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Analytical Solution of A Differential Equation that Predicts the Weather Condition by Lorenz Equations Using Homotopy Perturbation Method

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Abstract

The Lorenz equation has made qualifying chaos possible which has inspired many mathematicians to research and study chaos [2]. Chaos theory is the branch of mathematics focused on the behaviour of dynamical systems that are highly sensitive to initial systems. Chaotic behaviour exists in many natural systems such as weather and climate. The deterministic nature of the system does not make behaviour predictable. This behaviour is known as deterministic chaos or simply. Approximate analytical solution of Lorenz equation is obtained by Homotopy perturbation method (HPM).Furthermore, in this work numerical simulation of the problem is also reported using Scilab/Matlab program. Agreements between analytical and numerical results are noted. The analytical result reported in this work is useful to understand the behavior of the system.

Keywords: Lorenz equations; Chaos; Homotopy perturbation method (HPM); Mathematical modeling; State variables.

1. INTRODUCTION

The Lorenz system is a system of ordinary differential equations first studied by Edward Lorenz. In particular, the Lorenz attractor is a set of chaotic solutions of the Lorenz system which when plotted resemble a butterfly or figure eight. Small differences in initial conditions yield widely diverging outcomes for such dynamical

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systems popularly referred to as the butterfly effect- rendering long -term prediction of the behavior impossible in general. The path that led to Lorenz to these equations began with an effort to find a simple model problem which the methods used for statistical weather forecasting would fail. The Lorenz equations are also connected to other physical phenomenon [1]. The Lorenz equation has made qualifying chaos possible which has inspired many mathematicians to research and study chaos [2]. Chaos theory is the branch of mathematics focused on the behavior of dynamical systems that are highly sensitive to initial systems. Chaotic behavior exists in many natural systems such as weather and climate. The deterministic nature of the system does not make behavior predictable. This behavior is known as deterministic chaos or

simply chaos. This behavior can be studied through analysis of chaotic mathematical model. Chaos theory has applications in several disciplines such as meteorology, sociology, physics, ecology, economics, biology etc. In chaotic systems, the uncertainty in forecast increases exponentially with the elapsed time. The model he introduced [3] can be thought of gross simplification of one feature of atmosphere namely the fluid motion driven by the thermal buoyancy known as convection. The model describes the convection motion of a fluid in a small, idealized "Rayleigh Benard cell. Curry [4] has shown that if the mode truncation is not done, but instead sufficient modes are retained to give numerical convergence, the chaos disappears. On the other hand McLauglin and Martin [5] has showed that chaos is obtained for three dimension version. The experimental system described by Lorenz equations is the Rititake dynamo a homopolar generator with the output fed back through inductors and resistors to the coil generating the magnetic field [6]. The coupled circuit and rotation equations can be reduced to the Lorenz form and experiments [7]. In March 1963, Lorenz wrote that he wanted to introduce, "ordinary differential equations whose solution example the simplest example of deterministic non periodic flow and finite amplitude convection". In his paper he examines the work of Barry Saltzman and John Rayleigh while incorporating several physical phenomena [8,9]. Lorenz used three variables to construct a simple model based on the 2 dimensional representation of earth's atmosphere. The purpose of this paper is to derive the approximate expressions of state variables of Lorenz equations using Homotopy perturbation method for all values of parameters.

2. Mathematical modeling and analysis

Let us consider the differential equation as follows

$$\frac{dx}{dt} = -ax + ay \tag{1}$$

$$\frac{dy}{dt} = bx - y - xz \tag{2}$$

$$\frac{dz}{dt} = -cz + xy \tag{3}$$

where *a*, *b*, and *c* are parameters. The initial conditions are

At $t = 0$; $x(t)=1$, $y(t)=1$, $z(t)=1$ (4)

3. Analytical solution of Lorenz equation using Homotopy Perturbation method

Nonlinear system of equations plays an important role in physics, chemistry and biology. Constructing of particular exact solution for these equations remains an important problem. In the past many authors mainly had paid attention to study the solution of nonlinear equations by using various methods. The Homotopy perturbation method has been worked out over a number of years by numerous authors. The Homotopy perturbation method (HPM) was proposed by He and was successfully applied to autonomous ordinary differential equations to nonlinear polycrystalline solids and other fields. In this method the solution procedure is very simple and only few iterations lead to high accurate solutions which are valid for whole solution domain. By solving the Eqns. (1) - (4), using Homotopy Perturbation method we obtain the analytical solutions of Lorenz equation as follows:

