Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 11 (2017), pp. 7947-7954

http://www.ripublication.com

On the Forcing Vertex Steiner Number of A Graph

J.John
Department of Mathematics, Government College of Engineering
Tirunelveli 7, India

A.Arockiamary
Department of Mathematics, Government College of Engineering
Tirunelveli 7, India

Abstract

Let$x$ be a vertex of G. For a minimum x-$Steiner\ set W$ of G, a subset $T \subseteq W$ is called a forcing subset for W, if W is the unique minimum x-$Steinerset$ containing T. A forcing subset for W of minimum cardinalityis a minimum forcing subset of W. The $forcingx$-$Steiner\ number$ of W, denoted by $f_{sx}(W)$, is the cardinality of a minimum forcing subset of W. The $forcingx$-$Steiner\ number$ of G, denoted by $f_{sx}(G)$, is $f_{sx}(G) = \min f_{sx}(W)$,where the minimum is taken over all minimum x-$SteinersetsW$ in G. Some general properties satisfied by these concepts are studied. The forcing vertex Steinernumber of some standard graphs are obtained. For every pair a, b of integerswith $0 \leq a \leq b$, there exists a connected graph G such that $f_s(G) = a$ and $f_{sx}(G) = b$for some vertex x in G.

KeywordsSteiner distance,Steiner number,forcing Steiner number, vertex Steiner number,forcing vertex Steiner number.

AMS Subject Classification : 05C12.

1. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and $q$respectively. The$distance\ d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v geodesic$. It is known that the distance is a metric on the vertex set of G. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the radius, and denoted by $radG$ and the maximum eccentricity is its diameter, and denoted by $diamG$ of G. For basic graph theoretic terminology, we refer to Harary [2]. For a nonempty set W of vertices in a connected graph G, the $Steiner\ distance\ d(W)$ of W is the minimum size of a connected subgraph of G containing W. Necessarily, each such subgraph is a tree and is called a $Steiner tree$ with respect to W or a $Steiner W\text{-}tree$. It is to be noted that $d(W) = d(u, v)$ when $W = \{u, v\}$. If v is an end vertex of a $Steiner W\text{-}tree$, then $v \in W$. Also, if $< W >$ is connected, then any $Steiner W\text{-}tree$ contains the elements of W only. The Steiner distance of a graph is introduced in [6]. The set of all vertices of G that lie on some $Steiner W\text{-}tree$ is denoted by $S(W)$. If $S(W) = V$, then W is called a $Steiner\ set$ of G. A $Steiner\ set$ of minimum cardinality is a minimum $Steiner\ set$ or simply a $s\text{-}set$ of G and this cardinality is the $Steiner\ number s(G)$ of G. If W is a $Steiner\ set$ of G and v a cut vertex of G, then v lies in every $Steiner W\text{-}tree$ of G and so $W \cup \{v\}$ is also a $Steiner\ set$ of G. The $Steiner\ number$ of a graph was introduced in [7] and further studied in [3,4,8,9,10,11,12,13]. Let G be a connected graph and W a minimum $Steiner\ set$ of G. A subset $T \subseteq W$ is called a forcing subset for W, if W is the unique minimum $Steinerset$ containing T. A forcing subset for W of minimum cardinality is a minimum forcing subset of W. The $forcing\ Steiner\ number$ of W, denoted by $f_s(W)$, is the cardinality of a minimum forcing subset of W. The $forcing\ Steiner\ number$ of G, denoted by $f_s(G)$, is $f_s(G) = \min f_s(W)$, where the minimum is taken over all minimum $Steiner\ sets\ W$ in G. A vertex v is called a simplicial vertex of a graph G if the subgraph induced by its neighbors is complete. Let x be a vertex of a connected graph G and $W \subset V(G)$ such that $x \notin W$. Then W is called an $x\text{-}Steinerset$ of G if every vertex of G lies on some $Steiner W \cup \{x\}$-tree of G. The minimum cardinality of an $x\text{-}Steiner\ set$ of G is defined as the $x\text{-}Steiner$number of G and denoted by $s_x(G)$. Any $x\text{-}Steinerset$ of cardinality $s_x(G)$ is called an s_x-set of G.

