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Exact Solutions for a Two-Parameter Rayleigh Distribution

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Abstract

In attempting to estimate parameters of a Rayleigh distribution numerical itarative methods or routines are frequently employed. In this work, an exact method on the constant minimization of the goal function is proposed. The scale and location parameter of the Rayleigh distribution are estimated by changing the parameter dimensional space of the original function. Linearization of the probability density function (PDF) is done through differential techniques. A numerical simulation analysis and real life data set of the strength of glass fiber are used to evaluate the performance of the proposed optimization differential method (ODM), maximum likelihood estimators (MLE) and method of moments estimator (MME). The probability density functions of the three methods are constructed using the estimated parameters from the methods. The ODM shows a better convergence as the sample data increase and can be adopted in practice.

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1. Introduction

The Rayleigh distribution was introduced by Lord Rayleigh (1880) in connection with problems in the field of acoustics [1]. From then on the Rayleigh distribution has been applied in many different areas of science and technology. The two-parameter Rayleigh Distribution is a special case of the three-parameter Weibull distribution [2], [3], [4], the distribution may be used to estimate real life data than the one-parameter Rayleigh distribution [5], [6]. The Rayleigh distribution has a number of applications in settings where magnitudes of normal variables are important [7], [8]. An application for the Rayleigh distribution is the analysis of wind velocity [9] into its orthogonal two-dimensional vector components [10]. Assuming that each component is uncorrelated, normally distributed with equal variance, σ^2, and zero mean, $\eta = 0$, then the overall wind speed (vector magnitude) will be characterized by a Rayleigh distribution. The parameter estimation of the Rayleigh distribution can be found in several areas of applications that is to say, magnetic resonance imaging (MRI) [11], Radar [8], Fiber strength [12], [13], Plant data [13], Clustering [14], communication theory and engineering [8], [15], [16], seismic analysis [17], [18].

The rest of the article is organized as follows: Section 2, describes the motivation of the proposed method. In Section 3, different methods of parameter estimation for the Rayleigh distribution are presented and the proposed method is discussed. In Section 4, a numerical simulation is considered to assess the performance and present results of the proposed method. Section 5 comparison of the proposed method (ODM) with existing parameter estimation methods on real life data is done. In Section 6, results and discussion of the study are presented and conclusions presented in Section 7.

2. Motivation

The estimation of parameters in all algebraic models is regarded as a fundamental step in statistical modeling for a class of problems. In this work estimating parameters of a Rayleigh distribution is considered. The specification for the two-parameter Rayleigh distribution is

$$f(x; \lambda, \eta) = \begin{cases} 2\lambda\,(x-\eta)\,\mathrm{e}^{-\lambda(x-\eta)^2} & \text{if} \quad x > \eta, \quad \lambda > 0; \\ 0 & \text{else where,} \end{cases} \tag{2.1}$$

where, the scale parameter $\lambda > 0$ and the location parameter $0 < \eta < \infty$. Due to the presence of the location parameter, the two-parameter Rayleigh distribution can be used more effectively to analyze real life data than one-parameter Rayleigh distribution [5].

Although extensive work has been done on the estimation of the one-parameter Rayleigh distribution [6], according to DEY *et al.* (2014) [19] not much attention has been paid to the two-parameter Rayleigh distribution. Even then, the existing parameter estimation methods like MLE, MME and Gaussian–Newton, use numerical iteration routines to approximate solutions of the required parameters. In this work a method that computes the exact solutions of the parameters is proposed and discussed.

3. Parameter Estimation

There exists different methods for estimating parameters for X distributed data that is assumed to be of a Rayleigh distribution. Some of these methods are maximum likelihood estimation [7], [8], [19], [16], [15], [4], [20], method of moments estimators [16], [19], Modified Moment Estimators [4], Local frequency ratio method [18], L-moment estimator [4], [19], least squares estimators [4], [19], weighted least squares estimators [16], [4], percentile based estimators, Bayes estimators [7], [21], [19], [20], simulation consistent estimators. In this section the conventional MLE and MME are discussed together with the ODM.