$$\begin{aligned} x(t) = {} & e^{-at} + \frac{a}{a-1}\left(\mathrm{e}^{-t} - e^{-at}\right) + \frac{abe^{-at}}{(1-a)}\left[t - \frac{e^{t(a-1)}}{(a-1)} + \frac{1}{(a-1)}\right] \\ & + \frac{ae^{-at}}{(1-a-c)}\left[\frac{e^{t(a-1)}}{(a-1)} + \frac{e^{-ct}}{c} - \frac{1}{(a-1)} - \frac{1}{c}\right] \end{aligned} \tag{5}$$

$$\begin{aligned} y(t) = {} & e^{-t} + \frac{b}{(1-a)}\left(\mathrm{e}^{-t} - e^{-at}\right) + \frac{1}{(1-a-c)}\left(e^{-t(a+c)} - e^{-t}\right) + \frac{abe^{-t}}{(a-1)}\left[t - \frac{e^{t(1-a)}}{(1-a)} + \frac{1}{(1-a)}\right] \\ & + \frac{e^{-t}}{(c-a-1)}\left[\frac{1}{2a} - \frac{1}{(1-a-c)} - \frac{e^{-2at}}{2a} + \frac{e^{t(1-a-c)}}{(1-a-c)}\right] \\ & + \frac{ae^{-t}}{(a-1)}\left[\frac{1}{c} - \frac{1}{(1-a-c)} - \frac{e^{-ct}}{c} + \frac{e^{t(1-a-c)}}{(1-a-c)}\right] \end{aligned} \tag{6}$$

$$z(t)=e^{-ct}+\frac{1}{(c-a-1)}\left(e^{-ct}-e^{-t(a+1)}\right)+\frac{be^{-ct}}{(a-1)}\left[\frac{e^{t(c-2a)}}{(c-2a)}+\frac{e^{t(a+1-c)}}{(a+1-c)}-\frac{1}{(c-2a)}-\frac{1}{(a+1-c)}\right]$$
$$+\frac{e^{-ct}}{(1-a-c)}\left[\frac{1}{(a+1-c)}-\frac{e^{-t(a+1-c)}}{(a+1-c)}+t\right] \quad (7)$$
$$+\frac{ae^{-ct}}{(a-1)}\left[\frac{1}{(c-2)}+\frac{1}{(a+1-c)}-\frac{e^{t(c-2)}}{(c-2)}-\frac{e^{-t(a+1-c)}}{(a+1-c)}\right]$$

4. DISCUSSION

Eqns. (5) - (7) are the simple analytical expressions of state variable for all values of parameters a, b and c. Figure (1), represents state variable x versus time t for fixed values of b=1.2 and c=0.2. Figure (2), represents state variable y versus time t for fixed value of b=1.5. Figure (3), represents state variable z versus time t for fixed values of a =2 and b = 0.01.

5. CONCLUSION

Approximate analytical solutions of the Lorenz equations are presented using Homotopy Perturbation method. A simple and a new method of estimating the state variables are derived. This solution procedure can be easily extended to all kinds of non-linear differential equations with various complex boundary conditions in enzyme – substrate reaction diffusion processes. The expressions provided in this work are useful to understand the behavior of the system.

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Appendix A: Basic concepts of the Homotopy Perturbation method

To explain this method let us consider the following function

$$A(w) - f(r) = 0; \quad r \in \Omega \tag{A.1}$$

With the boundary conditions of $B\left(w, \frac{\partial w}{\partial n}\right) = 0 \; ; \quad r \in \Gamma$ (A.2)

Where A, B, $f\,(r)$ and Γ are a general differential operator, a boundary operator, a known analytic function and the boundary of the domain Ω respectively. Generally speaking the operator A can be divided into a linear part L and a nonlinear part N. Eqn. (A.1) can be written as

$$L(w) + N(w) - f(r) = 0 \tag{A.3}$$

We construct a homotopy $z(r,p) : \Omega \times [0,1] \rightarrow R$ which satisfies

$$H(z,p) = (1-p)[L(w_0) - L(z)] + p[A(z) - f(r)] = 0, p \in [0,1], r \in \Omega$$

$$or$$

$$H(z,p) = L(w_0) - L(z) + pL(w_0) + p[N(z) - f(r)] = 0 \tag{A.4}$$

where p ∈ [0,1] is an embedding parameter, while w_0 is an initial approximation of Eqn.(A.1),which satisfies the boundary conditions. Obviously, from Eqn.(A.4) we will have

$$H(z,0) = L(z) + L(w_0) = 0 \tag{A.5}$$

$$H(z,1) = A(z) + f(r) \tag{A.6}$$

The changing process of p from zero to unity is just that of $z(r, p)$ from w_0 to $w(r)$ .In Topology, this is called deformation, while $L(z) - L(w_0)$ and $A(z) - f(r)$ are called Homotopy. According to the HPM, we can first use embedding parameter p as a small parameter, and assume that the solutions of Eqns. (A.3) and (A.4) can be written as a power series in p.

