Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 11 (2017), pp. 7969–7989

http://www.ripublication.com/gjpam.htm

Controllability Results for Nonlinear Higher Order Fractional Delay Dynamical Systems with Distributed Delays in Control

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Abstract

This paper deals with the controllability result on nonlinear higher order fractional delay dynamical systems with distributed delays in control. A formula for the solution expression of the system is derived by using Laplace transform. A necessary and sufficient condition for the controllability of linear fractional delay dynamical system with distributed delays in control is established, and a sufficient condition for the corresponding nonlinear system has obtained. Examples has given to verify the results.

AMS subject classification:
Keywords: Fractional dynamical systems, Controllability, Distributed delays, Laplace transform, Mittag-Leffler Matrix function.

1. Introduction

Since during the 17^{th} century, the concept of fractional derivatives and fractional integrals (fractional calculus) is one of the old mathematical areas. It has a long history which has from 1965 Leibniz who introduced the notion of "$\frac{1}{2}$ order derivative" in a letter

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to L' Hospital. Now a days, fractional calculus take an increasing part in science and engineering also. Their applications are in different areas, applied sciences including electrochemistry, biophysics, mechanics, control theory, signal processing etc., [14, 18, 22, 23, 25].

All the future developments around the world depends upon not only on the present state, but also determined by the entire prehistory. Such systems arise in automatic control, economics, medicine, biology and other areas. Mathematical justification of these developments can be complete with the help of differential equations with delays. These kinds of analytical solutions of linear delay differential equations has been made by many researchers [9, 12, 13, 24, 29]. Lot of Mathematical models has described dynamical systems with delays in control, or both the state and control. So, it is important to study the properties of systems with delays.

The controllability of dynamical systems is one of the most fundamental concepts in mathematical control theory. It means that it is possibility to steer a control dynamical system from an initial state into a final state by using an aid admissible controls. Controllability of nonlinear integer order systems in both finite and infinite dimensional spaces has well established [1, 21, 26]. In modern era, various controllability problems for different types of fractional order dynamical systems have been presented several authors [2, 3, 4, 5]. A related study on the controllability of both linear and nonlinear fractional dynamical systems and fractional delay dynamical systems are different types of delays in control variables were analyzed [8, 15, 16, 17]. The importance of practical applications which distributed delay system viz., logistics, traffic flow, microorganism, hematopoiesis and spaceflight industry systems [10, 28, 30]. Controllability of fractional dynamical systems with distributed delays has discussed by many authors [6, 7, 8, 19, 20].

However, there are no works concerning the controllability of higher order fractional delay dynamical systems with distributed delays in control. To address this lacuna, the present study has carried out for controllability of nonlinear higher order fractional delay dynamical systems with distributed delays in control variable. Then the necessary and sufficient conditions for the controllability of linear fractional delay dynamical system with distributed delays in control established by using the controllability Grammian matrix which has defined by Mittag-Leffler matrix function. Further, the corresponding nonlinear higher order fractional delay dynamical systems with distributed delays in control has examined by applying the Schaefer's fixed point theorem. The respective numerical examples are reported to illustrate the results.

2. Preliminaries

In this section we shall provide some basic definitions.

Definition 2.1. [27] Let f be a real-or complex-valued function of the variable $t > 0$ and let s be a real or complex parameter. The Laplace transform of f is defined as

$$F(s) = \int_0^\infty e^{-st} f(t)dt. \quad \text{for} \quad Re(s) > 0$$

Definition 2.2. [18] The Caputo fractional derivative of order $\alpha > 0,\ n-1 < \alpha \leq n$, is defined as

$$^{C}D^{\alpha}_{0_+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds.$$

where $f^n(s) = \dfrac{d^n f}{ds^n}$. In particular, if $1 < \alpha < 2$ then

$$^{C}D^{\alpha}_{0_+} f(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} f''(s)ds.$$

where the function $f(t)$ has absolutely continuous derivative up to order $n-1$. For the brevity, the Caputo fractional derivative $^{C}D^{\alpha}_{0_+}$ is taken as $^{C}D^{\alpha}$.

The Laplace transform of Caputo derivative is

$$L[^{c}D^{\alpha}x(t)](s) = s^{\alpha}L[x(t)](s) - \sum_{k=0}^{n-1} x^{k}(0)s^{\alpha-1-k}, \quad n-1 < \alpha \leq n.$$

The Mittag-Leffler functions of various type are defined as [21]

$$E_{\alpha}(z) = E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad z \in C, \quad Re(\alpha) > 0,$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z, \beta \in C, \quad Re(\alpha) > 0,$$

$$E^{\gamma}_{\alpha,\beta}(-\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\gamma)_k(-\lambda)^k}{k!\Gamma(\alpha k+\beta)} t^{\alpha k}$$

where $(\gamma)_n$ is a Pochhamer symbol which is defined as $\gamma(\gamma+1)\cdots(\gamma+n-1)$ and $(\gamma)_n = \dfrac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$. The Laplace transforms of Mittag–Leffler functions are defined as

$$L[E_{\alpha,1}(\pm\lambda t^{\alpha})](s) = \frac{s^{\alpha-1}}{(s^{\alpha} \pm \lambda)}, \quad Re(\alpha) > 0,$$

$$L[t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^{\alpha})](s) = \frac{s^{\alpha-\beta}}{(s^{\alpha} \pm \lambda)}, \quad Re(\alpha) > 0, \quad Re(\beta) > 0,$$

$$L[t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\pm\lambda t^{\alpha})](s) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha} \pm \lambda)^{\gamma}}, \quad Re(s) > 0, \quad Re(\beta) > 0, \quad |\lambda s^{-\alpha}| < 1.$$

3. Solution Representation

Consider a fractional delay differential equation of the form

$$^{C}D^{\alpha}x(t) = Ax(t) + Bx(t-h) + f(t), \quad 1 < \alpha \leq 2 \tag{3.1}$$

$$x(t) = \phi(t), \quad x'(t) = \phi'(t), \quad -h < t \leq 0,$$

where A and B are $n \times n$ matrices, $x \in R^n$, $\phi(t), \phi'(t)$ is a continuous function on $[-h, 0]$ and f is a real-valued continuous function in R^n. By taking Laplace transform on both sides of (3.1) we get where $F(s) = \int_0^\infty e^{-st} f(t)dt$,

then

$$\begin{aligned} X(s) = &\left[\frac{s^{\alpha-1}}{s^\alpha I - A - Be^{-hs}}\right]\phi(0) + \left[\frac{s^{\alpha-2}}{s^\alpha I - A - Be^{-hs}}\right]\phi'(0) \\ &+ B\left[\frac{e^{-hs}}{s^\alpha I - A - Be^{-hs}}\right] * \int_{-h}^{0} e^{-s\tau}\phi(\tau)d\tau + \left[\frac{F(s)}{s^\alpha I - A - Be^{-hs}}\right]. \end{aligned}$$