Throughout the following G denotes a connected graph.

The following theorems are used in the sequel.

Theorem 1.1.[7] Each simplicial vertex of a graph G belongs to every Steiner set of G. In particular, each end-vertex of G belongs to every Steiner set of G.

Theorem 1.2.[14] Let G be a connected graph and W be the set of all Steiner vertices of G. Then $f_s(G) \leq s(G) - |W|$.

Theorem 1.3.[14]For a complete graph $G = K_p(p \geq 2)$ or a non-trivial tree $G = T, f_s(G) = 0$.

Theorem 1.4.[11]Every simplicial vertex of G other than the vertex x (whether x is extreme or not) belongs to every x- Steiner set for any vertex$x$in$G$.

Theorem 1.5.[11]For any vertex x in $G, s(G) \leq s_x(G) + 1$.

Theorem 1.6.[12]Let Gbe a connected graph, x a vertex of G and W the set of all x-Steiner vertices of G. Then $f_{sx}(G) \leq s_x(G) - |W|$.

Theorem 1.7.[12]For a complete graph $G = K_p(p \geq 2)$ or a non-trivial tree G, $f_{sx}(G) = 0$ for any vertex x in G.

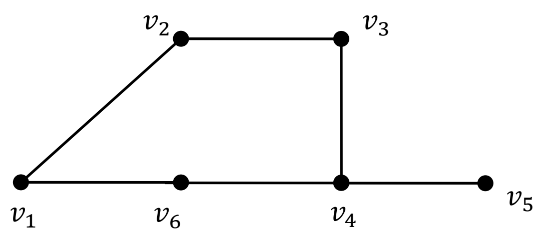
Theorem 1.8.[12] Let G be a connected graph and x a cut vertex of G. Then $f_s(G) = f_{sx}(G)$.

2 ON THE FORCING VERTEX STEINER NUMBER OF A GRAPH

Definition 2.1. Let x be a vertex of G.For a minimum x-Steiner set W of G, a subset$T \subseteq W$ is called a *forcing subset* for W, if $W$is the unique minimum x-Steiner set containing T. A forcing subset for $W$of minimum cardinality is a minimum forcing subset of W. The *forcing x-Steiner number* of W, denoted by $f_{sx}(W)$, is the cardinality of a minimum forcing subset of W. The forcing x-Steiner number of G, denoted by $f_{sx}(G)$, is$f_{sx}(G) = \min\{f_{sx}(W)\}$, where the minimum is taken over all minimum *x-Steiner* sets Win G.

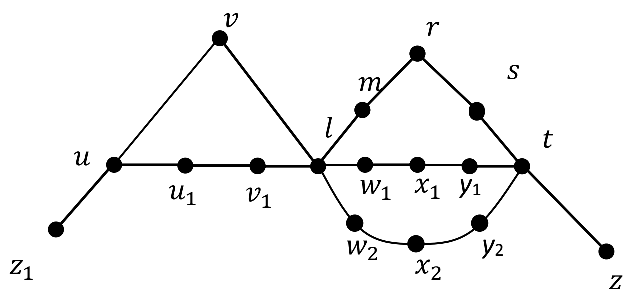
Remark2.2. From Theorem 1.5, there is a relationship between Steiner number and the vertex Steiner number. The following example shows there is no relationship between the forcing Steiner number and the forcing vertex Steiner number of a graph.

Example 2.3. For the graph G given in Figure 2.1,$W_1 = \{v_1, v_2, v_5\}, W_2 = \{v_1, v_3, v_5\}$ and $W_3 = \{v_2, v_5, v_6\}$are the s-sets of G such that $f_s(W_1) = 2, f_s(W_2) = f_s(W_3) = 1$ so that $f_s(G) = 1$. For the vertex $x = v_3, W = \{v_1, v_5\}$ is the unique s_x-set of G so that $f_{sx}(G) = 0$. Therefore $f_{sx}(G) < f_s(G)$.



