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New fixed point theorems for t_f type contractive conditions

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Abstract

In this paper, we prove some fixed point theorems for T_F type contractive conditions in the framework of complete metric spaces. The results obtained extend and generalize well-known comparable results in the literature.

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1. Introduction and Preliminaries

In 1922, Banach [5] proved the following famous fixed point theorem:
Suppose that (X, d) is a complete metric space and a self-map T of X satisfies $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$ where $\lambda \in [0, 1)$; that is, T is a contractive mapping. Then T has a unique fixed point. Subsequently, many authors have obtained various extensions

and generalizations of Banach's theorem by considering contractive mappings on many different metric spaces.

In 2010, Moradi and Beiranvand introduced the concept of the the T_F-contraction mappings as follows:

Definition 1.1. [6] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ also is convergent. T is said to be subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ has a convergent subsequence.

Definition 1.2. [6] Let (X, d) be a metric space and $f, T : X \to X$ be two mappings. A mapping f is said to be a T_F-contraction if there exists $\alpha \in [0, 1)$ such that for all $x, y \in X$,

$$F(d(Tfx, Tfy)) \leq \alpha F(d(Tx, Ty)),$$

(1) $F : [0, \infty) \to [0, \infty)$, F is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$.

(2) T is one-to-one and graph closed (or subsequentially convergent and continuous).

Definition 1.3. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be graph closed if for every sequence $\{x_n\}$ such that $lim_{n\to\infty} Tx_n = a$ then for some $b \in X$, $Tb = a$. For example the identity function on X is graph closed.

Moradi and Beiranvand [6] proved that if f is a T_F-contraction mapping then, f has a unique fixed point in the complete metric space (X, d).

In this work, we prove some fixed point theorems for T_F-contractive conditions on complete metric spaces. Our results extend various comparable results of Moradi and Beiranvand [6] and Kir and Kiziltunc [4].

2. Main Results

Theorem 2.1. Let (X, d) be a complete metric space and $T, f : X \to X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If $a, b \in [0, 1)$ *and* $x, y \in X$,

$$F(d(Tfx, Tfy)) \leq a[F(d(Tx, Ty))] + b[F(d(Tx, Tfx)) + F(d(Ty, Tfy))] \quad (2.1)$$

where, $F : [0, \infty) \to [0, \infty)$ is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$. Then, f has a unique fixed point in X. Also, if T is sequentially convergent then the every $x_0 \in X$ the sequence of iterates $\left\{f^n x_0\right\}$ converges to the fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point and $x_n = fx_{n-1} = f^n x_0$

$$\begin{aligned} F(d(Tx_n, Tx_{n+1})) &= F(d(Tfx_{n-1}, Tfx_n)) \\ &\leq a[F(d(Tx_{n-1}, Tx_n))] \\ &\quad + b[F(d(Tx_{n-1}, Tfx_{n-1})) + F(d(Tx_n, Tfx_n))] \\ &= a[F(d(Tx_{n-1}, Tx_n))] \\ &\quad + b[F(d(Tx_{n-1}, Tx_n)) + F(d(Tx_n, Tx_{n+1}))] \\ &\leq \frac{a+b}{1-b} F(d(Tx_{n-1}, Tx_n)) \end{aligned}$$

or

$$F(d(Tx_n, Tx_{n+1})) \leq \lambda F(d(Tx_{n-1}, Tx_n)) \tag{2.2}$$

where $\lambda = \dfrac{a+b}{1-b}$. Also, by continuing the process (2.2), we obtain that,

$$F(d(Tx_n, Tx_{n+1})) \leq \lambda^n F(d(Tx_0, Tx_1)). \tag{2.3}$$

Letting $n \to \infty$ in (2.3), we obtain that

$$F(d(Tx_n, Tx_{n+1})) \to 0^+ \quad \text{as} \quad n \to \infty. \tag{2.4}$$

Again using (2.3), for all $m, n \in \mathbb{N}$, taking $m > n$ we have

$$F(d(Tx_n, Tx_m)) = F(d(Tf^n x_0, Tf^m x_0)) \leq \lambda^n F(d(Tx_0, Tf^{m-n} x_0)). \tag{2.5}$$