Using inverse Laplace transform and convolution of Laplace transform we get

$$\begin{aligned} x(t) = &L^{-1}\left[s^{\alpha-1}(s^\alpha I - A - Be^{-hs})^{-1}\right](t)\phi(0) + L^{-1}\left[s^{\alpha-2}(s^\alpha I - A - Be^{-hs})^{-1}\right](t)\phi'(0) \\ &+ BL^{-1}\left[\frac{s^{\alpha-1}}{s^\alpha I - A - Be^{-hs}} s^{1-\alpha}\right](t) * L^{-1}\left[e^{-hs}\int_{-h}^{0} e^{-st}\phi(\tau)d\tau\right](t) \\ &+ L^{-1}[F(s)](t) * L^{-1}\left[\frac{s^{\alpha-1}}{s^\alpha I - A - Be^{-hs}} s^{1-\alpha}\right](t). \end{aligned}$$

Let

$$X_\alpha(t) = L^{-1}[s^{\alpha-1}(s^\alpha I - A - Be^{-hs})^{-1}](t),$$

$$X_{\alpha,2}(t) = t^{-1}L^{-1}\left[s^{\alpha-2}(s^\alpha I - A - Be^{-hs})^{-1}\right](t)$$

and

$$\begin{aligned} L^{-1}\left[\frac{s^{\alpha-1}}{s^\alpha I - A - Be^{-hs}} s^{1-\alpha}\right](t) = X_\alpha(t) * \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} &= \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} X_\alpha(s)ds. \\ &= t^{\alpha-1}X_{\alpha,\alpha}(t) \end{aligned}$$

where

$$X_{\alpha,\alpha}(t) = t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} X_\alpha(s)ds.$$

Define $\omega(t) : [-h, \infty) \to [0, 1]$ by $\omega(t) = 0$ for $t \geq 0$ and $\omega(t) = 1$ for $t < 0$. The function $\phi(t)$ is extended to $(-h, \infty)$ by defining $\phi(t) = \phi(0)$ for $t \geq 0$, then

$$\begin{aligned} e^{-hs}\int_{-h}^{0} e^{-st}\phi(t)dt &= e^{-hs}\int_0^h e^{-s(-h+\lambda)}\phi(-h+\lambda)d\lambda \\ &= \int_0^\infty e^{-s\lambda}\phi(-h+\lambda)\omega(-h+\lambda)d\lambda \\ &= L[\phi(-h+.)\omega(-h+.)](t). \end{aligned}$$

Thus,

$$
\begin{aligned}
x(t) =& X_\alpha(t)\phi(0) + tX_{\alpha,2}(t)\phi'(0) + B\int_0^t (t-s)^{\alpha-1}X_{\alpha,\alpha}(t-s)\phi(s-h)\omega(s-h)ds \\
&+ \int_0^t (t-s)^{\alpha-1}X_{\alpha,\alpha}(t-s)f(s)ds
\end{aligned}
$$

and so

$$
\begin{aligned}
x(t) =& X_\alpha(t)\phi(0) + tX_{\alpha,2}(t)\phi'(0) + B\int_0^h (t-s)^{\alpha-1}X_{\alpha,\alpha}(t-s)\phi(s-h)ds \\
&+ \int_o^t (t-s)^{\alpha-1}X_{\alpha,\alpha}(t-s)f(s)ds
\end{aligned}
$$

Hence

$$
\begin{aligned}
x(t) =& X_\alpha(t)\phi(0) + tX_{\alpha,2}(t)\phi'(0) + B\int_{-h}^0 (t-s-h)^{\alpha-1}X_{\alpha,\alpha}(t-s-h)\phi(s)ds \\
&+ \int_o^t (t-s)^{\alpha-1}X_{\alpha,\alpha}(t-s)f(s)ds,
\end{aligned}
\tag{3.2}
$$

4. Linear syetem

Consider the linear fractional delay dynamical system with distributed delays in control of the form

$$
\begin{aligned}
{}^C D^\alpha x(t) =& Ax(t) + Bx(t-h) + \int_{-h}^0 d_\tau C(t,\tau)u(t+\tau), \quad t \in J = [0,T], \\
x(t) =& \phi(t), \quad x'(t) = \phi'(t), \quad -h < t \le 0, \\
u(t) =& \psi(t), \quad -\tau < t \le 0,
\end{aligned}
\tag{4.1}
$$

where $x \in R^n, u \in R^m$, A and B are $n \times n$ matrices, $C(t,\tau)$ is an $n \times m$ matrix, continuous in t for fixed τ and is of bounded variation in τ on $[-h, 0]$ for each $t \in J$. The integral term is in the Lebesgue Stieltjes sense, for function $u : [-h, T] \to R^m$ and $t \in J$.

The general solution of system (4.1) can be written as

$$
\begin{aligned}
x(t) =& X_\alpha(t)\phi(0) + tX_{\alpha,2}(t)\phi'(0) + B\int_{-h}^0 (t-s-h)^{\alpha-1}X_{\alpha,\alpha}(t-s-h)\phi(s)ds \\
&+ \int_o^t (t-s)^{\alpha-1}X_{\alpha,\alpha}(t-s)\Big[\int_{-h}^0 d_\tau C(s,\tau)u(s+\tau)\Big]ds
\end{aligned}
\tag{4.2}
$$

Now using the well known result of unsymmetric Fubini [11] theorem and change of order of integration to the last term, we have

$$x(t) = X_\alpha(t)\phi(0) + tX_{\alpha,2}(t)\phi'(0) + B\int_{-h}^{0}(t-s-h)^{\alpha-1}X_{\alpha,\alpha}(t-s-h)\phi(s)ds + \int_{-h}^{0}dC_\tau\Big[\int_{-h}^{0}(t-s)^{\alpha-1}X_{\alpha,\alpha}(t-s)C(s,\tau)u(s+\tau)ds\Big]$$

$$x(t) = X_\alpha(t)\phi(0) + tX_{\alpha,2}(t)\phi'(0) + B\int_{-h}^{0}(t-s-h)^{\alpha-1}X_{\alpha,\alpha}(t-s-h)\phi(s)ds + \int_{-h}^{0}dC_\tau\Big[\int_{\tau}^{0}(t-(s-\tau))^{\alpha-1}X_{\alpha,\alpha}(t-(s-\tau)C(s-\tau,\tau)\psi(s)ds\Big] + \int_{-h}^{0}dC_\tau\Big[\int_{0}^{t+\tau}(t-(s-\tau))^{\alpha-1}X_{\alpha,\alpha}(t-(s-\tau))C(s-\tau,\tau)u(s)ds\Big]$$

$$x(t) = x(t;\phi) + \int_{-h}^{0}dC_\tau\Big[\int_{\tau}^{0}(t-(s-\tau))^{\alpha-1}X_{\alpha,\alpha}(t-(s-\tau)C(s-\tau,\tau)\psi(s)ds\Big] + \int_{0}^{t}\Big[\int_{-h}^{0}(t-(s-\tau))^{\alpha-1}X_{\alpha,\alpha}(t-(s-\tau))d_\tau C_t(s-\tau,\tau)\Big]u(s)ds \tag{4.3}$$

where

$$x(t;\phi) = X_\alpha(t)\phi(0) + tX_{\alpha,2}(t)\phi'(0) + B\int_{-h}^{0}(t-s-h)^{\alpha-1}X_{\alpha,\alpha}(t-s-h)\phi(s)ds$$

$$C_t(s,\tau) = \begin{cases} C(s,\tau), & s \le t \\ 0, & s > t \end{cases}$$

and dC_τ denotes the integration of Lebesgue Stielties sense with respect to the variable τ in the function $C(t,\tau)$.