***G* Figure 2.1**

Also for the graph G given in Figure 2.2, $S_1 = \{z_1, z, u_1\}, S_2 = \{z_1, z, v_1\}$ are the s- sets of Gsuch that $f_s(S_1) = 1, f_s(S_2) = 1$, so that $f_s(G) = 1$. For the vertex $x = s, W_1 = \{z_1, z, u_1, w_1, w_2\}, W_2 = \{z_1, z, u_1, y_1, y_2\}, W_3 = \{z_1, z, v_1, w_1, w_2\}$, $W_4 = \{z_1, z, v_1, y_1, y_2\}, W_5 = \{z_1, z, u_1, w_1, y_2\}$, $W_6 = \{z_1, z, u_1, w_2, y_1\} W_7 = \{z_1, z, v_1, y_1, w_2\}$,$W_8 = \{z_1, z, v_1, w_1, y_2\}$are the s_x- sets of G such that $f_{sx}(W_1) = f_{sx}(W_2) = f_{sx}(W_3) = f_{sx}(W_4) = f_{sx}(W_5) = f_{sx}(W_6) = f_{sx}(W_7) = f_{sx}(W_8) = 5$ so that $f_{sx}(G) = 5$. Therefore$f_s(G) < f_{sx}(G)$.



***G*Figure 2.2**

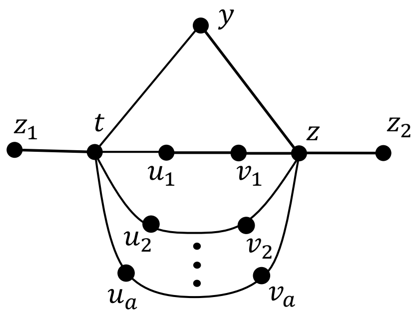
So we have the following realization result.

Theorem 2.4. For every pair a, b of integers with $0 \leq a \leq b$, there exists a connected graph Gsuch that $f_s(G) = a$ and $f_{sx}(G) = b$for some vertex x in G.

Proof. Case1. $a = b$. If $a = 0, b = 0$, let$G = K_a$. Then by Theorems 1.3 and 1.7, $f_s(G) = 0 = a$ and $f_{sx}(G) = 0 = b$. Now, assume that $a \geq 1$. Let $P: t, y, z$ be a path on three vertices and $P_i: u_i, v_i (1 \leq i \leq a)$be a copy of path on two vertices. Let H be the graph obtained from P and $P_i (1 \leq i \leq a)$ by joining the vertices $t$and u_i and the vertices $z$and v_i $(1 \leq i \leq a)$. Now, let G be the graph in Figure 2.3 obtained from H by adding *2* edges $tz_1$and zz_2. Let $Z = \{z_1, z_2\}$and $H_i = \{u_i, v_i\}(1 \leq i \leq a)$. Let $x = t$.

First, we show that $s(G) = a + 2$. Since the vertices u_i, v_ido not lie on any Steiner-Z tree of$G$, it is clear that Z is not a Steiner set of G. We observe that every s-set of G must contain exactly one vertex from each$H_i = \{u_i, v_i\}(1 \leq i \leq a)$. Thus, $s(G) \geq a + 2$. On the other hand, since the set $W = Z \cup \{v_1, v_2, \ldots, v_a\}$ is a Steiner set of G, it follows that $s(G) \leq |W| = a + 2$. Hence $s(G) = a + 2$.

Next, we show$f_s(G) = a$. By Theorem 1.1, every Steiner set of Gcontains Z and so it follows from Theorem 1.2 that $f_s(G) \leq s(G) - |Z| = a$. Now, since $s(G) = a + 2$ and every s-set of Gcontains Z, it is easily seen that every s-set S is of the form $Z \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in H_i (1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then there is a vertex $c_j (1 \leq j \leq a)$ such that $c_j \notin T$. Let $d_j$be a vertex of $H_j$distinct from c_j. Then $S_1 = (S - \{c_j\}) \cup \{d_j\}$ is a s-set properly containing$T$. Thus S is not the unique s-set containing $T$and so T is not a forcing subset of S. This is true for all s-sets of G and so$f_s(G) = a$. Since t is a cut vertex of G, by Theorem 1.8,$f_{sx}(G) = b$.