3.1. Maximum Likelihood Estimator (MLE)

Assume a random sample x_i for $i = 1, \ldots, n$ of n observations for a Rayleigh X distributed random data with a PDF of the form Equation (2.1) is obtainable. The likelihood function of x_i on the support of X is given by

$$(\lambda, \eta) = C + n \ln \lambda + \sum_{i=1}^{n} \ln (x_i - \eta) - \lambda \sum_{i=1}^{n} (x_i - \eta)^2. \tag{3.1}$$

The two normal equations are then

$$\frac{\partial\, l\,(\lambda, \eta)}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} (x_i - \eta)^2 = 0, \qquad \text{and} \tag{3.2}$$

$$\frac{\partial\, l\,(\lambda, \eta)}{\partial \eta} = -\sum_{i=1}^{n} (x_i - \eta)^{-1} + 2\lambda \sum_{i=1}^{n} (x_i - \eta) = 0. \tag{3.3}$$

It can be observed that the normal equations, Equations (3.2 and 3.3) can not be solved explicitly to obtain exact solutions for the scale (λ) and location (η) parameters, hence the application of numerical iterative approaches.

From Equation (3.2), we obtain the maximum likelihood estimation (MLE) of λ as a function of η, say $\widehat{\lambda}\,(\eta)$, as

$$\widehat{\lambda}\,(\eta) = \frac{n}{\sum_{i=1}^{n} (x_i - \eta)^2}. \tag{3.4}$$

Substituting $\widehat{\lambda}(\eta)$ in Equation (3.1), we obtain the log-likelihood function of η without the additive constant as

$$g(\eta) = l\left(\widehat{\lambda}(\eta), \eta\right) = n \ln n - n \ln \left(\sum_{i=1}^{n} (x_i - \eta)^2 \right) + \sum_{i=1}^{n} \ln (x_i - \eta) - n. \tag{3.5}$$

Therefore, the MLE of η, say $\widehat{\eta}_{MLE}$, can be obtained by maximizing Equation (3.5) with respect to η. It can be shown that the maximum of Equation (3.5) can be obtained as a fixed point solution of the following equation

$$h(\eta) = \eta, \tag{3.6}$$

where

$$h(\eta) = 2\sum_{i=1}^{n}(x_i - \eta)^2 \times \sum_{i=1}^{n}(x_i - \eta) \times \sum_{i=1}^{n}(x_i - \eta)^{-1}. \tag{3.7}$$

Once $\widehat{\eta}_{MLE}$ is obtained, the MLE of λ, $\widehat{\lambda}_{MLE} = \widehat{\lambda}(\widehat{\eta}_{MLE})$ can be easily obtained. Observe that very simple iterative technique namely $h(\eta^{(j)}) = \eta^{(j+1)}$, where $\eta^{(j)}$ is the j-th iterate, can be used to solve Equation (3.6).

The variances and distributional properties of $\widehat{\eta}_{MLE}$ and $\widehat{\lambda}_{MLE}$, are not in explicit forms, it is expected that the exact distributions of the MLEs will not be possible to obtain. We therefore, mainly rely on the asymptotic properties of the MLEs.

3.2. Method of Moment Estimators (MME)

The MMEs of the two-parameter Rayleight distribution can be obtained as

$$\widehat{\lambda}_{MME} = \frac{1}{s^2}\left[1 - \Gamma^2\left(\frac{3}{2}\right)\right] \tag{3.8}$$

and

$$\widehat{\eta}_{MME} = \bar{x} - \widehat{\lambda}_{MME}^{-\frac{1}{2}}\,\Gamma\left(\frac{3}{2}\right), \tag{3.9}$$

where $\Gamma(n) = (n-1)$, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ are sample mean and sample variance respectively.

3.3. Optimized Differential Method (ODM)

Proposition 3.1. Nonlinear least-squares problems can be linearized in parameters, through differential techniques by changing their parameter dimensional space.