For our convenience, let us take

$$G(t,s) = \int_{-h}^{0}(t-(s-\tau))^{\alpha-1}X_{\alpha,\alpha}(t-(s-\tau))d_\tau C_t(s-\tau,\tau) \tag{4.4}$$

Define the controllability Grammian Matrix

$$W(0,T) = \int_0^T G(T,s)G^*(T,s)ds \tag{4.5}$$

where the $*$ indicates the matrix transpose.

Theorem 4.1. The linear control system (4.1) is controllable on $[0, T]$ if and only if the controllability Grammian matrix $W = \int_0^T G(T, s)G^*(T, s)ds$ is positive definite, for some $T > 0$.

Proof. We first show that the necessity. Suppose that W is not positive definite. Then W is singular and so its inverse is does not exist, and there exists a nonzero y such that

$$0 = y^* W y = y^* \int_0^T G(T, s)G^*(T, s)y ds$$

and hence, for $s \in [0, T]$

$$y^* G(T, s) = y^* \int_{-h}^0 (T - (s - \tau))^{\alpha-1} X_{\alpha,\alpha}(T - (s - \tau))d_\tau C_T(s - \tau, \tau) = 0 \quad (4.6)$$

Now choose $y = x(T; 0)$ and take $\psi(s) = 0$. Since system (4.1) is controllable, there exists a control $u \in C(J)$ such taht if steers the complete state $y(0) = \{\phi(0), \phi'(0), \psi(s)\}$ to the origin in the interval J, it follow that

$$\begin{aligned} x(T) =& x(T; \phi) + \int_{-h}^0 dC_\tau \left[\int_\tau^0 (T - (s - \tau))^{\alpha-1} X_{\alpha,\alpha}(T - (s - \tau))C(s - \tau, \tau)\psi(s)ds\right] \\ &+ \int_0^T \left[\int_{-h}^0 (T - (s - \tau))^{\alpha-1} X_{\alpha,\alpha}(T - (s - \tau))d_\tau C_T(s - \tau, \tau)\right] u(s)ds \\ =& y + \int_0^T \left[\int_{-h}^0 (T - (s - \tau))^{\alpha-1} X_{\alpha,\alpha}(T - (s - \tau))d_\tau C_T(s - \tau, \tau)\right] u(s)ds \\ =& 0 \end{aligned}$$

Thus,

$$0 = y^* y + \int_0^T y^* G(T, s)u(s)ds \quad (4.7)$$

It follows from (4.6) and (4.7) that $y^* y = 0$. This is a contradiction to $y \neq 0$. Thus W is non singular.

Next, we show that the sufficiency. Suppose that W is positive definite, that is, it is nonsingular and so its inverse is well-defined. Define the control function as

$$\begin{aligned} u(t) = G^*(T; s)W^{-1} &\left[x_1 - x(T; \phi) - \int_{-h}^0 dC_\tau \right. \\ &\times \left.\left[\int_\tau^0 (T - (s - \tau))^{\alpha-1} X_{\alpha,\alpha}(T - (s - \tau))C(s - \tau, \tau)\psi(s)ds\right]\right] \quad (4.8) \end{aligned}$$

where the complete state $y(0) = \{\phi(0), \phi'(0), \psi(s)\}$ and the vector $x_1 \in R^n$ are choosen arbitrarily. Substituting $u(t)$ in (4.3) and using (4.4) we have

$$
\begin{aligned}
x(T) &= x(T;\phi) \\
&\quad + \int_{-h}^{0} dC_\tau \left[\int_{\tau}^{0} (T-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(T-(s-\tau)) C(s-\tau,\tau)\psi(s)ds \right] \\
&\quad + \int_{0}^{T} \left[\int_{-h}^{0} (T-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(T-(s-\tau)) d_\tau C_T(s-\tau,\tau) \right] \\
&\times \left[\int_{-h}^{0} (T-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(T-(s-\tau)) d_\tau C_T(s-\tau,\tau) \right]^* \\
&\times W^{-1} \Bigg[x_1 - x(T;\phi) - \int_{-h}^{0} dC_\tau \\
&\quad \times \left[\int_{\tau}^{0} (T-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(T-(s-\tau)) C(s-\tau,\tau)\psi(s)ds \right] \Bigg] ds \\
&= x_1
\end{aligned}
$$

This means that control $u(t)$ transfer the initial state $y(0)$ to the desired vector $x_1 \in R^n$ at time T. Hence the system (4.1) is controllable. ■

5. Nonlinear systems

Consider the nonlinear fractional delay dynamical system with distributed delays in control represented by the fractional differential equation of the form

$$
\begin{aligned}
{}^C D^\alpha x(t) &= Ax(t) + Bx(t-h) \\
&\qquad + \int_{-h}^{0} d_\tau C(t,\tau) u(t+\tau) + f(t, x(t), {}^C D^\beta x(t), u(t)), \quad t \in J, \\
x(t) &= \phi(t), \quad x'(t) = \phi'(t), \quad -h < t \le 0, \ 1 < \alpha \le 2, \qquad (5.1) \\
u(t) &= \psi(t), \quad -\tau < t \le 0, \ 0 < \beta \le 1,
\end{aligned}
$$

where $x \in R^n, u \in R^m$, A and B are $n \times n$ matrices, $C(t,\tau)$ is an $n \times m$ matrix, continuous in t for fixed τ and is of bounded variation in τ on $[-h, 0]$ for each $t \in J$. The integral term is in the Lebesgue Stieltjes sense, for function $u : [-h, T] \to R^m$ and $t \in J$. The nonlinear function $f : [0, T] \times R^n \times R^n \times R^m \to R^n$ is continuous.