Letting $m, n \to \infty$, we have $F(d(Tx_n, Tx_m)) \to 0^+$. From the Definition (1.2), it is evident that F is nondecreasing continuous from the right and hence $d(Tx_n, Tx_m) \to 0^+$ as $m, n \to \infty$. Thus, we hold that $\{Tx_n\}$ is a Cauchy sequence in the metric space (X, d). Taking into account the completeness of X, we obtain that there exists $\vartheta \in X$ such that

$$\lim_{n\to\infty} Tx_n = \vartheta. \tag{2.6}$$

Note that T is subsequentially convergent, then $\{x_n\}$ has a convergent subsequence, so there is $u \in X$ such that

$$\lim_{k\to\infty} x_{n(k)} = u. \tag{2.7}$$

Also, T is continuous and $x_{n(k)} \to u$, therefore,

$$\lim_{k\to\infty} Tx_{n(k)} = Tu. \tag{2.8}$$

Note that $\left\{Tx_{n(k)}\right\}$ is a subsequence of $\{Tx_n\}$. So $Tu = \vartheta$. Now, we will show that $u \in X$ is a fixed point of f. Indeed, we have

$$\begin{aligned} F(d(Tu, Tfu)) &\leq F(d(Tu, Tx_{n(k)}) + d(Tx_{n(k)}, Tfu)) \\ &= F(d(Tu, Tx_{n(k)}) + d(Tf^{n(k)} x_0, Tfu)) \\ &\leq F(d(Tu, Tx_{n(k)}) + a[F(d(Tx_0, Tu))] \end{aligned}$$

$$+b[F(d(Tx_0, Tf^{n(k)}x_0)) + F(d(Tu, Tfu))] \quad (2.9)$$

Letting $k \to \infty$ in (2.9), we have

$$F(d(Tu, Tfu)) \leq 0. \quad (2.10)$$

The above inequality is contradiction unless $d(Tu, Tfu) = 0$. Thus, we obtained $Tu = Tfu$. Also, T being one-to-one, we obtain $fu = u$. Thus, we obtain that $u \in X$ is a fixed point of f. Uniqueness of the fixed point as follows. Assume u' is another fixed point of f that is $fu' = u'$ and

$$\begin{aligned} F(d(Tu, Tu')) &= F(d(Tfu, Tfu')) \\ &\leq a[F(d(Tu, Tu'))] + b[F(d(Tu, Tfu)) + F(d(Tu', Tfu'))] \end{aligned}$$

$$= a[F(d(Tu, Tu'))] + b[F(d(Tu, Tu)) + F(d(Tu', Tu'))] \quad (2.11)$$

The inequality (2.11) is a contradiction unless $F(d(Tu, Tu')) = 0$. Thus, we obtain $Tu = Tu'$ and taking in to account that T is one-to-one, we obtain $u = u'$. Thus, we obtain that the fixed point is unique. Also, if we assume that T is sequentially convergent, by replacing $\{n\}$ with $\{n(k)\}$ we conclude that

$$lim_{n \to \infty} x_n = u. \quad (2.12)$$

Thus, the inequality (2.12) shows that $\{x_n\}$ converges to the fixed point of f. ■

Corollary 2.2. Let (X, d) be a complete metric space and $T, f : X \to X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If $\alpha, \beta, \gamma \in [0, 1)$ and $x, y \in X$,

$$F(d(Tfx, Tfy)) \leq \alpha[F(d(Tx, Ty))] + \beta[F(d(Tx, Tfx))] + \gamma[F(d(Ty, Tfy))]$$

where, $F : [0, \infty) \to [0, \infty)$ is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$. Then f has a unique fixed point in X. Also, if T is sequentially convergent then the every $x_0 \in X$ the sequence of iterates $\left\{f^n x_0\right\}$ converges to the fixed point.