$$z = z_0 + pz_1 + p^2 z_2 + \ldots\ldots \tag{A.7}$$

setting p=1 results in the approximate solution of Eqn.(A.1)

$$w = \lim_{p \to 1} z = z_0 + z_1 + z_2 + \ldots\ldots \tag{A.8}$$

The combination of the Perturbation method and the Homotopy method is called the HPM, which eliminates the drawbacks of the traditional Perturbation methods while keeping all its advantages.

Appendix B: Analytical solutions for the dimensionless concentrations

In this appendix, we indicate how Eqns. (5) – (7) in this paper has been derived.

To find the solutions of Eqn. (1), we construct Homotopy as follows, from Eqn.(A.1),

$$(1-p)\left[\frac{dx}{dt}+ax\right]+p\left[\frac{dx}{dt}+ax-ay\right]=0 \quad \text{(B.1)}$$

The initial approximations are as follows

$$x = x_0 + px_1 + p^2x_2 + ... \quad \text{(B.2)}$$

$$P^0 : \frac{dx_0}{dt}+ax_0 = 0 \quad \text{(B.3)}$$

$$P^1 : \frac{dx_1}{dt}+ax_1 - ay_0 = 0 \quad \text{(B.4)}$$

$$P^2 : \frac{dx_2}{dt}+ax_2 - ay_1 = 0 \quad \text{(B.5)}$$

Solving the above Eqns. (B.1) to (B.5), we get

$$x_0(t) = e^{-at} \quad \text{(B.6)}$$

$$x_1(t) = \frac{a}{(a-1)}\left[e^{-t} - e^{-at}\right] \quad \text{(B.7)}$$

$$x_2(t) = \frac{abe^{-at}}{(1-a)}\left[t - \frac{e^{t(a-1)}}{(a-1)} + \frac{1}{(a-1)}\right] + \frac{ae^{-at}}{(1-a-c)}\left[\frac{e^{t(a-1)}}{(a-1)} + \frac{e^{-ct}}{c} - \frac{1}{(a-1)} - \frac{1}{c}\right] \quad \text{(B.8)}$$

Adding the above Eqns. (B.6) to (B.8), we get Eqn. (5)

Similarly we can obtain Eqns. (6) and (7).

Appendix C

```
function p4num
options= odeset ('RelTol',1e-6,'Stats','on');
%initial conditions
Xo= [1; 1; 1];
tspan = [0,1];
tic
[t,X] = ode45(@TestFunction,tspan,Xo,options);
toc
```

```
figure
hold on
%plot(t, X(:,1),'-')
plot(t, X(:,2),'-')
%plot(t, X(:,3),'-')
legend('x1','x2','x3')
ylabel('x')
xlabel('t')
return
function [dx_dt]= TestFunction(t,x)
a=2;b=0.01;c=1;
dx_dt(1) = -a*x(1)+a*x(2);
dx_dt(2) =b*x(1)-x(2)-x(1)*x(3);
dx_dt(3) =-c*x(3)+x(1)*x(2);
dx_dt = dx_dt';
return
```

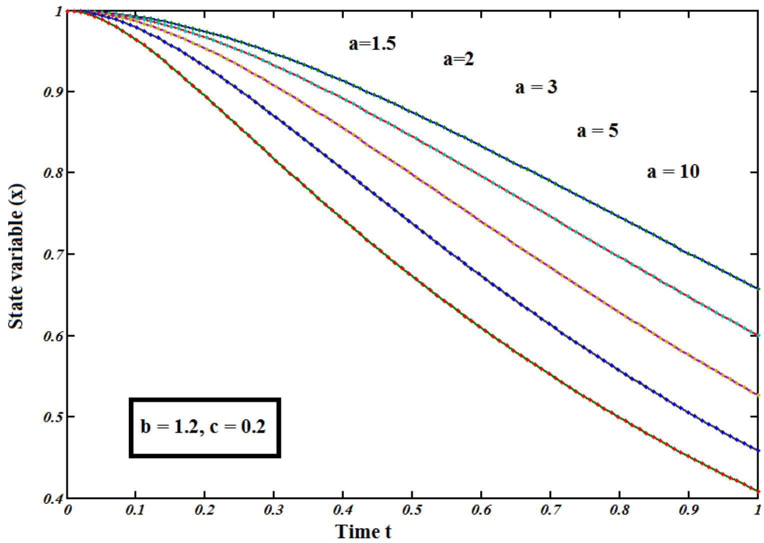

Figure 1: Plot of State variable (x) versus time t various values of the parameters a.