Then the solution of the system (5.1) can be expressed in the following form

$$
\begin{aligned}
x(t) =& x(t;\phi) + \int_{o}^{t} (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) f(s, x(s), {}^C D^\beta x(s), u(s)) ds \\
&+ \int_{o}^{t} (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) \left[\int_{-h}^{0} d_\tau C(s,\tau) u(s+\tau) \right] ds
\end{aligned}
$$

Using the well known result of unsymmetric Fuubini theorem [11] and change of order of integration to the last term, we have

$$\begin{aligned} x(t) =& x(t;\phi) + \int_o^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) f(s, x(s), {}^C D^{\beta} x(s), u(s)) ds \\ &+ \int_{-h}^0 dC_\tau \Big[\int_\tau^0 (t-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(t-(s-\tau)) C(s-\tau,\tau)\psi(s) ds \Big] \\ &+ \int_0^t \Big[\int_{-h}^0 (t-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(t-(s-\tau)) d_\tau C_t(s-\tau,\tau) \Big] u(s) ds \end{aligned}$$

Where

$$C_t(s,\tau) = \begin{cases} C(s,\tau), & s \le t \\ 0, & s > t \end{cases}$$

and dC_τ denotes the integration of Lebesgue Stieltjes sense with respect to the variable τ in the function $C(t,\tau)$. For brevity, let us introduce the notation

$$\begin{aligned} &\sigma(y(0), x_1; z, v) \\ &= x_1 - x(T;\phi) - \int_0^T (T-s)^{\alpha-1} X_{\alpha,\alpha}(T-s) f(s, x(s), {}^C D^{\beta} x(s), v(s)) ds \\ &- \int_{-h}^0 dC_\tau \Big[\int_\tau^0 (T-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(T-(s-\tau)) C(s-\tau,\tau)\psi(s) ds \Big] \end{aligned}$$

Define the control function

$$u(t) = G^*(T,s) W^{-1} \sigma(y(0), x_1; z, v)$$

Where the complete state $y(0) = \phi(0), \phi'(0), \psi(s)$ and the vector $x_1 \in R^n$ are choosen arbitrarily and $*$ denotes the matrix transpose.

We assume the following hypotheses.

(**H1**) For each $t \in J$, the function $f(t,\cdot,\cdot) : R^n \times R^n \to R^n$ is continuous and the function $f(\cdot,x,y) : J \to R^n$ is strongly measurable for each $x, y \in R^n$.

(**H2**) For every positive constant k there exists $h_k \in L^1(J)$ such that

$$\sup_{\|x\|,\|y\|,\|u\| \le k} \|f(t,x,y,u)\| \le h_k(t), \text{ for almost all } t \in J.$$

(**H3**) There exits a continuous function $m_1 : J \to [0,\infty)$ such that

$$\|f(t,x,y,u)\| \le m_1(t)\Omega(\|x\| + \|y\| + \|u\|), \ t \in J, \ x, \ y \in R^n, u \in R^m$$

where $\Omega : (0,\infty) \to (0,\infty)$ is a continuous nondecreasing function and

$$\int_0^T m(s) ds < \int_r^\infty \frac{ds}{\Omega(s)}$$

(**H4**) There exists a constant $M > 0$ and a continuous function $m_2 : J \to (0, \infty)$ such that

$$\frac{k_2 t^{1-\beta}}{\Gamma(2-\beta)} + \frac{n_6}{\Gamma(2-\beta)} \int_0^t (t-\xi)^{1-\beta} m_1(\xi) \Omega\left(\omega(\xi)\right) d\xi \leq M m_2(t) \Omega(\omega(t))$$

where,

$$\begin{aligned}
n_1 &= \sup\left\{\|x(t;\phi)\|,\ t \in J\right\},\ \ n_2 = \sup\left\{\left\|X_{\alpha,\alpha}(t-s)\right\|,\ t,s \in J\right\} \\
n_3 &= \sup\left\{\left\|X_{\alpha,\alpha}(t-\tau-s)\right\|,\ t,\tau,s \in J\right\},\ \ n_4 = \sup\left\{\left\|x'(t;\phi)\right\|,\ t \in J\right\} \\
n_5 &= \sup\left\{\left\|\int_{-h}^0 dC_\tau\left[\int_\tau^0 (t-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(t-(s-\tau)) C(s-\tau,\tau)\psi(s)ds\right]\right\|,\ t \in J\right\} \\
n_6 &= \sup\left\{\left\|X_{\alpha_1,\alpha_1}(t-s)\right\|,\ t,s \in J\right\} \\
n_7 &= \sup\left\{\left\|\int_{-h}^0 dC_\tau\left[\int_\tau^0 (t-(s-\tau))^{\alpha-1} X_{\alpha_1,\alpha_1}(t-(s-\tau)) C(s-\tau,\tau)\psi(s)ds\right]\right\|,\ t \in J\right\} \\
n_8 &= \sup\|G(T,t)\|,\ \ n_9 = \sup\left\|G^*(T,t)\right\| \\
n_{10} &= n_4 + n_7 + t n_8 n_9 \left\|W^{-1}\right\| \left[\|x_1\| + n_1 + n_5 + n_2 \int_0^T (T-\xi)^{\alpha-1} m_1(\xi)\right. \\
&\quad \left. \times \left(\|x(\xi)\| + \left\|{}^C D^\beta x(\xi)| + \|u(\xi)\|\right\| + \|u(\xi)\|\right) d\xi\right] \\
r &= n_1 + n_5 + n_3 \int_0^t \|G(T,t)\| \left\|G^*(T,t)\right\| \left\|W^{-1}\right\| \left[\|x_1\| + n_1 + n_5\right. \\
&\quad \left. + n_2 \int_0^T (T-\xi)^{\alpha-1} m_1(\xi)\left(\|x(\xi)\| + \left\|{}^C D^\beta x(\xi)\right\| + \|u(\xi)\|\right) d\xi\right] \\
m(t) &= \max\left\{n_3 m_1(t), M m_2(t)\right\}.
\end{aligned}$$

Theorem 5.1. Assume that hypotheses $(H1) - (H4)$ hold and suppose that the linear system (4.1) is controllable. Then the nonlinear system (5.1) is controllable on J.

Proof. Consider the space $X = \{x : x' \in (J, R^n)$ and ${}^C D^\beta x \in C(J, R^n)\}$ with norm $\|x\|^* = \max\left\{\|x\|, \left\|{}^C D^\beta\right\|\right\}$. Using the hypothesis, for an arbitrary function $x(\cdot)$ define the control for

$$u(t) = G^*(T,s) W^{-1} \sigma(y(0), x_1; z, v)$$

Now we shall show that the nonlinear operator $F : X \to X$ has a fixed point. When using the control function, the nonlinear operator F is defined by

$$\begin{aligned}
(Fx)(t) &= x(t;\phi) + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) f(s, x(s), {}^C D^\beta x(s), u(s)) ds \\
&\quad + \int_{-h}^0 dC_\tau\left[\int_\tau^0 (t-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(t-(s-\tau)) C(s-\tau,\tau)\psi(s)ds\right] \\
&\quad + \int_0^t \left[\int_{-h}^0 (t-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(t-(s-\tau)) d_\tau C_t(s-\tau,\tau) u(s) ds\right]
\end{aligned}$$

This fixed point is then a solution of (5.1). Substituting the control $u(t)$ in the above equation we get

$$
\begin{aligned}
&(Fx)(t)\\
&= x(t;\phi) + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) f(s, x(s), {}^C D^\beta x(s), u(s))ds\\
&+ \int_{-h}^0 dC_\tau \Big[\int_\tau^0 (t-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(t-(s-\tau)) C(s-\tau,\tau)\psi(s)ds \Big]\\
&+ \int_0^t \Big[\int_{-h}^0 (t-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(t-(s-\tau)) d_\tau C_t(s-\tau,\tau) \Big]\\
&\times \Big[\int_{-h}^0 (t-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(t-(s-\tau)) d_\tau C_t(s-\tau,\tau) \Big]^* W^{-1} \sigma(y(0), x_1; z, v)
\end{aligned}
$$

Clearly, $(Fx)(T) = x_1$ which implies that the control u steers the system from the initial state x_0 to x_1 in time T, provided we obtain a fixed point of the nonlinear operator F.