Theorem 3.2. For any random variable X that is assumed to follow a Rayleigh distribution $f(x;\lambda,\eta) = 2\lambda(x-\eta)e^{-\lambda(x-\eta)^2}$ for $x > \eta$ and $\lambda > 0$, the scale parameter λ, and location parameter η, can be estimated by

$$\min_{i\to n}\sum_{i=1}^{n}\left[f_i(\xi)\left(x_i^2 4\lambda^2 - x_i 8\lambda^2\eta + 4\lambda\eta^2 - 6\lambda\right) - \frac{d^2}{dx_i^2}\{f_i(\xi)\}\right]^2 \longrightarrow 0,$$

where $\xi = (x; \lambda, \eta)$.

Proof. It follows from the work done by KIKAWA *et al.* [22], [23]. Assuming that a random sample $x = \sum_{i=1}^{N} x_i$ is available, using numerical differentiation, on Equation (2.1) the first derivative is

$$\frac{\partial f}{\partial x} = f(\xi)\left[\frac{1}{x-\eta} - 2\lambda(x-\eta)\right], \tag{3.10}$$

and the second order derivative is

$$\frac{\partial^2 f}{\partial x^2} = f(\xi)\left(x^2 4\lambda^2 - x8\lambda^2\eta + 4\lambda\eta^2 - 6\lambda\right) \tag{3.11}$$

With the assumption that the random sample, $x = \{x_1, x_2, \ldots, x_n\}$ is available and is assumed to follow a Rayleigh distribution then the scale λ, and location η, parameters for the distribution can be obtained by minimizing the goal function

$$\min_{i\to n} \sum_{i=1}^{n}\left[f_i(\xi)\left(x_i^2 4\lambda^2 - x_i 8\lambda^2\eta + 4\lambda\eta^2 - 6\lambda\right) - \frac{d^2}{dx_i^2}\{f_i(\xi)\}\right]^2 \longrightarrow 0, \tag{3.12}$$

where $\xi = (x; \lambda, \eta)$.

Introducing, $\phi_1 = 4\lambda^2$, and $\phi_2 = \eta$, the goal function, Equation 3.12 becomes

$$\min_{i\to n} \sum_{i=1}^{n}\left[f_i(\xi)\left(x_i^2\phi_1 - x_i 2\phi_1\phi_2 + 2\sqrt{\phi_1}\phi_2^2 - 3\sqrt{\phi_1}\right) - \frac{d^2}{dx_i^2}\{f_i(\xi)\}\right]^2 \longrightarrow 0, \tag{3.13}$$

where $\xi = (x; \phi_1, \phi_2)$.

The goal function Equation 3.13 is obtained from the observed data for the Rayleigh distribution.

The estimation of the scale parameter λ is

$$\widehat{\lambda} = \frac{1}{2}\sqrt{\phi_1}, \tag{3.14}$$

and for the location parameter η,

$$\widehat{\eta} = \phi_2. \tag{3.15}$$

Hence Equations (3.14 and 3.15) give the exact solution approximations required, using the Rayleigh distribution data via the ODM method. ■