Proof. The symmetry property of d and the above inequality imply that

$$F(d(Tfx, Tfy)) \leq \alpha[F(d(Tx, Ty))] + \frac{\beta + \gamma}{2}[F(d(Tx, Tfx)) + F(d(Ty, Tfy))].$$

By substituting $\alpha = a$ and $\dfrac{\beta + \gamma}{2} = b$ in above inequality we obtain the required result as given in Theorem 2.1. ■

Corollary 2.3. Let (X, d) be a complete metric space and $T, f : X \to X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If $a, b \in [0, 1)$ and $x, y \in X$,

$$F(d(Tfx, Tfy)) \leq a[F(d(Tx, Ty))] + b[F(d(Tx, Tfy)) + F(d(Ty, Tfx))]$$

where, $F : [0, \infty) \to [0, \infty)$ is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$. Then f has a unique fixed point in X. Also, if T is sequentially convergent then the every $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.

Proof. The proof of this corollary is similar as the Theorem 2.1. ■

Theorem 2.4. Let (X, d) be a complete metric space and $T, f : X \to X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If $a, b, c \in [0, 1)$ *and* $x, y \in X$,

$$\begin{aligned} F(d(Tfx, Tfy)) \leq\ & a[F(d(Tx, Ty))] + b[F(d(Tx, Tfy)) + F(d(Ty, Tfx))] \\ & + c[F(d(Tx, Tfx)) + F(d(Ty, Tfy))] \end{aligned}$$

where, $F : [0, \infty) \to [0, \infty)$ is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$. Then f has a unique fixed point in X. Also, if T is sequentially convergent then the every $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.

Proof. The proof of this Theorem is similar as the Theorem 2.1. ■

Theorem 2.5. Let (X, d) be a complete metric space and $T, f : X \to X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. For all $x, y \in X$,

$$\begin{aligned} F(d(Tfx, Tfy)) \leq\ & a(x, y)[F(d(Tx, Ty))] \\ & + b(x, y)[F(d(Tx, Tfy)) + F(d(Ty, Tfx))] \\ & + c(x, y)[F(d(Tx, Tfx)) + F(d(Ty, Tfy))] \qquad (2.13) \end{aligned}$$

where $a(x, y), b(x, y), c(x, y) \geq 0$ and

$$\sup_{(x,y)\in X} [a(x, y) + 2b(x, y) + 2c(x, y)] \leq \lambda < 1. \qquad (2.14)$$

$F : [0, \infty) \to [0, \infty)$ is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$. Then, f has a unique fixed point in X. Also, if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point and $x_n = fx_{n-1} = f^n x_0$

$$\begin{aligned} F(d(Tx_n, Tx_{n+1})) &= F(d(Tfx_{n-1}, Tfx_n)) \\ &\leq a(x_{n-1}, x_n)[F(d(Tx_{n-1}, Tx_n))] + b(x_{n-1}, x_n)[F(d(Tx_{n-1}, Tfx_n)) \\ &\quad + F(d(Tx_n, Tfx_{n-1}))] + c(x_{n-1}, x_n)[F(d(Tx_{n-1}, Tfx_{n-1})) \\ &\quad + F(d(Tx_n, Tfx_n))] \\ &= a(x_{n-1}, x_n)[F(d(Tx_{n-1}, Tx_n))] + b(x_{n-1}, x_n)[F(d(Tx_{n-1}, Tx_{n+1})) \\ &\quad + F(d(Tx_n, Tx_n))] + c(x_{n-1}, x_n)[F(d(Tx_{n-1}, Tx_n)) + F(d(Tx_n, Tx_{n+1}))] \\ &\leq (a + b + c)(x_{n-1}, x_n)[F(d(Tx_{n-1}, Tx_n))] \\ &\quad + (b + c)(x_{n-1}, x_n)[F(d(Tx_n, Tx_{n+1}))] \\ &\leq \frac{a(x_{n-1}, x_n) + b(x_{n-1}, x_n) + c(x_{n-1}, x_n)}{1 - b(x_{n-1}, x_n) - c(x_{n-1}, x_n)} F(d(Tx_{n-1}, Tx_n)). \end{aligned}$$