The first step is to attain a priori bound of the set $\zeta(F) = \{x \in X : x = \lambda F x$ for some $\lambda \in (0, 1)\}$.

Let $x \in \zeta(F)$, then $x = \lambda F x$ for some $0 < \lambda < 1$. Then for each $t \in J$, we have

$$
\begin{aligned}
x(t) =& \lambda x(t;\phi) + \lambda \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) f(s, x(s), {}^C D^\beta x(s), u(s))ds\\
&+ \lambda \int_{-h}^0 dC_\tau \Big[\int_\tau^0 (t-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(t-(s-\tau)) C(s-\tau,\tau)\psi(s)ds \Big]\\
&+ \lambda \int_0^t \Big[\int_{-h}^0 (t-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(t-(s-\tau)) d_\tau C_t(s-\tau,\tau) \Big]\\
&\times \Big[\int_{-h}^0 (t-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(t-(s-\tau)) d_\tau C_t(s-\tau,\tau) \Big]^* W^{-1} \Big[x_1 - x(T;\phi)\\
&- \int_0^T (T-\xi)^{\alpha-1} X_{\alpha,\alpha}(T-\xi) f(\xi, x(\xi), {}^C D^\beta x(\xi), u(\xi))d\xi\\
&- \int_{-h}^0 dC_\tau \Big[\int_\tau^0 (T-(\xi-\tau))^{\alpha-1} X_{\alpha,\alpha}(T-(\xi-\tau)) C(\xi-\tau,\tau)\psi(\xi)d\xi \Big]\Big] ds
\end{aligned}
$$

$$
\begin{aligned}
\|x(t)\| \leq n_1 + n_5 + n_2 \int_0^t (t-s)^{\alpha-1} m_1(s)\Omega \left(\|x(s)\| + \left\|{}^C D^\beta x(s)\right\|\right. \\
\left. + \|u(s)\|\right) ds + \int_0^t \|G(T,s)\| \left\|G^*(T,s)\right\| \\
\times \left\|W^{-1}\right\| \Bigg[\|x_1\| + n_1 + n_5 + n_2 \int_0^T (T-\xi)^{\alpha-1} m_1(\xi)\Omega \left(\|x(\xi)\|\right. \\
\left. + \left\|{}^C D^\beta x(\xi)\right| + \|u(\xi)\|\right\|\right) d\xi \Bigg] ds \\
\equiv k_1 + n_2 \int_0^t (t-s)^{\alpha-1} m_1(s)\Omega \left(\|x(s)\| + \left\|{}^C D^\beta x(s)\right\| + \|u(s)\|\right) ds
\end{aligned}
$$

$r_1(t)$ represented by the right hand side of the above inequality, we get $r_1(0) = k_1$, $\|x(t)\| \leq r_1(t)$ and

$$
r_1'(t) = \frac{n_2 t^\alpha}{\alpha} m_1(t)\Omega \left(\|x(t)\| + \left\|{}^C D^\beta x(t)\right\| + \|u(t)\|\right).
$$

Also,

$$
\begin{aligned}
x(t) =& \lambda x'(t;\phi) + \lambda \int_0^t (t-s)^{\alpha-1} X_{\alpha_1,\alpha_1}(t-s) f(s, x(s), {}^C D^\beta x(s), u(s)) ds \\
&+ \lambda \int_{-h}^0 dC_\tau \Bigg[\int_\tau^0 (t-(s-\tau))^{\alpha-1} X_{\alpha_1,\alpha_1}(t-(s-\tau)) C(s-\tau,\tau)\psi(s) ds \Bigg] \\
&+ \lambda \int_0^t \Bigg[\int_{-h}^0 (t-(s-\tau))^{\alpha-1} X_{\alpha_1,\alpha_1}(t-(s-\tau)) d_\tau C_t(s-\tau,\tau) \Bigg] \\
&\times \Bigg[\int_{-h}^0 (t-(s-\tau))^{\alpha-1} X_{\alpha_1,\alpha_1}(t-(s-\tau)) d_\tau C_t(s-\tau,\tau) \Bigg]^* W^{-1} \Bigg[x_1 - x(T;\phi) \\
&- \int_0^T (T-\xi)^{\alpha-1} X_{\alpha,\alpha}(T-\xi) f(\xi, x(\xi), {}^C D^\beta x(\xi), u(\xi)) d\xi \\
&- \int_{-h}^0 dC_\tau \Bigg[\int_\tau^0 (T-(\xi-\tau))^{\alpha-1} X_{\alpha,\alpha}(T-(\xi-\tau)) C(\xi-\tau,\tau)\psi(\xi) d\xi \Bigg] \Bigg] ds
\end{aligned}
$$

$$\begin{aligned}
\|x(t)\| &\le n_5 + n_7 + n_6 \int_0^t (t-s)^{\alpha-1} m_1(s)\Omega\left(\|x(s)\| + \left\|{}^C D^{\beta} x(s)\right\| + \|u(s)\|\right) ds \\
&\quad + \int_0^t \|G(T,s)\| \left\|G^*(T,s)\right\| \\
&\quad \times \left\|W^{-1}\right\| \Bigg[\|x_1\| + n_1 + n_5 + n_2 \int_0^T (T-\xi)^{\alpha-1} m_1(\xi)\Omega\,(\|x(\xi)\| \\
&\quad + \left\|{}^C D^{\beta} x(\xi)| + \|u(\xi)\|\right\|\big) d\xi \Bigg] ds \\
&\equiv k_2 + n_6 \int_0^t (t-s)^{\alpha-1} m_1(s)\Omega\left(\|x(s)\| + \left\|{}^C D^{\beta} x(s)\right\| + \|u(s)\|\right) ds
\end{aligned}$$

Thus,

$$\begin{aligned}
\left\|{}^C D^{\beta} x(t)\right\| &\le \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \left\|x'(s)\right\| ds \\
&\le \frac{k_2 t^{1-\beta}}{\Gamma(2-\beta)} + \frac{n_6}{\Gamma(2-\beta)} \int_0^t (t-\xi)^{1-\beta} m_1(\xi)\Omega\,(\|x(\xi)\| \\
&\quad + \left\|{}^C D^{\beta} x(\xi)\right\| + \|u(\xi)\|\big) d\xi.
\end{aligned}$$