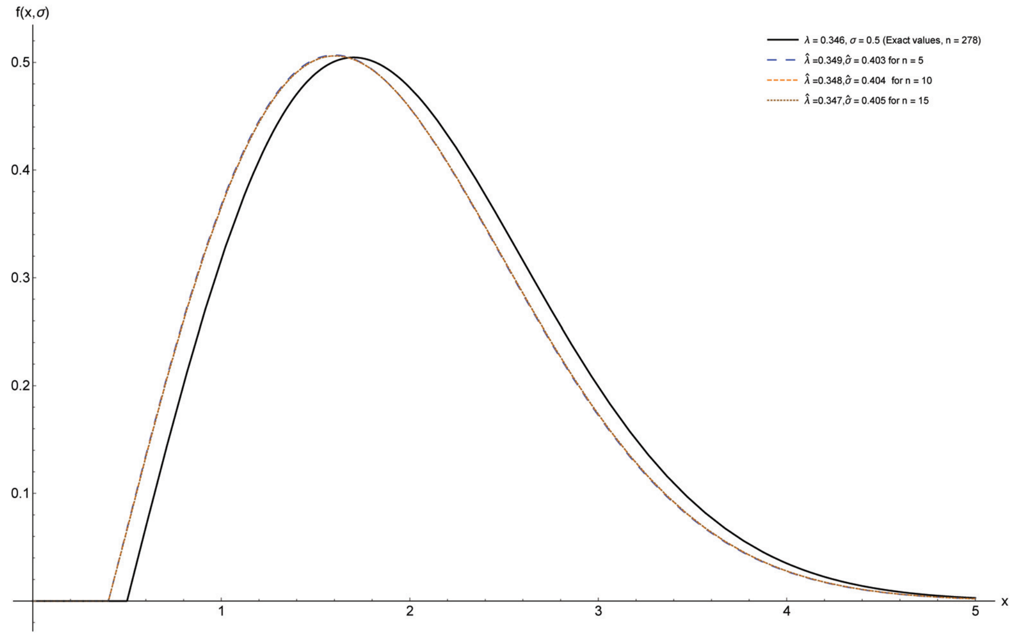

Figure 1: The scale and location parameter estimation for a Rayleigh Distribution using the ODM method for different samples sizes $n = 5,\ 10,\ 15$ respectively

4. Numerical Simulations

A simulation of Rayleigh distribution $R(\lambda, \eta)$, Equation (2.1), where the scale $\lambda = 0.346$, and the location $\eta = 0.5$, is performed. Estimation of the scale parameter using the ODM is represented by Figures 1 and 2.

Figures 1 and 2, represent the results of the ODM method, to determine the scale and location parameter of the numerical simulation of a two-parameter Rayleigh distribution with exact scale parameter $\lambda = 0.346$, and location parameter $\eta = 0.5$. Using the ODM method it can be shown that with a sample of $n = 5,\ 10,\ 15$ (small), Figure 1, the scale parameter $\lambda = \widehat{\lambda}$ is approximated better than the location parameter $\eta = \widehat{\eta}$. The difference in the estimation for the small sample, that is $n = 5,\ 10,\ 15$, is very minimal. Using the ODM it can be shown that with a sample of $n = 20,\ 25$ (moderate) and $n = 30$ (large), Figure 2, the scale parameter $\lambda = \widehat{\lambda}$ and location parameter $\eta = \widehat{\eta}$, are approximated. In this case the ODM method gives a better approximation for the scale parameter than the location parameter. The difference in the estimation for the moderate and large samples, that is $n = 20,\ 25,\ 30$, is very minimal. If the sample size is increased say $n > 30$ the estimate of the scale and location parameter are improved, Figure 3.

The detailed results of the numerical simulation are discussed and presented in Section 6. The numerical simulations were done in Mathematica®.