Using (2.14), we have $\dfrac{a(x,y)+b(x,y)+c(x,y)}{1-b(x,y)-c(x,y)} \leq \lambda$ for all $x, y \in X$. Thus, from above we have,

$$F(d(Tx_n, Tx_{n+1})) \leq \lambda F(d(Tx_{n-1}, Tx_n)). \tag{2.15}$$

where $\dfrac{a(x,y)+b(x,y)+c(x,y)}{1-b(x,y)-c(x,y)} \leq \lambda < 1$ for all $x, y \in X$. Therefore, for all n,

$$F(d(Tx_n, Tx_{n+1})) \leq \lambda^n F(d(Tx_0, Tx_1)). \tag{2.16}$$

Letting $n \to \infty$ in (2.16), we obtain that

$$F(d(Tx_n, Tx_{n+1})) \to 0^+ \, as \, n \to \infty. \tag{2.17}$$

Again using (2.16), for all $m, n \in \mathbb{N}$, taking $m > n$, we have

$$F(d(Tx_n, Tx_m)) = F(d(Tf^n x_0, Tf^m x_0)) \leq \lambda^n F(d(Tx_0, Tf^{m-n} x_0)). \tag{2.18}$$

Letting $m, n \to \infty$ we have

$$F(d(Tx_n, Tx_m)) \to 0^+ \, as \, m, n \to \infty. \tag{2.19}$$

From the Definition (1.2), it is evident that F is nondecreasing continuous from the right and hence $d(Tx_n, Tx_m) \to 0^+ as\, m, n \to \infty$. Thus, we hold that $\{Tx_n\}$ is a Cauchy sequence in the metric space (X, d). Taking into account the completeness of X, we obtain that there exists $\vartheta \in X$ such that

$$lim_{n\to\infty} Tx_n = \vartheta. \tag{2.20}$$

Note that T is subsequentially convergent, then $\{x_n\}$ has a convergent subsequence, so there is $u \in X$ such that

$$lim_{k\to\infty} x_{n(k)} = u. \tag{2.21}$$

Also, T is continuous and $x_{n(k)} \to u$, therefore,

$$lim_{k\to\infty} Tx_{n(k)} = Tu. \tag{2.22}$$

Note that $\left\{Tx_{n(k)}\right\}$ is a subsequence of $\{Tx_n\}$ so $Tu = \vartheta$. Now we will show that $u \in X$ is a fixed point of f. Indeed we have

$$\begin{aligned}
F(d(Tu, Tfu)) &\leq F(d(Tu, Tx_{n(k)}) + d(Tx_{n(k)}, Tfu)) \\
&= F(d(Tu, Tx_{n(k)}) + d(Tf^{n(k)}x_0, Tfu)) \\
&\leq F(d(Tu, Tx_{n(k)})) + a(x_0, u)[F(d(Tx_0, Tu))] + b(x_0, u)[F(d(Tx_0, Tfu)) \\
&\quad + F(d(Tu, Tf^{n(k)}x_0))] + c(x_0, u)[F(d(Tx_0, Tf^{n(k)}x_0)) + F(d(Tu, Tfu))] \\
&= F(d(Tu, Tx_{n(k)})) + a(x_0, u)[F(d(Tx_0, Tx_{n(k)}))] + b(x_0, u)[F(d(Tx_0, Tfu)) \\
&\quad + F(d(Tu, Tx_{n(k)}))] + c(x_0, u)[F(d(Tx_0, Tx_{n(k)})) + F(d(Tu, Tfu))] \\
&\leq F(d(Tu, Tx_{n(k)})) + a(x_0, u)[F(d(Tx_0, Tx_{n(k)}))] + b(x_0, u)[F(d(Tx_0, Tx_{n(k)}))
\end{aligned}$$

$$+F(d(Tx_{n(k)}, Tfu))] + c(x_0, u)[F(d(Tx_0, Tx_{n(k)})) + F(d(Tx_{n(k)}, Tfu))]. \tag{2.23}$$