$r_2(t)$ represented by the right hand side of the above inequality, we have $r_2(0) = 0$, $\left\|{}^C D^{\beta} x(t)\right\| \le r_2(t)$

$$\begin{aligned}
r_2'(t) = \frac{(1-\beta)k_2 t^{-\beta}}{(1-\beta)\Gamma(1-\beta)} + \frac{n_6(\alpha-\beta-1)}{(1-\beta)\Gamma(1-\beta)} \int_0^t (t-\xi)^{-\beta} m_1(\xi)\Omega\,(\|x(\xi)\| \\
+ \left\|{}^C D^{\beta} x(\xi)\right\| + \|u(\xi)\|\big) d\xi.
\end{aligned}$$

Let $\omega(t) = r_1(t) + r_2(t), t \in J$. Then $\omega(0) = r_1(0) + r_2(0) = r$ and

$$\omega'(t) = r_1'(t) + r_2'(t) \le m(t)\Omega(\omega(t)).$$

Which implies that for each $t \in J$,

$$\int_{\omega(0)}^{\omega(t)} \frac{ds}{\Omega(s)} \le \int_0^T m(s)ds < \int_r^{\infty} \frac{ds}{\Omega(s)}.$$

Using this inequality implies that there exists a constant K such that

$$\omega(t) = r_1(t) + r_2(t) \le K, t \in J.$$

Then $\|x(t)\| \le r_1(t)$ and $\left\|{}^C D^{\beta} x(t)\right\| \le r_2(t), t \in J$, and hence

$$\|x\|^* = \max\left\{\|x\|, \left\|{}^C D^{\beta} x(t)\right\|\right\} \le K$$

and the set $\zeta(F)$ is bounded.

Next we shall prove that the operator $F : X \to X$ is completely continuous.

Let $B_m = \{x \in X; \|x\|^* \leq m\}$. We first show that F maps bounded sets B_m into equicontinuous family. Let $x \in B_m$ and $t_1, t_2 \in J$. Then, if $0 < t_1 < t_2 \leq T$,

$$\begin{aligned}
&\|(Fx)(t_2) - (Fx)(t_1)\| \\
&\leq \|x(t_2;\phi) - x(t_1;\phi)\| + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \left\|X_{\alpha,\alpha}(t_2 - s)\right\| h_m(s)ds \\
&+ \int_0^{t_1} \left\|(t_2 - s)^{\alpha-1} X_{\alpha,\alpha}(t_2 - s) - (t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s)\right\| h_m(s)ds \\
&+ \int_{-h}^0 dC_\tau \left[\int_\tau^0 ||(t_2 - (s - \tau))^{\alpha-1} X_{\alpha,\alpha}(t_2 - (s - \tau)) \right. \\
&\left. - (t_1 - (s - \tau))^{\alpha-1} X_{\alpha,\alpha}(t_1 - (s - \tau))|| \, \|C(s - \tau, \tau)\| \, \|\psi(s)\| \, ds \right] \\
&+ \int_{t_1}^{t_2} \left[\int_{-h}^0 \left\|(t_2 - (s - \tau))^{\alpha-1} X_{\alpha,\alpha}(t_2 - (s - \tau))\right\| d_\tau C_{t_2}(s - \tau, \tau) \right] \|u(s)\| \, ds \\
&+ \int_0^{t_1} \left[\int_{-h}^0 ||(t_2 - (s - \tau))^{\alpha-1} X_{\alpha,\alpha}(t_2 - (s - \tau)) - (t_1 - (s - \tau))^{\alpha-1} \right. \\
&\left. \times X_{\alpha,\alpha}(t_1 - (s - \tau))|| \left\|C_{t_2}(s - \tau, \tau) - C_{t_1}(s - \tau, \tau)\right\| d_\tau \right] \|u(s)\| \, ds \qquad (5.2)
\end{aligned}$$

and

$$\begin{aligned}
\|u(t_2) - u(t_1)\| &= \left\|G^*(T, t_2) - G^*(T, t_1)\right\| \left\|W^{-1}\right\| \\
&\times \left[\|x_1\| + n_1 + n_5 + n_2 \int_0^T \left\|(T - s)^{\alpha-1}\right\| h_m(s)ds \right] \qquad (5.3)
\end{aligned}$$

and hence

$$\left\|(Fx)'(t)\right\| \leq n_{10} + n_6 \int_0^t (t - s)^{\alpha-1} h_m(s)ds$$

Thus,

$$\begin{aligned}
&\left\|{}^C D^\beta (Fx)(t_2) - {}^C D^\beta (Fx)(t_1)\right\| \\
&= \left\| \frac{1}{\Gamma(1-\beta)} \int_0^{t_2} (t_2 - s)^{-\beta} (Fx)'(s)ds - \frac{1}{\Gamma(1-\beta)} \int_0^{t_1} (t_1 - s)^{-\beta} (Fx)'(s)ds \right\| \\
&\leq \frac{n_{10}}{\Gamma(2-\beta)} (t_2^{1-\beta} - t_1^{1-\beta}) + \frac{n_6}{\Gamma(1-\beta)} \int_{t_1}^{t_2} (t_2 - s)^{-\beta} \left(\int_0^s (s - \xi) h_m(\xi) d\xi \right) ds \\
&+ \frac{n_6}{\Gamma(2-\beta)} \int_0^{t_1} ((t_2 - s)^{1-\beta} - (t_2 - t_1)^{1-\beta} - (t_1 - s)^{1-\beta}) \\
&\times \left(\int_0^s (s - \xi) h_m(\xi) d\xi \right) ds \qquad (5.4)
\end{aligned}$$

The right hand side of (5.2),(5.3) and (5.4) tend to zero as $t_2 \to t_1$. Then F maps B_m into an equicontinuous family of functions and hence the family FB_m is uniformly bounded.

Next we show that F is a compact operator. It suffices to show that the closure of FB_m is compact. Let $0 \le t \le T$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $x \in B_m$ we define

$$\begin{aligned}(F_\epsilon x)(t) =& x(t;\phi) + \int_o^{t-\epsilon} (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) f(s, x(s), {}^C D^\beta x(s), u(s)) ds \\ &+ \int_{-h}^0 dC_\tau \Big[\int_\tau^0 (t-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(t-(s-\tau)) C(s-\tau,\tau)\psi(s) ds \Big] \\ &+ \int_0^{t-\epsilon} \Big[\int_{-h}^0 (t-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(t-(s-\tau)) d_\tau C_t(s-\tau,\tau) \Big] G^*(T,s) \\ &\times W^{-1} \Big[x_1 - x(T;\phi) - \int_0^T (T-\xi)^{\alpha-1} X_{\alpha,\alpha}(T-\xi) f(\xi, x(\xi), {}^C D^\beta x(\xi), u(\xi)) d\xi \\ &- \int_{-h}^0 dC_\tau \Big[\int_\tau^0 (T-(\xi-\tau))^{\alpha-1} X_{\alpha,\alpha}(T-(\xi-\tau)) C(\xi-\tau,\tau)\psi(\xi) d\xi \Big] \Big] ds\end{aligned}$$