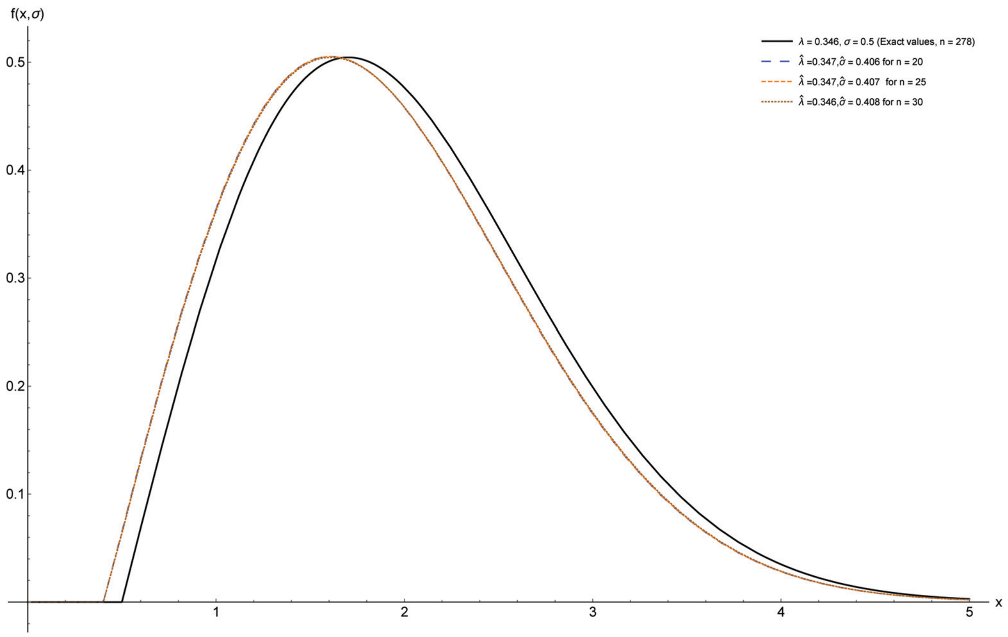

Figure 2: The scale and location parameter estimation for a Rayleigh Distribution using the ODM method for different samples sizes $n = 20,\ 25,\ 30$ respectively

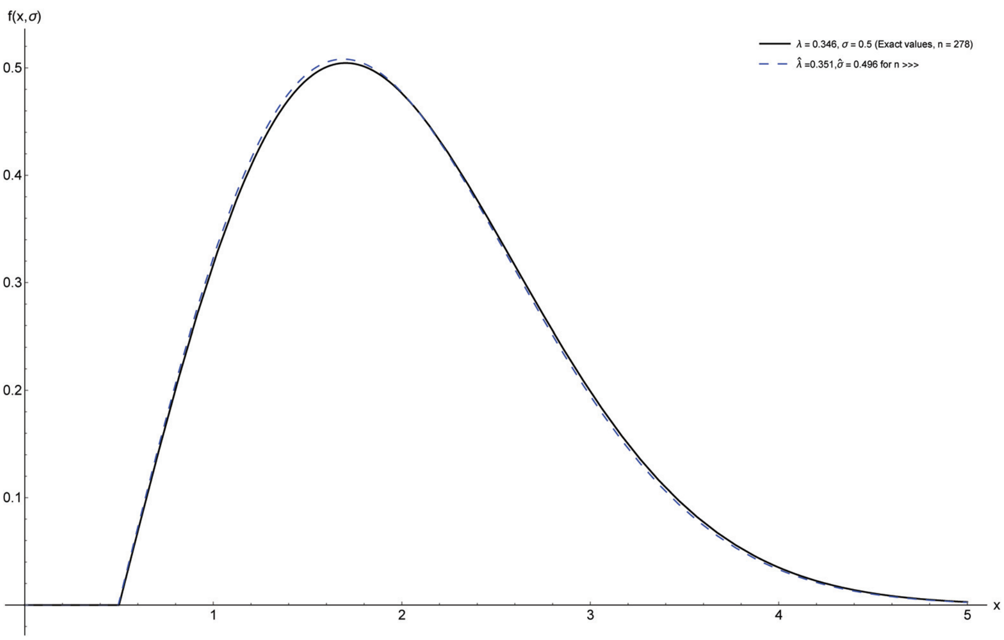

Figure 3: The scale and location parameter estimation for a Rayleigh Distribution using the ODM method for different samples size $n > 30$

4.1. Results for the Numerical Simulation

Table 1 shows estimates of both the scale and location parameters, λ, η respectively, using the ODM method for different sample sizes (n) with a scale parameter $\lambda = 0.346$, and location parameter $\eta = 0.5$. It can be observed that the accuracy of the estimates increases with increase in sample size, which is consistent with statistical estimation theory.

Table 1: Parameter estimation for the two-parameter Rayleigh distribution model using the ODM method

Scale Parameter $\lambda = 0.346$, Location Parameter $\eta = 0.5$		
Sample	Estimated Value Scale Parameter	Estimated Value Location Parameter
$n = 5$	$\widehat{\lambda}_5 = 0.3486$	$\widehat{\eta}_5 = 0.4032$
$n = 10$	$\widehat{\lambda}_{10} = 0.3482$	$\widehat{\eta}_{10} = 0.4043$
$n = 15$	$\widehat{\lambda}_{15} = 0.3478$	$\widehat{\eta}_{15} = 0.4054$
$n = 20$	$\widehat{\lambda}_{20} = 0.3474$	$\widehat{\eta}_{20} = 0.4063$
$n = 25$	$\widehat{\lambda}_{25} = 0.3469$	$\widehat{\eta}_{25} = 0.4071$
$n = 30$	$\widehat{\lambda}_{30} = 0.3465$	$\widehat{\eta}_{30} = 0.4077$
$\vdots$	$\vdots$	$\vdots$
$n > 30$	$\widehat{\lambda} = 0.3509$	$\widehat{\eta} = 0.4962$

5. Real Life Data

In this section the real life data for the strength of fiber glass, is considered. The ODM, MLE and MME approaches are employed to estimate both the scale and location parameters of the real life data and reconstruction of the data PDF pattern is done, see Figure 4.