***G*Figure 2.3**

Case2.$1 < a < b$.

Let $P: u, v, w$ be a path on three vertices and let $P_i: u_i, v_i$ $(1 \leq i \leq a)$ be a copy of path on two vertices. Let Hbe the graph obtained from P and

$P_i(1 \leq i \leq a)$ by adding new vertices $z_1, z_2, \ldots, z_{c-a-1}$ and joining the vertex $u$with each z_i $(1 \leq i \leq c-a-1)$ and with each $u_i(1 \leq i \leq a)$ and also joining the vertex w with each $v_i(1 \leq i \leq a)$. Let $P_1: l, m, r, s, t$ be a path on five vertices and let $P'_i: w_i, x_i, y_i(1 \leq i \leq b-a)$ be a copy of path on three vertices. Let K be the graph obtained from P_1 and P'_i by adding new vertex z and joining the vertex $z$with t and each y_i $(1 \leq i \leq b-a)$ with t and also joining each $w_i(1 \leq i \leq b-a)$ with l . Let Gbe the graph obtained from $H$and K by identifying w of $H$and l of K. The graph G is given in Figure 2.4. Let $Z = \{z_1, z_2, \ldots, z_{c-a-1}, z\}$be the set of end-vertices of G. Let $H_i = \{u_i, v_i\}(1 \leq i \leq a)$ and$Q_j = \{w_j, y_j\}(1 \leq j \leq b-a)$. Let $x = s$.

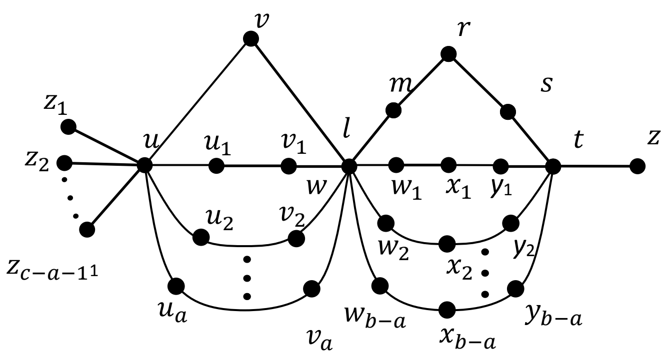
First, show that$s(G) = c$. Let S be any Steiner set of$G$. Then by Theorem 1.1, $Z \subseteq S$. It is clear that Z is not a Steiner set of$G$. We observe that every Steiner set of G must contain exactly one vertex from each $H_i(1 \leq i \leq a)$ and so $s(G) \geq c-a+a = c$. On the other hand, since the set $S_1 = Z \cup \{u_1, u_2, \ldots, u_a\}$is a Steiner set of G, it follows that $s(G) \leq |S_1| = c$. Thus $s(G) = c$.

Next, we show that $f_s(G) = a$. Since every s-set of G contains Z, it follows from Theorem 1.2 that $f_s(G) \leq s(G) - |Z| = c - (c-a) = a$. Now, since $s(G) = c$and every s-sets of G contains Z, it is easily seen that every s-set S of G is of the form$Z \cup \{c_1, c_2, \ldots, c_a\}$where $c_i \in H_i(1 \leq i \leq a)$. Let Tbe any proper subset of $S$with $|T| < a$. Then there is a vertex $x \in S$such that $x \notin T$. Suppose that $x = c_j$, let e_j be a vertex of$H_j$distinct from c_j.Then $S_2 = (S - \{c_j\}) \cup \{e_j\}$ is a s-set properly containing T. Thus S is not the unique s-set containing T so that T is not a forcing subset of S. This is true for all s-sets of G and so it follows that $f_s(G) = a$.

Now, we show that $s_x(G) = c+b-a$.Let S be any x-Steiner set of G. Then by Theorem 1.4, $Z \subseteq S$.It is clear that $Z$is not an x-Steiner set of G. We observe that every s_x- set of G must contain exactly one vertex from each $H_i(1 \leq i \leq a)$ and exactly one vertex from each$Q_j(1 \leq j \leq b-a)$. Thus $s_x(G) \geq c+b-a$. On the other hand $S_3 = Z \cup \{u_1, u_2, \ldots, u_a\} \cup \{y_1, y_2, \ldots, y_{b-a}\}$ is an x-Steiner set of $G$and so$s_x(G) \leq c+b-a$. Hence$s_x(G) = c+b-a$.