Solid lines represent numerical solutions whereas the dotted line represents analytical solutions.

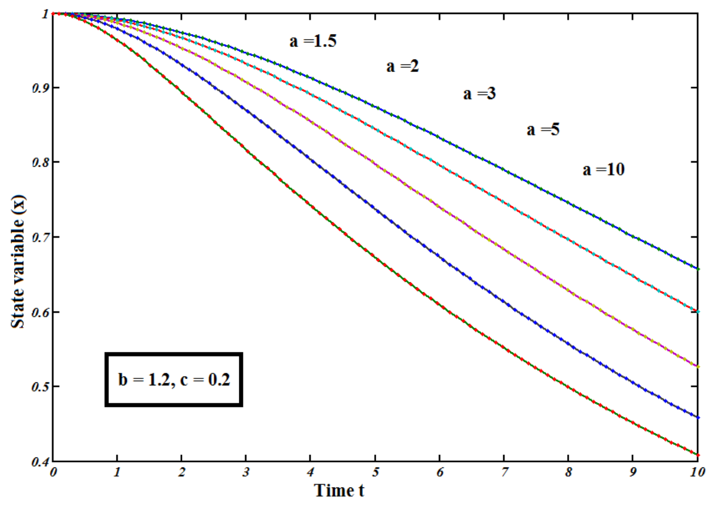

Figure 2: Plot of State variables (*x*) versus time *t* various values of the parameters *a*.

Solid lines represent numerical solutions whereas the dotted line represents analytical solutions.

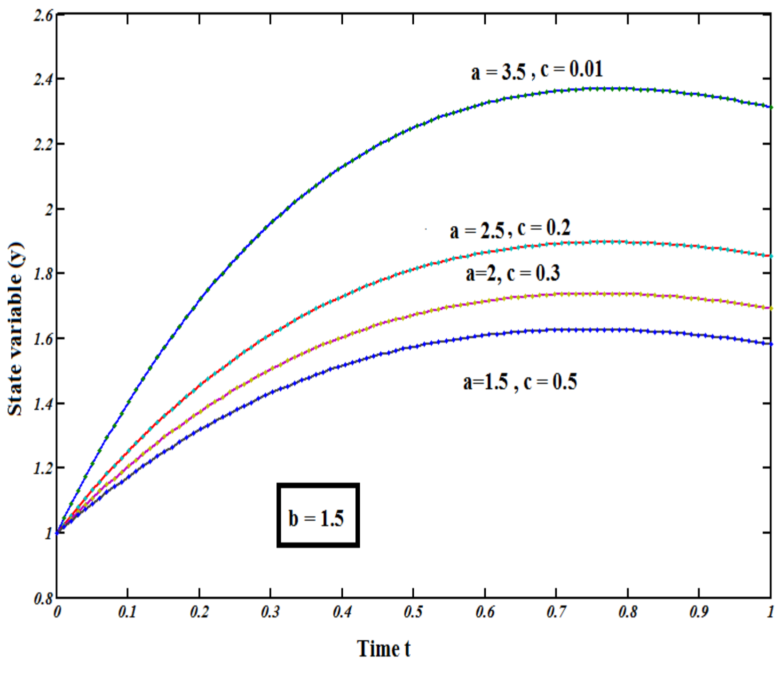

Figure 3: Plot of State variables (*y*) versus time *t* various values of the parameters *a* and *c*.

Solid lines represent numerical solutions whereas the dotted line represents analytical solutions.

