

As a consequence of the above discussion we give an algorithm to get a graph G whose M-set distance eigenvalues are $\sqrt{n}$, $-\sqrt{n}$, 0 of multiplicity 1, 1, $(p - 2)$ respectively, where n is a positive nonprime integer where $n = k.m$ and $p = k + m$.

Algorithm 1:

Start

Step 1: Input two positive integers k and m

Step 2: Input positive nonprime integer n.

Step 3: Define $n = k \cdot m$ and $p = k + m$.

Step 4: Construct a graph $\langle M\rangle$ on k vertices.

Step 5: Consider m isolates and call it $\overline{K_m}$.

Step 6: Form G with p vertices and edges defined as $e = u_i v_j$ with $u_i \in \langle M\rangle$ and $v_j \in \overline{K_m}$, for all $1 \le i \le k$, $1 \le j \le m$.

Step 7: Display the eigenvalues with their multiplicities.

Step 8: Return G.

Illustration 1.

Construct a graph G with eigenvalues $\sqrt{12}$, $-\sqrt{12}$, 0 of multiplicity 1, 1, $(p-2)$ where $12 = k.m$ and $p = k + m$. Obviously, 12 has three different 2-integer factor decompositions. $12 = 3 \times 4, 12 = 6 \times 2$, $12 = 1 \times 12$

Consider $12 = 3 \times 4$.

Step 1: Input two positive integers 3 and 4.

Step 2: Input positive nonprime integer 12

Step 3: Define $12 = 3 \cdot 4$ and $7 = 3 + 4$.

Step 4: Construct a graph $\langle M\rangle$ on 3 vertices.

Step 5: Consider 4 isolates and call it $\overline{K_4}$.

Step 6: Form G with 7 vertices and edges are defined as $e = u_i v_j$ with $u_i \in \langle M\rangle$ and $v_j \in \overline{K_m}$, for all $1 \le i \le 3$, $1 \le j \le 4$.

Step 7: Eigenvalues of G are $\sqrt{12}$, $-\sqrt{12}$, 0 of multiplicity 1, 1, 5 respectively.

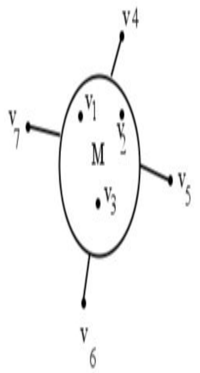

Figure 3:

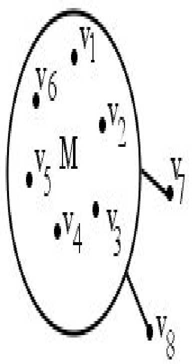

Figure 4:

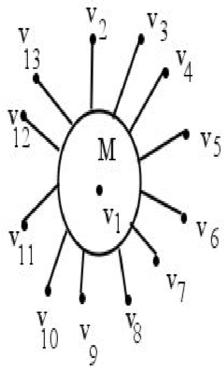

Figure 5:

Step 8: G is the required graph as in Figure 3.
Similarly, we can get the graphs as in Figure 4 and Figure 5 for the decompositions $12 = 6 \times 2$ ($|M| = 6$), $12 = 1 \times 12$ ($|M| = 1$) respectively.

The above algorithm can be used to find M-set equienergetic nonisomorphic graphs.

Corollary 2.9. Given a positive integer n, two nonisomorphic M-set cospectral graphs can be constructed with the eigenvalues $\sqrt{n}$, $-\sqrt{n}$, 0 of multiplicity 1, 1, $(p-2)$ where p is the order of the graphs.

Proof. Let $n = k.m$ where k and m are distinct positive integers. Let $p = k + m$ Even if n is a prime number, we can consider $n = n.1$ as the 2-integer decomposition. Then, by the previous theorem. we can construct a graph G with M-set eigenvalues $\sqrt{n}$, $-\sqrt{n}$, 0 of multiplicity. 1, 1, $(p-2)$ where $|M| = k$.

$$G = \langle M \rangle + \overline{K_m}.$$

Also, we can construct another graph H with the same eigenvalues as that of G using the previous theorem.

$$H = \langle M_1 \rangle + \overline{K_k}.$$

where $|M_1| = m$. If $k \neq m$ then G and H are nonisomorphic graphs. If $k = m$ then we can choose nonisomorphic graphs M and M_1 on k vertices to get nonisomorphic graphs G and H. ■

We can rewrite the Algorithm 1 for the above procedure.

Algorithm 2: Consider a positive integer n such that $n = k \times m$ where k and m are positive integers
Start
Step 1: Input two positive integers k and m
Step 2: Input positive nonprime integer n.
Step 3: Define $n = k \cdot m$ and $p = k + m$.
Step 4: Construct a graph $\langle M \rangle$ on k vertices.