Next, we show that $f_{sx}(G) = b$. Since every s_x-set of G contains Z,it follows from Theorem 1.6 that $f_{sx}(G) \leq s_x(G) - |Z| = (c+b-a) - (c-a) = b$. Now, since $s_x(G) = c+b-a$ and every s_x-set of Gcontains$Z$, it is easily seen that every s_x-set S of G is of the form $Z \cup \{c_1, c_2, \ldots, c_a\} \cup \{d_1, d_2, \ldots, d_{b-a}\}$, where $c_i \in H_i(1 \leq i \leq a)$ and $d_j \in Q_j(1 \leq j \leq b-a)$. Let Tbe any proper subset of S with $|T| < b$. Then there is a vertex $y \in S$such that $y \notin T$. If $y = c_i$, then let e_i be a vertex of H_i distinct from c_i. Then $S' = (S -$

$\{c_i\}) \cup \{e_i\}$ is a s_x-set properly containing T. If $y = d_j$ then let f_j be a vertex of Q_j distinct from d_j. Then $S'' = (S - \{d_j\}) \cup \{f_j\}$ is a s_x-set properly containing T. Thus $S$is not the unique s_x-set containing T and so T is not a forcing subset of S. This is true for all s_x-sets of $G$and so $f_{sx}(G) = b$. ■



G**Figure 2.4**

We leave the following problem as an open problem.

Problem 2.5. For every pair a, b of integers with $0 \leq a \leq b$, does there exists a connected graph Gsuch that $f_{sx}(G) = a$ and $f_s(G) = b$ for some vertex x in G?

REFERENCES

[1] F. Buckley, F. Harary and L.V. Quintas, Extremal results on the Geodetic Number of a graph, *Scientia A*2 (1988) 17–26

[2] F. Buckley and F. Harary, Distance *in Graphs*, Addition-Wesley, Redwood City, CA, 1990.

[3] B.Bresar M ChangatSteiner intervals geodesic intervals, betweenness,*Discrete Mathematics* 309 (20) 2009 6114-6125.

[4] M. Changat, A.K.Lakshmikuttyamma, J.Mathews, A note on 3- Steiner intervals and betweenness, *Discrete Mathematics* 311(22) 2011, 2601-2609.

[5] P.A. Ostrand Graphs with specified radius and diameter, *Discrete Mathematics*, 4(1973) 71 – 75

[6] G. Chartrand, Oellermann, Ortred, Tian Song, Song Ling ,Zou and Hung Kin,Steiner distance in graphs, *Caspopis pro pestovaniMathematiky* (114) (1989) 399-410

[7] G. Chartrand and P. Zhang, The Steiner number of a graph, *Discrete Mathematics* 242 (2002) 41 – 54.

[8] Carmen Hernando, Tao Jiang, Merce Mora, Ignacio. M. Pelayo and Carlos Seara, On the Steiner, geodetic and hull number of graphs, *Discrete Mathematics* 293 (2005) 139 −154.

[9] R. Eballe, S. Canoy, Jr., Steiner sets in the join and composition of graphs, *CongressusNumerantium*, 170 (2004) 65 – 73.

[10] K.M. Kathiresan, S. Arockiaraj, R. Gurusamy and K. Amutha, On the Steiner radial number of graphs, *LNCS* 7643, 2012, 65-72.

[11] J.John and A.Arockiamary, The vertex Steiner number of a Graph (Paper Communicated)

[12] J.John and A.Arockiamary, The forcing vertex Steiner number of a Graph (Paper Communicated)

[13] A. P. Santhakumaran and J. John, The Upper Steiner Number of a Graph, *Graph Theory Notes of New York* LIX, (2010) 9 – 14.

[14] A. P. Santhakumaran and J. John, The forcing Steiner number of a graph. *DiscussionesMathematicae Graph Theory* 31 (1)(2011) 171-181.