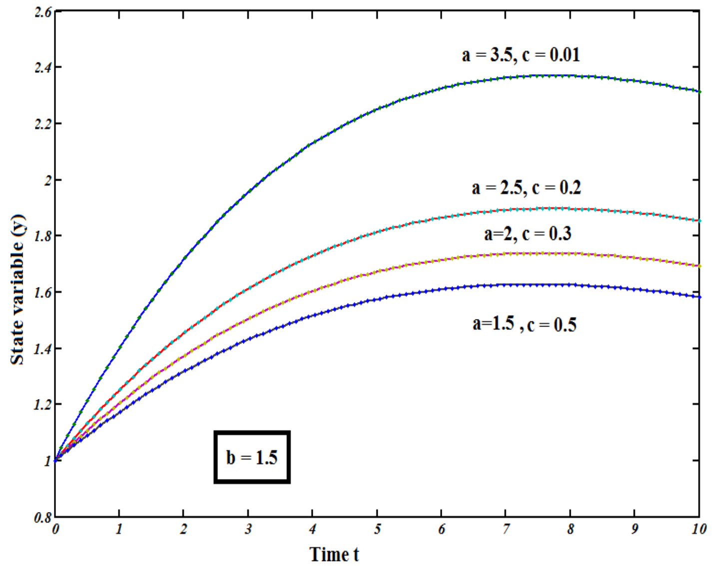

Figure 4: Plot of State variables (y) versus time t various values of the parameters a and c.

Solid lines represent numerical solutions whereas the dotted line represents analytical solutions.

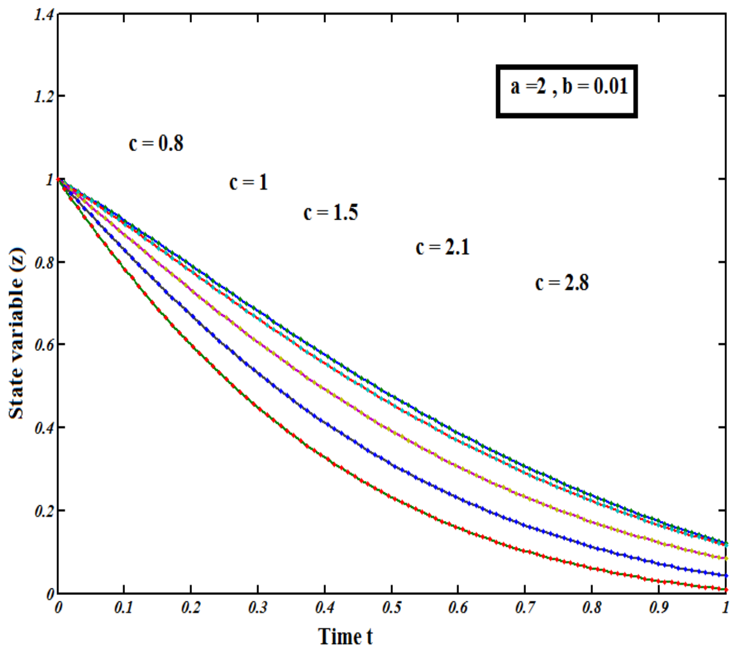

Figure 5: Plot of State variables (z) versus time t various values of the parameters a and b.

Solid lines represent numerical solutions whereas the dotted line represents analytical solutions.

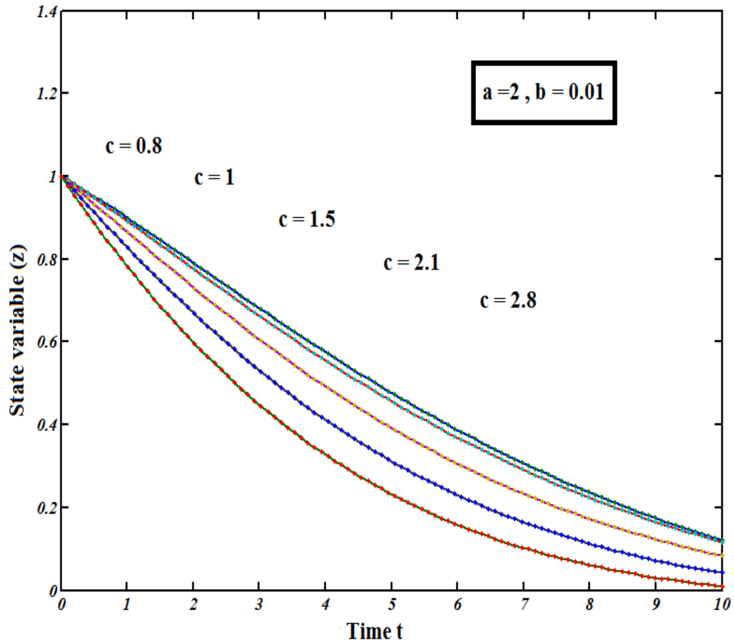

Figure 6: Plot of State variables (z) versus time t various values of the parameters a and b.

Solid lines represent numerical solutions whereas the dotted line represents analytical solutions.