Step 5: Consider m isolates and call it $\overline{K_m}$.
Step 6: Form G with p vertices and edges defined as $e = u_i v_j$ with $u_i \in \langle M \rangle$ and $v_j \in \overline{K_m}$, for all $1 \leq i \leq k$, $1 \leq j \leq m$.
Step 7: Display the eigenvalues with their multiplicities.
Step 8: Return G.
Step 9: Form a graph H by taking a graph $\langle M_1 \rangle$ on m vertices and joining it with $\overline{K_k}$.

$$H = \langle M \rangle + \overline{K_k}.$$

Step 10: Display eigenvalues of H with their multiplicities.
Step 11: Return H.
Step 12: Compare the eigenvalues of G and H. Obviously, G and H are nonisomorphic but are cospectral having the eigenvalues $\sqrt{n}$, $-\sqrt{n}$, 0 of multiplicity 1, 1, $(p-2)$ where p is the order of the graphs.

Illustration 2.
Consider $10 = 2 \times 5$.
Start
Step 1: Input two positive integers 2 and 5
Step 2: Input positive nonprime integer 10.
Step 3: $10 = 2 \cdot 5$ and $7 = 2 + 5$.
Step 4: Construct a graph $\langle M \rangle$ on 2 vertices.
Step 5: Consider 5 isolates and call it $\overline{K_5}$.
Step 6: Form G with 7 vertices and edges defined as $e = u_i v_j$ with $u_i \in \langle M \rangle$ and $v_j \in \overline{K_5}$, for all $1 \leq i \leq 2$, $1 \leq j \leq 5$.
Step 7: Display the eigenvalues with their multiplicities.
Step 8: The required graph is G (Figure 6).
Step 9: Form a graph H (Figure 7) by taking a graph $\langle M_1 \rangle$ on m vertices and joining it with $\overline{K_k}$.

$$H = \langle M \rangle + \overline{K_k}.$$

Obviously, G and H are nonisomorphic but are cospectral having the eigenvalues $\sqrt{10}$, $-\sqrt{10}$, 0 of multiplicity 1, 1, 5.

Theorem 2.10. The set of polynomials of the form $x^p - k(p-k)x^{p-2}$ where p and k are positive integers with $p \leq k$ can be realized as the characteristic polynomial of a M-set distance matrix of a simple connected graph G on p vertices. Two nonisomorphic graphs having the same characteristic polynomial $x^p - k(p-k)x^{p-2}$ can be obtained if $p \neq 2k$.

Proof. Let $P(x) = x^p - k(p-k)x^{p-2}$ be a given polynomial. Then, this is the characteristic polynomial of the M-set distance matrix

$$\begin{bmatrix} 0_{k\times k} & 1_{k\times(p-k)} \\ 1_{(p-k)\times k} & 1_{(p-k)\times(p-k)} \end{bmatrix}$$

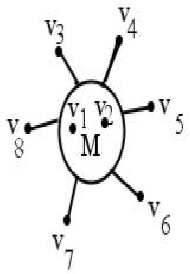

Figure 6:

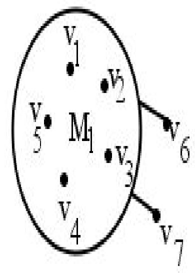

Figure 7:

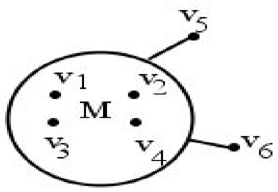

Figure 8:

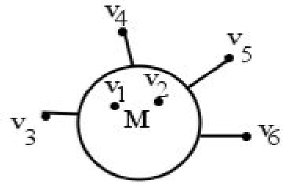

Figure 9:

The corresponding graph G can be obtained by sequentially joining a graph M on k vertices with $\overline{K_{p-k}}$ i.e. $G = \langle M \rangle + \overline{K_{p-k}}$. If $p \neq 2k$ then, another graph H can be obtained by sequentially joining a graph M on $(p-k)$ vertices with $\overline{K_k}$. i.e. $H = \langle M \rangle + \overline{K_k}$ where $|M| = p - k$ Now, G and H are nonisomorphic graphs having the same marker set distance polynomial $P(x)$. ■

Note: An algorithm can be written for the above result.
Here we give an example.

Example 2.11. Let $P(x) = x^6 - 8x^5$ be the given polynomial. Then, $P(x)$ can be written as $x^6 - 4(6-4)x^5$.
Take any graph M on 4 vertices.
Join this sequentially with $\overline{K_2}$ to get a graph G (Figure 8).

$$G = \langle M \rangle + \overline{K_2}$$

G is the graph with marker set M, whose M-set distance matrix is

$$\begin{bmatrix} 0_{4\times4} & 1_{4\times2} \\ 1_{2\times4} & 1_{2\times4}. \end{bmatrix}$$

and the corresponding characteristic polynomial is $x^6 - 8x^5$. Also, another nonisomorphic graph H (Figure 9) having the same marker set distance characteristic polynomial can be obtained by interchanging 2 and 4 in the graph G.

$$H = \langle M \rangle + \overline{K_4}.$$

where $|M| = 2$.

3. Conclusion

Some of the properties of M-set cospectrality have been explored in this paper. We would like to continue to study the distance degree sequence of a marker set in a graph.

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