Letting $k \to \infty$ in (2.23), we have

$$F(d(Tu, Tfu)) \leq 0. \tag{2.24}$$

The above inequality (2.24) is contradiction unless $d(Tu, Tfu) = 0$. Thus we obtained $Tu = Tfu$. Also T being one-to-one, we obtain $fu = u$. Thus, we obtain that $u \in X$ is a fixed point of f. Now we show that the fixed point is unique. Assume u' is another fixed point of f that is $fu' = u'$ and

$$\begin{aligned} F(d(Tu, Tu')) &= F(d(Tfu, Tfu')) \leq a(u, u')[F(d(Tu, Tu'))] \\ &\quad +b(u, u')[F(d(Tu, Tfu')) + F(d(Tu', Tfu))] \\ &\quad +c(u, u')[F(d(Tu, Tfu)) + F(d(Tu', Tfu'))] \\ &\leq (a + 2b)(u, u')[F(d(Tu, Tu'))]. \end{aligned} \tag{2.25}$$

The inequality (2.25) is a contradiction unless $F(d(Tu, Tu')) = 0$. Thus, we obtain $Tu = Tu'$ and taking into account that T is one-to-one, we obtain $u = u'$. Thus, we obtain that the fixed point is unique. Also, if we assume that T is sequentially convergent, by replacing $\{n\}$with $\{n(k)\}$ we conclude that

$$lim_{n\to\infty} x_n = u. \tag{2.26}$$

Thus, the inequality (2.26) shows that $\{x_n\}$ converges to the fixed point of f. Thus, the proof, is completed. ■

3. Conclusion

In this paper, we prove some fixed point theorems by using the T_F type contractive conditions in a metric space. Our theorems extend some recent results of Kir and Kiziltunc [4] and Moradi and Beiranvand [6].

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Competing Interests

The authors declare that they have no competing interests.

References

[1] A. Branciari, A fixed point theorem for mapping satisfying a general contractive condition of integral type, Int. J.M. and M. Sciences, 29(9)(2002), 531–536.

[2] B.E. Rhoades, some theorems on weakly contractive maps, Nonlinear Anal., 47 (2001), 2683–2693.

[3] B.S. Choudhury, Unique fixed point theorem for weakly c-contractive mappings, Kathmandu Univ. J. Sic, Eng. and Tech., 5(2009), 6–13.

[4] M.Kir, H. Kiziltunc, T_F Type contractive conditions for Kannan and Chatterjea fixed point theorems, Adv. Fixed Point Theory, 4(2014), No. 1, 140–148.

[5] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integerales, Fundamenta Mathematicae, 3(1922), 133–181.

[6] S. Moradi, A. Beiranvand, Fixed point of T_F-contractive single-valued mappings, Iranian J. Math. Sci. and Inform., Vol.5, No. 2 (2010), 25–32.

[7] S. Moradi, A. Davood, New extension of Kannan fixed point theorem on complete metric and generalized metric spaces, Int. J. Math. Anal., 5(2011), 2313–2320.

[8] S. Kutukcu and S. Sharma, A fixed point theorem in fuzzy metric spaces, Int. J. of Math. Analysis, 1(18)(2007), 861–872.

[9] S.K. Chatterjea, Fixed point theorems, C.R. Acad. Bulgare Sci., 25(1972), 727–730.