5.1. Strength of Glass Fiber Data

The data taken from SMITH and NAYLOR (1987) [24], studied the strength of 1.5cm glass fibers measured at the national Physical Laboratory, England, The data comprises of 63 observations: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5,

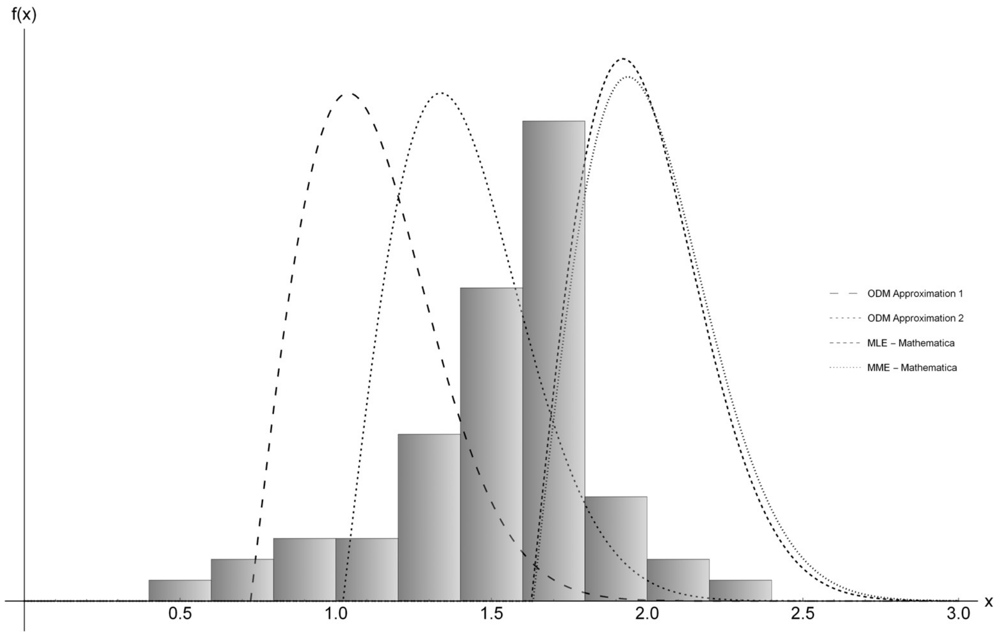

Figure 4: The estimation of the location and scale parameters for the strength of glass fiber data, using ODM method, MLE and MME method

1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89.

5.1.1 Results: Strength of Glass Fibre Data Using ODM Method

The scale parameter for the strength of glass fiber was estimated using the proposed method and found to be $\widehat{\lambda} = 5.073$, using MLE, $\widehat{\lambda} = 5.781$, and MME, $\widehat{\lambda} = 5.402$. The estimated location parameter for this distribution (strength of glass fiber) for the ODM method was computed to be $\widehat{\eta} = 1.024$ and for the MLE, $\widehat{\eta} = 1.628$, and MME, $\widehat{\eta} = 1.634$.[2]

From the results and Figure 4, it is shown that the proposed method gives better estimates for both the scale and location parameter. The ODM pattern, see Figure 4, takes into consideration majority of the data points as it is centrally positioned as opposed to the patterns of the MLE and MME which are greatly skewed to the right. Hence leaving out a greater part of the data points on the left. The MLE and MME ignore a majority of the data thus poorly estimate the strength of glass fiber data. All the methods have estimated the scale parameter for the data to within an accuracy of $\pm 6.090\,\%$. That is a considerably low error margin of accuracy on the scale parameter. The location parameter has a significantly high discrepancy and thus the ODM can be considered for the estimation of the location and scale parameter for the strength of glass fiber data.