To acquire the bounded and equicontinuous property of F_ϵ we can use the similar methods. Hence we have,

$$S_\epsilon(t) = \{(F_\epsilon x)(t), x \in B_m\}$$

is relatively compact in X for every $0 < \epsilon < t$, and hence for every $x \in B_m$,

$$\begin{aligned}&\|(Fx)(t) - (F_\epsilon x)(t)\| \\ &\le \int_{t-\epsilon}^t \left\| (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) \right\| h_m(s) ds \\ &+ \int_{t-\epsilon}^t \left[\int_{-h}^0 \left\| (t-(s-\tau))^{\alpha-1} X_{\alpha,\alpha}(t-(s-\tau)) d_\tau C(t-\tau,\tau) \right\| \right] \\ &\times \left\| G^*(t,s) \right\| \left\| W^{-1} \right\| \left[\|x_1\| + n_1 + n_5 + n_2 \int_0^T \left\| (T-\xi)^{\alpha-1} \right\| h_m(\xi) d\xi \right] ds\end{aligned}$$

Also,

$$\begin{aligned}&\left\| (Fx)'(t) - (F_\epsilon x)'(t) \right\| \\ &\le \int_{t-\epsilon}^t \left\| (t-s)^{\alpha-1} X_{\alpha_1,\alpha_1}(t-s) \right\| h_m(s) ds \\ &+ \int_{t-\epsilon}^t \left[\int_{-h}^0 \left\| (t-(s-\tau))^{\alpha-1} X_{\alpha_1,\alpha_1}(t-(s-\tau)) d_\tau C(s-\tau,\tau) \right\| \right] \\ &\times \left\| G^*(t,s) \right\| \left\| W^{-1} \right\| \left[\|x_1\| + n_1 + n_5 + n_2 \int_0^T \left\| (T-\xi)^{\alpha-1} \right\| h_m(\xi) d\xi \right]\end{aligned}$$

Since $\|(Fx)(t)-(F_\epsilon x)(t)\| \to 0$ and $\left\|(Fx)'(t)-(F_\epsilon x)'(t)\right\| \to 0$ as $\epsilon \to 0$, therefore,

$$\begin{aligned}&\left\|{}^C D^\beta (Fx)(t) -{}^C D^\beta (F_\epsilon x)(t)\right\| \\ &\le \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\left\|(Fx)'(t)-(F_\epsilon x)'(t)\right\| ds, \quad \to 0, \text{ as } \quad \epsilon \to 0.\end{aligned}$$

Hence, the relatively compact sets $S_\epsilon(t)$ are arbitrary close to the set $\{(F_\epsilon x)(t), x \in B_m\}$. By the Arzela-ascoli theorem the set $\{(F_\epsilon x)(t), x \in B_m\}$ is compact in X.

Finally to show that F is continuous. Let $\{x_n\}$ be a sequence in X such that $\|x_n - x|\| \to 0$ and $\|u_n - u|\| \to 0$ as $n \to \infty$. Then there is an integer k such that $\|x_n\| \le k$, $\left\|{}^C D^\beta x_n\right\| \le k$ for each n and $t \in J$. So, $\|x\| \le k$, $\left\|{}^C D^\beta x\right\| \le k$ and $x,\ {}^C D^\beta x \in X$. By $(H1)$,

$$f(t, x_n(t), {}^C D^\beta x_n(t), u_n(t)) \to f(t, x(t), {}^C D^\beta x(t), u(t))$$

for all $t \in J$ and since

$$\left\| f(t, x_n(t), {}^C D^\beta x_n(t), u_n(t)) - f(t, x(t), {}^C D^\beta x(t), u(t))\right| \le 2h_k(t),$$

Hence, the dominated convergence theorem that

$$\begin{aligned}&\|(Fx_n)(t)-(Fx)(t)\| \\ &\le \int_0^T \Big\|(t-s)^{\alpha-1}X_{\alpha,\alpha}(t-s)[f(s,x_n(s),{}^C D^\beta x_n(s),u_n(s)) \\ &-f(s,x(s),{}^C D^\beta x(s),u(s))]\Big\| ds \\ &+\int_0^T \Big\|\Big\|\Big[\int_{-h}^0 (t-(s-\tau))^{\alpha-1}X_{\alpha,\alpha}(t-(s-\tau))d_\tau C_t(s-\tau,\tau)\Big] \\ &\times\Big[\int_{-h}^0 (t-(s-\tau))^{\alpha-1}X_{\alpha,\alpha}(t-(s-\tau))d_\tau C_t(s-\tau,\tau)\Big]^* \\ &\times W^{-1}\Big[\int_0^T (T-\xi)^{\alpha-1}X_{\alpha,\alpha}(T-\xi)f(\xi,x_n(\xi),{}^C D^\beta x_n(\xi),u_n(\xi))d\xi \\ &+\int_{-h}^0 dC_\tau\Big[\int_{-\tau}^0 (T-(\xi-\tau))^{\alpha-1}X_{\alpha,\alpha}(T-(\xi-\tau))C(\xi-\tau,\tau)\psi(\xi)d\xi\Big]\Big]\Big\|\Big\| ds \quad \to 0,\end{aligned}$$

as $n \to \infty$.

and

$$\begin{aligned}\|u_n(t)-u(t)\| &= \left\|G^*(T,t_n)-G^*(T,t)\right\|\left\|W^{-1}\right\| \\ &\times\left[\|x_1\|+n_1+n_5+n_2\int_0^T \left\|(T-s)^{\alpha-1}\right\| h_m(s)ds\right] \to 0,\end{aligned}$$

as $n \to \infty$.

Also,

$$
\begin{aligned}
&\left\|(Fx_n)'(t) - (Fx)'(t)\right\| \\
&\quad \leq \int_0^T \Big\|(t-s)^{\alpha-1}X_{\alpha_1,\alpha_1}(t-s)[f(s, x_n(s), {}^C D^\beta x_n(s), u_n(s)) \\
&\quad - f(s, x(s), {}^C D^\beta x(s), u(s))]\Big\| ds \\
&\quad + \int_0^T \Big\|\Big[\int_{-h}^0 (t-(s-\tau))^{\alpha-1}X_{\alpha_1,\alpha_1}(t-(s-\tau))d_\tau C_t(s-\tau,\tau)\Big] \\
&\times \Big[\int_{-h}^0 (t-(s-\tau))^{\alpha-1}X_{\alpha_1,\alpha_1}(t-(s-\tau))d_\tau C_t(s-\tau,\tau)\Big]^* \\
&\times W^{-1}\Big[\int_0^T (T-\xi)^{\alpha-1}X_{\alpha_1,\alpha_1}(T-\xi)f(\xi, x_n(\xi), {}^C D^\beta x_n(\xi), u_n(\xi))d\xi \\
&+ \int_{-h}^0 dC_\tau\Big[\int_{-\tau}^0 (T-(\xi-\tau))^{\alpha-1}X_{\alpha,\alpha}(T-(\xi-\tau))C(\xi-\tau,\tau)\psi(\xi)d\xi\Big]\Big]\Big\| ds, \quad \to 0,
\end{aligned}
$$

as $n \to \infty$.