[2]The MLE and MME was obtained using Mathematica®

6. Summary, Results and Future work

In this section the proposed method (ODM), MLE and MME approaches are compared to obtain their estimation capabilities.

Table 2: Summary of the performance of the proposed method and existing MLE, MME techniques on the estimated value for the Scale $\lambda = \widehat{\lambda}$ and Location Parameter $\eta = \widehat{\eta}$

Original values $\lambda = 0.346$ and $\eta = 0.5$						
Sample	ODM		MLE		MME	
	$\widehat{\lambda}$	$\widehat{\eta}$	$\widehat{\lambda}$	$\widehat{\eta}$	$\widehat{\lambda}$	$\widehat{\eta}$
$n = 5$	0.3486	0.4032	0.0425	0.0091	0.0000	0.0494
$n = 10$	0.3482	0.4043	0.1167	0.0137	-0.2422	0.3479
$n = 15$	0.3478	0.4054	0.1522	0.02791	-0.3009	0.4386
$n = 20$	0.3474	0.4063	25.0528	-24.7945	-0.3437	0.5122
$n = 25$	0.3469	0.4071	0.4942	-0.2096	-0.3699	0.5679
$n = 30$	0.3465	0.4077	0.1895	0.0137	-0.3809	0.6119
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$n > 30$	0.3509	0.4962	0.5695	0.1280	-1.3633	1.4673

From Table 3 the mean absolute errors for the different estimation methods are presented. It is observed that the proposed method presents a relatively smaller error for the scale and location parameters respectively. The MLE and MME present a bigger errors for the estimated parameters.

Thus in this case the ODM has the lowest mean absolute percentage error margin compared with the other estimation methods (MLE and MME, respectively), Table 4. the ODE in the overall error estimation and the MLE has an overall bigger marginal error compared with the ODE and MME methods. The proposed method performs better than the existing methods in the numerical analysis. Thus the ODM method can be adopted to estimate the location and scale parameters of Rayleigh distributed samples.

7. Conclusion

It has been shown that the proposed method estimates the scale and location parameter of the Rayleigh X distribution, Sections 4 and 6. The second order derivative of the X random variable was obtained and the least squares for the derivative of the X random variable was obtained. The proposed method thus works best with the X random variable being $X \geq 5$, Section 6 and Tables 1 and 2. Thus the proposed method may be imple-

Table 3: The mean absolute percentage error analysis of the proposed method and existing MLE, MME techniques on the estimation of the original values, scale parameter, $\lambda = 0.346$ and location parameter $\eta = 0.5$

Error analysis for the Scale $\lambda = \widehat{\lambda}$ and Location Parameter $\eta = \widehat{\eta}$						
Sample	ODM$_{error}$		MLE$_{error}$		MME$_{error}$	
	$\widehat{\lambda}_{error}$	$\widehat{\eta}_{error}$	$\widehat{\lambda}_{error}$	$\widehat{\eta}_{error}$	$\widehat{\lambda}_{error}$	$\widehat{\eta}_{error}$
$n = 5$	0.75	19.36	87.72	98.18	100.00	90.12
$n = 10$	0.64	19.14	66.27	97.26	170.00	30.42
$n = 15$	0.52	18.92	56.01	94.42	186.97	12.28
$n = 20$	0.40	18.74	7140.69	5058.9	199.34	2.44
$n = 25$	0.26	18.58	42.83	141.92	206.91	13.58
$n = 30$	0.14	18.46	45.23	97.26	210.09	22.38
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n > 30$	1.42	0.76	64.59	74.40	494.00	193.46

Table 4: The average mean absolute percentage error for the estimation methods, ODM, MLE and MME

Method	Scale parameter error	Location parameter error
ODM	± 0.59	± 16.28
MLE	± 1071.91	± 808.91
MME	± 223.90	± 52.10

mented into a program to estimate the scale and location parameter of a two-parameter Rayleigh distribution.

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