Therefore F is continuous. Hence, the set $\zeta(F) = \{x \in X; x = \lambda Fx, \lambda \in (0, 1)\}$ is bounded as we proved in the first step. By Schaefer's theorem, the operator F has a fixed point in X. This fixed point is then the solution of (5.1). Hence the system (5.1) is controllable on $[0, T]$. ■

6. Example

In this section, we present a numerical examples to illustrate the theoretical results.

Example 6.1. Consider the linear fractional delay dynamical system with distributed delays in control of the form

$$
{}^C D^{\frac{3}{2}} x(t) = Ax(t) + Bx(t-1) + \int_{-1}^0 d_\tau C(t,\tau)u(t+\tau) \tag{6.1}
$$

where $\alpha = \dfrac{3}{2}$, $h = 1, x(t) = \phi(t)$, $x'(t) = \phi'(t) \in R^2$, $u(t) = \psi(t)$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,
$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} -cos(t+\tau) & sin(t+\tau) \\ -sin(t+\tau) & -cos(t+\tau) \end{pmatrix}$.

The solution of the system (6.1) can be written as

$$\begin{aligned} x(t) =& E_{\frac{3}{2},1}(A(t^{\frac{3}{2}})) + E_{\frac{3}{2},2}(A(t^{\frac{3}{2}})) + B\int_{-1}^{0}(t-(s-1))^{\frac{1}{2}}E_{\frac{3}{2},\frac{3}{2}}(A(t-s-1)^{\frac{1}{2}})\phi(s)ds \\ &+ \int_{-1}^{0} dC_\tau \int_{\tau}^{0}(t-(s-\tau))^{\frac{1}{2}}E_{\frac{3}{2},\frac{3}{2}}(A(t-s-1)^{\frac{1}{2}})C(s-\tau,\tau)\psi(s)ds \\ &+ \int_{0}^{t}\int_{-1}^{0}(t-(s-\tau))^{\frac{1}{2}}E_{\frac{3}{2},\frac{3}{2}}(A(t-s-1)^{\frac{1}{2}})C_(t)(s-\tau,\tau)u(s)ds \end{aligned}$$

The Mittag-Leffler functions of the matrices are given by

$$E_{\frac{3}{2}}(At^{\frac{3}{2}}) = \begin{pmatrix} \sum_{j=0}^{\infty}\frac{(-1)^j t^{3j}}{\Gamma(1+3j)} & \sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3}{2}(2j+1)}}{\Gamma(1+\frac{3}{2}(2j+1))} \\ -\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3}{2}(2j+1)}}{\Gamma(1+\frac{3}{2}(2j+1))} & \sum_{j=0}^{\infty}\frac{(-1)^j t^{3j}}{\Gamma(1+3j)} \end{pmatrix}$$

Further,

$$\begin{aligned} &E_{\frac{3}{2},\frac{3}{2}}(A(T-(s-\tau))^{\frac{3}{2}}) \\ &= \begin{pmatrix} \sum_{j=0}^{\infty}\frac{(-1)^j(T-(s-\tau))^{3j}}{\Gamma(\frac{3}{2}(2j+1))} & \sum_{j=0}^{\infty}\frac{(-1)^j(T-(s-\tau))^{\frac{3}{2}(2j+1)}}{\Gamma(3(j+1))} \\ -\sum_{j=0}^{\infty}\frac{(-1)^j(T-(s-\tau))^{\frac{3}{2}(2j+1)}}{\Gamma(3(j+1))} & \sum_{j=0}^{\infty}\frac{(-1)^j(T-(s-\tau))^{3j}}{\Gamma(\frac{3}{2}(2j+1))} \end{pmatrix} \end{aligned}$$

and

$$(T-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(T-(s-\tau))^{\alpha}) = \begin{pmatrix} cos_\alpha(t) & sin_\alpha(t) \\ -sin_\alpha(t) & cos_\alpha(t) \end{pmatrix}$$

where

$$cos_\alpha(t) = \sum_{j=0}^{\infty}\frac{(-1)^j(T-(s-\tau))^{\frac{1}{2}(2j+1)}}{\Gamma(\frac{3}{2}(2j+1))}$$

$$sin_\alpha(t) = \sum_{j=0}^{\infty}\frac{(-1)^j(T-(s-\tau))^{2(j+1)}}{\Gamma(3(j+1))}$$

$$\begin{aligned} G(T,s) &= \int_{-1}^{0}(T-(s-\tau))^{\alpha-1}E_{\alpha,\alpha}(A(T-(s-\tau))^{\alpha})d_\tau C_T(s-\tau,\tau) \\ &= \begin{pmatrix} P(s) & -Q(s) \\ Q(s) & P(s) \end{pmatrix} \end{aligned}$$

where

$$P(s) = \int_{-1}^{0} \big[sin_\alpha(T)sin(T+\tau) - cos_\alpha(T)cos(T+\tau)\big]d\tau$$
$$Q(s) = \int_{-1}^{0} \big[cos_\alpha(T)sin(T+\tau) + sin_\alpha(T)cos(T+\tau)\big]d\tau$$

Thus, simple matrix calculation we obtain the controllability matrix

$$\begin{aligned} W(0,T) &= \int_0^T G(T,s)G^*(T,s)ds \\ &= \int_0^T \big[P^2(s)+Q^2(s)\big]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ds \end{aligned}$$

is positive definite for any $T > 0$. Therefore, by Theorem 4.1 the linear system (6.1) is controllable.

Example 6.2. Consider the nonlinear fractional delay dynamical system with distributed delays in control of the form

$$\begin{aligned} {}^C D^{\frac{3}{2}} x(t) &= Ax(t) + Bx(t-1) \\ &\quad + \int_{-1}^{0} d_\tau C(t,\tau)u(t+\tau) + f\begin{pmatrix} \dfrac{x_1(t)}{1 + x_1^2(t) + {}^C D^\beta x_1(t) + u_1(t)} \\ \dfrac{x_2(t)}{1 + x_2^2(t) + {}^C D^\beta x_2(t) + u_2(t)} \end{pmatrix} \end{aligned} \tag{6.2}$$

where $\alpha = \frac{3}{2}$, $h = 1, x(t) = \phi(t),\ x'(t) = \phi'(t) \in R^2,\ \ u(t) = \psi(t),\ \ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,
$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} -cos(t+\tau) & sin(t+\tau) \\ -sin(t+\tau) & -cos(t+\tau) \end{pmatrix}$.
It is easy to verify that the nonlinear function f satisfies the condition in Theorem 5.1, and hence the fractional delay dynamical system (6.2) is controllable on $[0, T]$.

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